Relaxation in the space of Bounded Hessian

Adrian Hagerty

July 11, 2017

Bounded Hessian

The space of Bounded Hessian, $BH(\Omega;\mathbb{R}^d)$ is defined as

$$
\begin{aligned}
BH(\Omega;\mathbb{R}^d) :&= \{u \in L^1(\Omega;\mathbb{R}^d) : Du \in BV(\Omega;\mathbb{R}^{d\times n})\} \\
&= \{u \in W^{1,1}(\Omega;\mathbb{R}^d) : D(\nabla u) \in \mathcal{M}(\Omega;\mathbb{R}^{d\times n\times n})\}
\end{aligned}
$$

Of particular note is that although a function $u \in BH(\Omega;\mathbb{R}^d)$ will not have sharp changes, called "jumps", it may have "kinks", or jumps in ∇u.

This property in particular makes the space BH the natural setting for problems in the fields of image processing and material science.

Image Processing

Bergounioux and Piffet, 2010: Modification of Rudin-Osher-Fatemi model for image denoising using functions of Bounded Hessian.

Decompose a noisy image $u_d \in L^2(\Omega)$ into $u_d = u + v$ via

$$F(v) = \frac{1}{2}\|u_d - v\|^2_{L^2(\Omega)} + \lambda|D(\nabla v)|(\Omega) + \delta\|v\|_{W^{1,1}(\Omega)}$$

- $v \in BH(\Omega)$ is a regularized second order part which minimizes $F(v)$.
- $u \in L^2(\Omega)$, $u = u_d - v$ represents noise or texture.

Avoids the so-called "staircasing effect" observed in ROF by disallowing jumps.

Plate Theory

Models of elastic-perfectly plastic materials:

- Demengel, 1984, 1989
- Carriero, Leaci, Tomarelli, 1992, 2004
- Bleyer, Carlier, Duval, Mirebeau, Peyré, 2016

Introduced by Demengel, involve energy

$$F(u) = \int_\Omega \psi(\nabla^2 u)$$

$$|\psi(H)| \leq C(1 + |H|)$$

Sequences with $F(u_n)$ bounded will be compact in BH, making it the natural setting for such problems.

Relaxation Problem

We approach the general problem of relaxation in BH.

$$F(u) := \int_\Omega f(x, \nabla^2 u)dx, \;\; u \in W^{2,1}(\Omega;\mathbb{R}^d)$$

Goal: Find integral representation of $\mathcal{F} : BH(\Omega;\mathbb{R}^d) \to \mathbb{R}$ via

$$\mathcal{F}(u) := \inf\left\{ \liminf_{n\to\infty} F(u_n) : u_n \xrightarrow{W^{1,1}} u;\;\; \nabla^2 u \overset{\star}{\rightharpoonup} D(\nabla u) \right\}$$

$\mathcal{A}$-free measures

Recently studied in the sense of $\mathcal{A}$-free measures introduced by Fonseca & Müller, 1999.

Rabasa, De Phillipis, & Rindler, 2017: Up to a BH density result,

$$\boxed{\mathcal{F}(u) = \int_\Omega \mathcal{Q}_2 f(x, \nabla^2 u))dx + \int_\Omega (\mathcal{Q}_2 f)^\infty \left(x, \frac{D_s(\nabla u)}{|D_s(\nabla u))|} \right) d|D_s(\nabla u)|}$$

where $\mathcal{Q}_2 f$ is the 2-quasiconvex envelope of f and $(\mathcal{Q}_2 f)^\infty$ is the recession function

$$\mathcal{Q}_2 f(x, H) := \inf \left\{ \int_Q f(x, H + \nabla^2 \phi(y))dy) : \phi \in W_0^{2,\infty}(Q; \mathbb{R}^d) \right\}$$

$$(\mathcal{Q}_2 f)^\infty (x, H) = \limsup_{t \to \infty} \frac{\mathcal{Q}_2 f(x, tH)}{t}.$$

BH Density Result

To apply Rabasa, De Phillipis & Rindler, we need the following result:

Proposition (BH Density)

For $u \in BH(\Omega; \mathbb{R}^d)$*, there exist* $u_n \in C^\infty(\Omega; \mathbb{R}^d)$ *such that* $u_n \xrightarrow{W^{1,1}} u$*,* $\nabla^2 u \xrightarrow{\langle\,\cdot\,\rangle} D(\nabla u)$

This is area-strict convergence, as discussed in Kristensen and Rindler in 2009. We say measures $\mu^n \xrightarrow{\langle\,\cdot\,\rangle} \mu \in \mathcal{M}(\Omega, \mathbb{R}^d)$ if $\mu^n \overset{\star}{\rightharpoonup} \mu$ and

$$\int_\Omega \sqrt{1 + |\mu^n_{ac}|^2} dx + |\mu^n_s|(\Omega) \to \int_\Omega \sqrt{1 + |\mu_{ac}|} dx + |\mu_s|(\Omega)$$

We say that functions $f_n \xrightarrow{\langle\,\cdot\,\rangle} \mu$ if $f_n \, \mathcal{L}^N \restriction \Omega \xrightarrow{\langle\,\cdot\,\rangle} \mu$

Remark

Area-strict convergence is strictly stronger than the notion of strict convergence.

Take $\chi_{(0,\frac{1}{2})}$ as a function on $I = (0,1)$. Extend it periodically to some $\widetilde{\chi}$ and define

$$\psi_n(x) := \widetilde{\chi}(nx).$$

Then, by the Riemann-Lebesgue Lemma we have $\psi_n \overset{\star}{\rightharpoonup} \psi = \frac{1}{2}$. It is clear that $|\psi_n|(I) = |\psi|(I) = \frac{1}{2}$, but

$$\int_I \sqrt{1+|\psi_n|^2}dx = \frac{1+\sqrt{2}}{2} > \frac{\sqrt{5}}{2} = \int_I \sqrt{1+|\psi|^2}dx.$$

The power of area-strict convergence is the following Reshetnyak type result:

Theorem (Kristensen, Rindler)

Let $f \in \mathbf{E}(\Omega \times \mathbb{R}^{m\times d})$. Then, the function

$$G(\mu) = \int_\Omega f(\mu_{ac})dx + \int_\Omega f^\infty\left(\frac{d\mu_s}{d|\mu_s|}\right)d|\mu_s|,\ \mu \in \mathcal{M}(\Omega;\mathbb{R}^d)$$

is continuous with respect to area-strict convergence.

Where $\mathbf{E}(\Omega \times \mathbb{R}^{d\times m})$ consists of functions $f \in C(\Omega \times \mathbb{R}^{d\times m})$ such that the transformed function

$$(x,\xi) \to (1-|\xi|)\ f(x,(1-|\xi|)^{-1}\xi), (x,\xi) \in \Omega \times B(0,1)$$

can be extended continuously to to $\overline{\Omega \times B(0,1)}$.

Results

Theorem (Hagerty)

For all $\mu \in \mathcal{M}(\mathbb{R}^N; \mathbb{R}^d)$ *such that* $|\mu|(\partial\Omega) = 0$*, there exist smooth functions* $\mu_\varepsilon := \mu * \phi_\varepsilon$*, where* ϕ_ε *are the standard mollifiers, such that* $\mu_\varepsilon \xrightarrow{\langle\ \cdot\ \rangle} \mu$ *in* Ω*.*

Corollary

We obtain the density result in BH *assuming some boundary regularity. (Currently* C^2 *but it is believed that this can be extended to Lipschitz.)*

A similar area-strict density result for BV functions is a corollary of existing BV relaxation results, Ambrosio & Dal Maso, 1992, Fonseca & Müller, 1993.

Proof of continuity:

Step 1: $g(\xi) := \sqrt{1+|\xi|^2}$ is convex with linear growth, so

$$\mu \to \int_\Omega g(\mu_{ac})dx + \int_\Omega g^\infty\left(\frac{d\mu_s}{d|\mu_s|}\right)d|\mu_s|$$

is lower semicontinuous with respect to strict convergence, ie $\mu_n \overset{\star}{\rightharpoonup} \mu$, $|\mu_n|(\Omega) \to |\mu|(\Omega)$.

Step 2: Use Jensen's inequality to establish the pointwise inequality

$$g(\mu_{ac} * \phi_\varepsilon(x)) \leq g(\mu_{ac}) * \phi_\varepsilon(x)$$

which allows us to employ the blow-up method (Fonseca & Müller 1993) in the support of $|\mu_{ac}|$

Singular part: A slightly more complicated inequality, also established via Jensen's inequality,

$$g(\mu_s * \phi_\varepsilon(x)) \leq \frac{1}{t_\varepsilon(x)} \int_{\mathbb{R}^N} g\left(t_\varepsilon(x) \frac{d\mu_s}{d|\mu_s|}(y)\right) \phi_\varepsilon(x-y) d|\mu_s|(y)$$

where

$$t_\varepsilon(x) := \int_{\mathbb{R}^N} \phi_\varepsilon(y-x) d|\mu_s|(y) \approx \frac{|\mu_s|(B(x,\varepsilon))}{\varepsilon^N}.$$

Since $\frac{|\mu_s|(B(x,\varepsilon))}{\varepsilon^N} \to \infty$ for $|\mu_s|$ almost every $x \in \Omega$, we can get $|\mu_s|$ a.e.

$$\begin{aligned}
\frac{1}{t_\varepsilon(x)} \int_{\mathbb{R}^N} g\left(t_\varepsilon(x) \frac{d\mu_s}{d|\mu_s|}(y)\right) \phi_\varepsilon(x-y) d|\mu_s|(y) \\
\approx \int_{\mathbb{R}^N} g^\infty\left(\frac{d\mu_s}{d|\mu_s|}(y)\right) \phi_\varepsilon(x-y) d|\mu_s|(y) \\
= g^\infty\left(\frac{d\mu_s}{d|\mu_s|}(\cdot)\right) |\mu_s| * \phi_\varepsilon(x)
\end{aligned}$$

Thank you for your attention!

▷ References

- Ambrosio, Dal Maso, 1992
- Bergounioux, Piffet, 2010
- Demengel, 1984, 1989
- Blyer, Carlier, Duval, Mirebeau, Peyré, 2016
- Carriero, Leaci, Tomarelli, 1992, 2004
- Fonseca, Müller, 1993, 1999
- Kristensen, Rindler, 2009
- Rabasa, De Phillipis, Rinlder, 2017