Spectral Gaps of Generalized Flag Complexes

Alan Lew
Technion – Israel Institute of Technology
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Graph Laplacian

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The Laplacian of G is the $V \times V$ matrix L_G:

$$L_G(u,v) = \begin{cases} \deg(u) & \text{if } u = v, \\ -1 & \text{if } uv \in E, \\ 0 & \text{otherwise.} \end{cases}$$

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$$\lambda_2 > 0 \Leftrightarrow G \text{ is connected.}$$

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X a simplicial complex on vertex set V.
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For $\sigma \in X(k+1)$, $\tau \in \sigma(k)$ ordered simplices:
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The Coboundary Operator $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k\phi(\sigma) = \sum_{\tau \in \sigma(k)} (\sigma : \tau)\phi(\tau).$$

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$Z^k(X) =$ *k*-cocycles $= \mathrm{Ker}(d_k)$.
$B^k(X) =$ *k*-coboundaries $= \mathrm{Im}(d_{k-1})$.
k-th reduced cohomology group of X:

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Adjoint of coboundary operator: $d_k^* : C^{k+1}(X) \to C^k(X)$

$$\langle d_k \phi, \psi \rangle = \langle \phi, d_k^* \psi \rangle.$$

Higher Laplacians

$$C^{k-1}(X) \underset{d_{k-1}^*}{\overset{d_{k-1}}{\rightleftarrows}} C^k(X) \underset{d_k^*}{\overset{d_k}{\rightleftarrows}} C^{k+1}(X)$$

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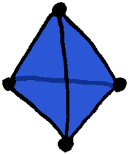
Matrix form of the k-Laplacian

$$L_k(\sigma, \tau) = \begin{cases} \deg(\sigma) + k + 1 & \text{if } \sigma = \tau, \\ (\sigma : \sigma \cap \tau)(\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k,\ \sigma \cup \tau \notin X, \\ 0 & \text{otherwise.} \end{cases}$$

Higher Laplacians

Example

X = boundary of a tetrahedron.



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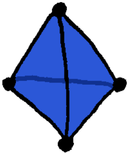

$$L_1(X) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

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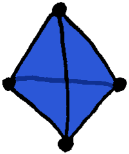

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Relation with the graph Laplacian

Let G = 1-skeleton of X. Then

$$L_0(X) = L_G + J,$$

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Simplicial Hodge Theorem

$$\mathrm{Ker}(L_k) \cong \tilde{\mathrm{H}}^k(X; \mathbb{R}).$$

In particular:

$$\mu_k > 0 \Leftrightarrow \tilde{\mathrm{H}}^k(X; \mathbb{R}) = 0.$$

Flag Complexes

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Vertex set: V, Simplices: all cliques of G.

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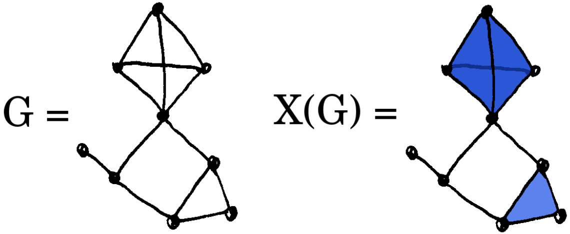

Spectral Gaps of Flag Complexes

$G = (V, E)$ a graph with $|V| = n$. Let $X = X(G)$.

Theorem[Aharoni-Berger-Meshulam]:

For $k \geq 1$

$$k\mu_k(X) \geq (k+1)\mu_{k-1}(X) - n.$$

In particular

$$\mu_k(X) \geq (k+1)\lambda_2(G) - kn.$$

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Corollary:

$$\lambda_2(G) > \frac{kn}{k+1} \implies \mu_k(X) > 0 \implies \tilde{\mathrm{H}}^k(X; \mathbb{R}) = 0.$$

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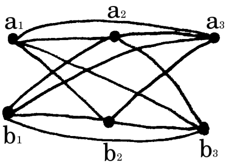
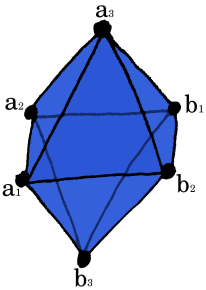
Extremal Example [Aharoni-Berger-Meshulam]:

Let $n = r\ell$ and let G be the Turán graph $T(n, r)$, i.e. the complete r-partite graph.

$G =$ 

$X(G) =$ 

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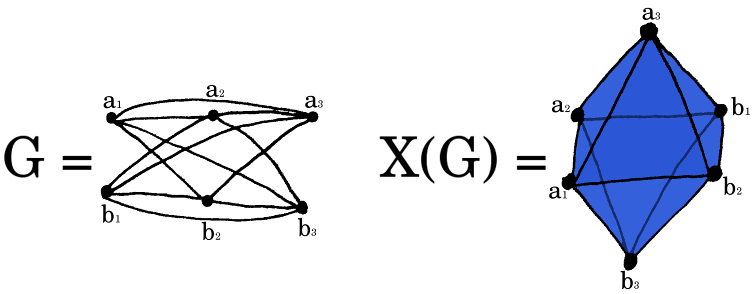

Then $\lambda_2(G) = \ell(r-1) = \frac{r-1}{r}n$, but $\tilde{\mathrm{H}}^{r-1}(X(G); \mathbb{R}) \neq 0$.

Generalized Flag Complexes

Missing Faces

X a simplicial complex on vertex set V.
$\tau \subset V$ is a missing face of X if $\tau \notin X$ but $\eta \in X$ for all $\eta \subsetneq \tau$.
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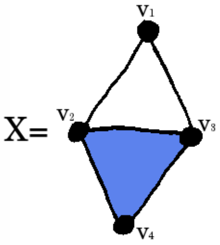

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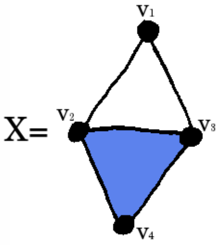

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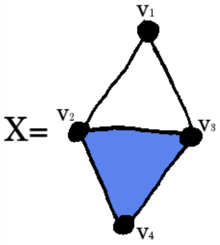

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X is a flag complex $\Leftrightarrow$ $h(X) = 1$
(missing faces= edges of the complement of G)

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Theorem:
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Corollary:

$\mu_{d-1}(X) > \left(1 - \binom{k+1}{d}^{-1}\right) n \implies \mu_k(X) > 0 \implies \tilde{\mathrm{H}}^k(X; \mathbb{R}) = 0.$

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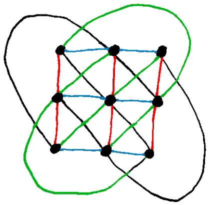

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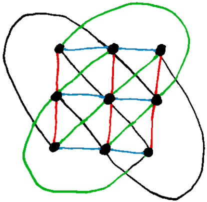

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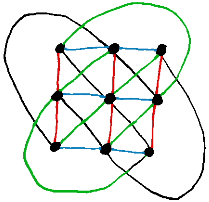

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Several more examples for $d = 2$ and $k \leq 4$, all arising from finite geometries.

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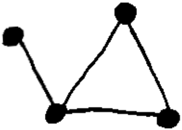

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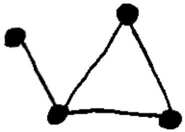

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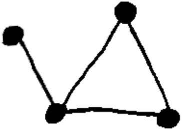

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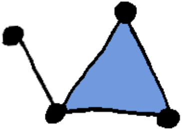

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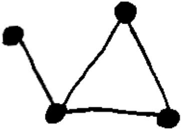

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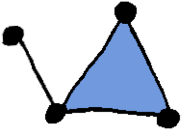

$\eta(X) = \infty$

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For example: A subset $S \subset V$ is totally dominating if every vertex $v \in V$ has a neighbor in S. Let $\tilde{\gamma}(G)$ be the minimal size of a totally dominating set.

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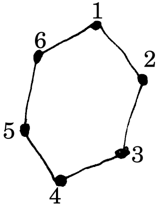

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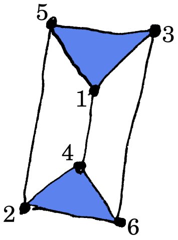

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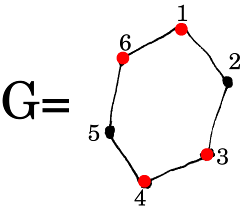

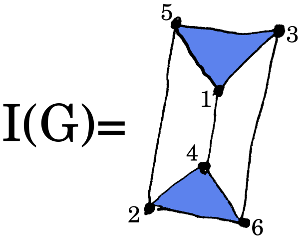

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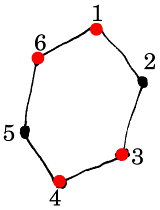
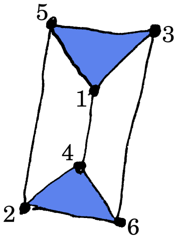
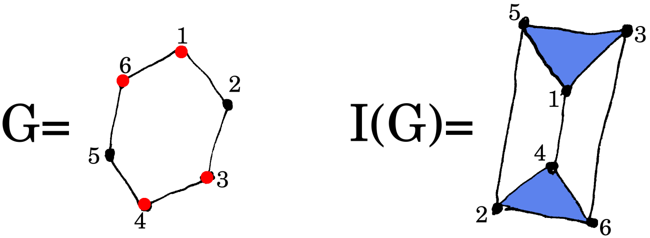
G= 

I(G)= 

$$\tilde{\gamma}(G) = 4$$

Connectivity of Independence Complexes

Example



$$\tilde{\gamma}(G) = 4$$

$$\eta(I(G)) = 2 = \tilde{\gamma}(G)/2.$$

Vector Domination of a Graph

A vector representation of G:
$P : V \to \mathbb{R}^\ell$ such that for any $v, w \in V$

$$P(v) \cdot P(w) \geq \begin{cases} 1 & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

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Identify P with an $\mathbb{R}^{|V| \times \ell}$ matrix.

A vector $\mathbf{0} \leq \alpha \in \mathbb{R}^V$ is dominating for P if $\alpha P P^T \geq \mathbf{1}$, i.e.

$$\sum_{v \in V} \alpha(v) P(v) \cdot P(u) \geq 1$$

for all $u \in V$.

Vector Domination of a Graph

The value of P:

$$|P| = \min\{\alpha \cdot \mathbf{1} : \alpha \text{ is dominating}\}$$
$$= \max\{\alpha \cdot \mathbf{1} : \alpha \geq \mathbf{0},\ \alpha P P^T \leq \mathbf{1}\}.$$

Define $\Gamma(G)$ to be the supremum of $|P|$ over all vector representations of G.

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Theorem[Aharoni-Berger-Meshulam]:

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Vector Domination of a Simplicial Complex

Let X be a simplicial complex on vertex set V.
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Colorful Simplices

Let $V_1, \ldots, V_m$ be a partition of V. A simplex $\sigma \in X$ is colorful if $|\sigma \cap V_i| = 1$ for all $i = 1, \ldots, m$.

Theorem[Aharoni-Haxell, Meshulam]:

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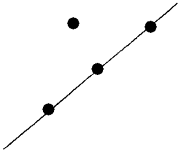

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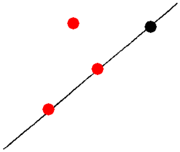

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$\varphi^*(S) \geq \varphi(S)$ (the characteristic function of any subset of S in general position is in fractional general position).

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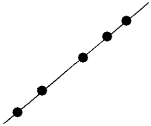

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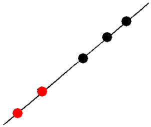

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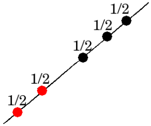

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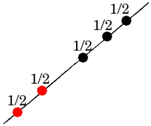

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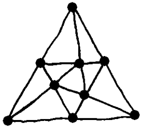

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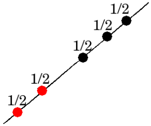

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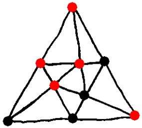

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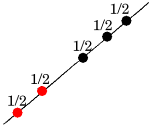

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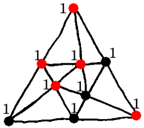

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Theorem:

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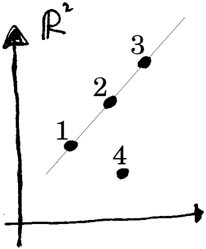

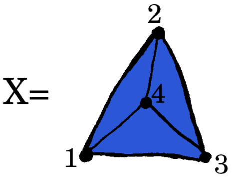

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$$\Gamma(X) \ge |P| \ge \varphi^*(V)/d.$$

Spectral Gaps and Minimal Degrees

X a simplicial complex on vertex set V, $|V| = n$, with $h(X) = d$.
Let $k \geq 0$.
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As a consequence:

Theorem[Adamaszek]:
$\tilde{\mathrm{H}}^k(X; \mathbb{R}) = 0$ for all $k > \frac{d}{d+1}n - 1$.

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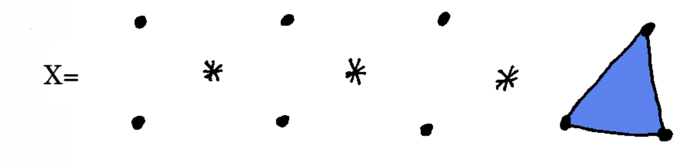

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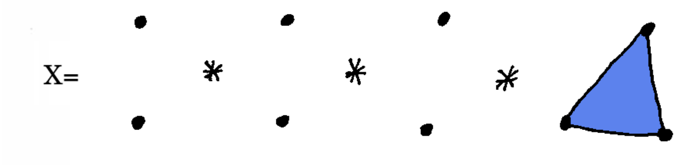

For example for $k = 2$:

$$\mu_2(X) = 3,$$
$$\delta_2(X) = 3.$$

Indeed $\mu_2(X) = 2(\delta_2(X) + 2 + 1) - n = 12 - 9 = 3.$

Uniqueness of Extremal Examples for Flag Complexes

For flag complexes ($h(X) = 1$) these are the only extremal examples:

Theorem:
Let X be a flag complex on vertex set V, $|V| = n$, such that $\mu_k(X) = 2(k+1) - n$ for some $k \geq 0$. Then

$$X \cong \underbrace{\Delta_1^{(0)} * \Delta_1^{(0)} * \cdots * \Delta_1^{(0)}}_{(n-k-1) \text{ times}} * \Delta_{2(k+1)-n-1}.$$