On graph irregularity strength

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Abstract

An assignment of positive integer weights to the edges of a simple graph G is called irregular if the weighted degrees of the vertices are all different. The irregularity strength, $s(G)$, is the maximal weight, minimized over all irregular assignments. In this paper we show, that $s(G) \leq c_1 n/\delta$, for graphs with maximum degree $\Delta \leq n^{1/2}$ and minimum degree δ, and $s(G) \leq c_2(\log n)n/\delta$, for graphs with $\Delta > n^{1/2}$, where c_1 and c_2 are explicit constants. To prove the result, we are using a combination of deterministic and probabilistic techniques.

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1 Introduction:

Perhaps the second oldest "fact" in graph theory is that in a simple graph, two vertices must have the same degree. This fact no longer holds for multigraphs. By an irregular multigraph we mean one in which each vertex has a different degree. Hence, a natural question would be: What is the least number of edges we would need to add to a graph in order to convert a simple graph into an irregular multigraph?

Another way to view this question is via an assignment of integer weights to the edges of the graph. Given a simple graph G of order n, an assignment $f : E(G) \to \{1, ..., w\} = [w]$ of positive integers weights to the edges of G is called *irregular* if the weighted degrees, $f(v) = \sum_{u \in N(v)} f(uv)$ of the vertices are all different. The *irregularity strength*, $s(G)$, is the maximal weight w, minimized over all irregular weight assignments, and is set to ∞ if no such assignment is possible. Clearly, $s(G) < \infty$ if and only if G contains no isolated edges and at most one isolated vertex.

The irregularity strength was introduced in [3] by Chartrand *et al.* . The irregularity strength of regular graphs was considered by Faudree and Lehel in [4]. They showed that if G is a d-regular graph of order n, $d \geq 2$, then $s(G) \leq \lceil n/2 \rceil + 9$, and they conjectured that $s(G) = \left\lceil \frac{n+d-1}{d} \right\rceil + c$ for some constant c. This conjecture comes from the lower bound $s(G) \geq \left\lceil \frac{n+d-1}{d} \right\rceil$. For general graphs with finite irregularity strength, Aigner and Triesch [1] showed that $s(G) \leq n - 1$ if G is connected and $s(G) \leq n + 1$ otherwise. Nierhoff [8] refined their method to show $s(G) \leq n - 1$ holds for all graphs with finite irregularity strength, except for K_3. We will provide an improvement of both the Faudree-Lehel bound and the Aigner-Triesch-Nierhoff bound in this paper.

For a review of other results and open problems in this area we refer the reader to a survey paper by Lehel [7].

In this paper all graphs are simple of order n. The degree of a vertex v is denoted by d_v or $deg(v)$, we shall denote the minimum degree of G by δ and the maximum degree by Δ. For terms not found here see [2] or [6]. Our upper bounds on $s(G)$ involve a function of n and δ or both δ and Δ, and are stated in the next Theorem.

Theorem 1 *Let G be a graph with no isolated vertices or edges.*

(a) *If $\Delta \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 7n\left(\frac{1}{\delta} + \frac{1}{\Delta}\right)$.*

(b) *If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq \Delta \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 60n/\delta$.*

(c) *If $\Delta \geq \lfloor n^{1/2} \rfloor + 1,\ \delta \geq \lceil 6 \log n \rceil$ then $s(G) \leq 336(\log n)n/\delta$.*

For regular graphs, we get the following Theorem with improved constants.

Theorem 2 *Let G be a d-regular graph with no isolated vertices or edges.*

(a) *If $d \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 10n/d + 1$.*

(b) *If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq d \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 48n/d + 1$.*

(c) *If $d \geq \lfloor n^{1/2} \rfloor + 1$, then $s(G) \leq 240(\log n)n/d + 1$.*

Observe that both (a) and (b) give bounds of the correct order of magnitude. If $\Delta \geq \lceil n^{1/2} \rceil + 1$ and $\delta < \lfloor 6 \ln n \rfloor$, Theorem 1 does not apply, but we can still make the following statement:

Theorem 3 *Let G be a graph with no isolated vertices or edges. If n is sufficiently large, then $s(G) \leq 14n/\delta^{1/2}$.*

To explain the main technique used to prove all results let us define

$$m_g = \max_{X \subseteq V(G)} \{|X| : g(v) = g(u) \text{ for all } v, u \in X\},$$

where g is defined as a weight assignment, i.e., $g : E(G) \to \{1, 2, \ldots, w\} = [w]$, for some integer w. In the deterministic part of our proof (see Lemma 4) we show that $s(G) \leq 3(w+1)m_g$. Next, we use probabilistic tools to establish bounds on m_g. Here the idea is to assign weights to edges from the set $\{1, 2\}$ or $\{1, 2, 3\}$, and show that for such weightings, there exist assignments with m_g of the order n/δ or $n \log n/\delta$ (see Lemmas 7, 8 and 9).

2 Deterministic Lemmas

The next two Lemmas will be fundamental to our results. Their proofs follow below.

Lemma 4 *Let G be a graph without isolated vertices or isolated edges. Let $g : E(G) \to [w]$ be a weight assignment. Then there exists an irregular assignment $f : E(G) \to \{2m_g, \ldots, (3w+1)m_g\}$.*

Lemma 5 *Let G be a d-regular graph without isolated vertices or isolated edges. Let $g : E(G) \to [w]$ be a weight assignment. Then there exists an irregular assignment $f : E(G) \to [(3w-1)m_g + 1]$.*

We begin with a lemma needed to prove Lemma 4. We will call a tree with at most one vertex of degree greater than two, and k vertices of degree one, a *generalized k-star.*

Lemma 6 *Let G be a graph without isolated vertices or isolated edges. Then G has a factor consisting of generalized stars of order at least three.*

Proof: Let T be a spanning tree of a component of G. Note that $|V(T)| \geq 3$ by our hypothesis. We show that T can be broken into disjoint generalized stars that together span $V(T)$. Then repeating this argument on each component produces the result.

To do this we induct on $|U|$, where $U = \{u \in V(T) | deg_T(u) \geq 3\}$. If $|U| \leq 1$ we are done, as T is itself a generalized star. Now assume the result holds on any tree T with $|U| = l \geq 1$ and suppose T is a tree with $|U| = l+1$. Now root T at $u \in U$ and select any vertex $v \in U, v \neq u$, such that the distance in T between u and v is maximum over all vertices of U. Let T_v be the subtree of T rooted at v and consider $T' = T \setminus T_v$. This tree has $|U| = l$ and by the induction hypothesis, we can find generalized stars in T' that span $V(T')$. Further, the tree T_v is, by our choice of v, a generalized star of order at least three. This star, together with the collection of stars that spans T', spans T, completing the proof. ∎

Proof of Lemma 4. Denote the weight class of a vertex $v \in V(G)$ as

$$C_v = \{u \in V(G) : g(u) = g(v)\}.$$

Define a new weight function $\hat{f} : E \to [3m_g w]$ by $\hat{f}(e) = 3m_g g(e)$. Note that the weight classes are unchanged under this function. Let $\mathcal{S}$ be a generalized star factor of G, guaranteed by Lemma 6. We select one generalized star S from $\mathcal{S}$. Let u be a vertex of maximum degree in S and suppose that S consists of t paths rooted at u. Let $u_1, u_2, \ldots, u_t$ be the neighbors of u in S. Consider the first branch (path) of S, say $v_1, v_2, \ldots, v_r$, where $v_1 = u_1$ and $r \geq 2$ (if such a branch of S exists). Now begin with the last edge $v_r v_{r-1}$. We change the weight of this edge as follows. Put $f(v_r v_{r-1}) = \hat{f}(v_r v_{r-1}) + x$, where x is selected from the set $L = \{0, -1, \ldots, -(m_g - 1)\}$ in such a way that $f(v_r)$, its new weighted degree, is different from the current weighted degrees of any vertex from $C_{v_r} \setminus \{v_r\}$. Since $|C_{v_r}| \leq m_g$, it is always possible to select an appropriate x. We now repeat this process to the edges $v_{r-1}v_{r-2}, v_{r-2}v_{r-3}, \ldots v_2 v_1$, thus making $f(v_{r-1}), f(v_{r-2}), \ldots, f(v_2)$ unique also. To complete the first phase, repeat the procedure on the paths emanating from $u_2, u_3, \ldots, u_t$, in this order.

It remains to adjust the weights of the star centered at u. So, we change the weights of the edges $uu_1, uu_2, \ldots, uu_{t-1}$, one by one, starting at uu_1. Let $f(uu_i) = \hat{f}(uu_i) + y_i$, where y_i is chosen from the set $L' = \{-m_g, -(m_g - 1), \ldots, m_g - 1, m_g\}$, in such a way that $f(u_i)$, $i = 1, 2, \ldots, t-1$, the new weighted degree of u_i, is different from the current weighted degrees of any vertex from $C_{u_i} \setminus \{u_i\}$ and, additionally, such that $\sum_{k=1}^{i} y_i$ belongs to the set $(L \cup \{-m_g\}) \setminus \{f(u_t v) - \hat{f}(u_t v)\}$, where v is the second vertex of the path starting in u_t (if no such vertex v exists, use instead $(L \cup \{-m_g\}) \setminus \{0\}$). Now we are left with uu_t. Observe that u and u_t have different weighted degrees at this time. Now let $f(uu_t) = \hat{f}(uu_t) + x$, where $x \in L' \setminus \{-m_g\}$, such that both $f(u)$ and $f(u_t)$ are unique in their respective classes. This is possible, since there are $2m_g$ options, and C_u and C_{u_t} can only block $2(m_g - 1)$ of these. Finally, repeat the process for all remaining stars $S \in \mathcal{S}$.

Now for every weight class C_u, all vertices have different weighted degrees under f. The weighted degrees were altered from $\hat{f}$ by total values from the range $\{-2m_g + 1, \ldots, m_g\}$, the different classes were at least $3m_g$ apart from each other under $\hat{f}$, so f is an irregular assignment to the set $\{2m_g, 2m_g + 1, \ldots, 3m_g w + m_g\}$. ∎

Proof of Lemma 5. Use Lemma 4 to get an irregular weight assignment $f' : E(G) \to \{2m_g, 2m_g + 1, \ldots, 3m_g w + m_g\}$. Now define $f : E(G) \to [(3w - 1)m_g + 1]$ by $f(e) = f'(e) - 2m_g + 1$. This assignment is irregular, since the weighted degree of every vertex is reduced by $d(2m_g - 1)$. ∎

3 Probabilistic Lemmas

The following two lemmas will be used to get bounds on the irregularity strength of graphs with maximal degree $\Delta \leq n^{1/2}$. Again, the proofs follow below.

Lemma 7 *Let G be a graph. If $\Delta \leq (n/\ln n)^{1/4}$, then $\exists g : E(G) \to \{1,2\}$ such that $m_g \leq \frac{n}{\delta} + \frac{n}{\Delta}$.*

Lemma 8 *Let G be a graph. If $\Delta \leq n^{1/2}$, then $\exists g :\ E(G) \to \{1,2,3\}$ such that $m_g \leq 6n/\delta$.*

The next lemma is used for graphs with $\Delta > n^{1/2}$.

Lemma 9 *Let G be a graph. If $n \geq 10$ and $\delta \geq 10 \log n$, then $\exists g :\ E(G) \to \{1,2\}$ such that $m_g \leq 48(\log n)n/\delta$.*

Finally, we state the lemma which provides bounds on m_g, without any restrictions on vertex degrees of a graph G, but for sufficiently large n only.

Lemma 10 *Let G be a graph. If n is sufficiently large, then $\exists g :\ E(G) \to \{1,2\}$ such that $m_g \leq 2n/\delta^{1/2}$.*

Since the proofs of both Lemma 7 and Lemma 9 use the same model of assigning weights to the edges, at random, we will present their proof together.

Proof of Lemmas 7 and 9.
Let $X_v, v \in V$ be independent random variables with uniform distibution over the interval $[0,1]$, and then for $e = uv \in E$, let

$$g(e) = \begin{cases} 2 & \text{if } X_u + X_v \geq 1 \\ 1 & \text{if } X_u + X_v < 1 \end{cases}.$$

For the non-negative integer $y \in \{0,1,\ldots,d_v\}$,

$$\mathbf{Pr}(g(v) = d_v + y) = \int_{x=0}^{1} \binom{d_v}{y} x^y (1-x)^{d_v - y} dx = \frac{1}{d_v + 1} \leq \frac{1}{\delta + 1}. \tag{1}$$

It follows for every y with $\delta \leq y \leq 2\Delta$ and $Z_y = |\{v \in V : g(v) = y\}|$ that

$$\mathbf{E}(Z_y) \leq \frac{n}{\delta + 1}. \tag{2}$$

To prove Lemma 7, we assume that G is a graph with maximum degree $\Delta \leq (n/\log n)^{1/4}$.

We apply the Hoeffding-Azuma inequality, see e.g. Janson, Łuczak and Ruciński [6]. Changing the value of an X_v can only change the value of Z_y by at most $\Delta + 1$. It follows that for $t > 0$,

$$\mathbf{Pr}(Z_y \geq \mathbf{E}(Z_y) + t) \leq \exp\left\{-\frac{t^2}{2n(\Delta+1)^2}\right\}. \tag{3}$$

Putting $t = \frac{n}{\Delta+1}$ and using (2) we see that

$$\mathbf{Pr}(Z_y \geq \mathbf{E}(Z_y) + t) < \frac{1}{2\Delta},$$

and thus

$$\mathbf{Pr}(\exists y : \; Z_y \geq \frac{n}{\delta} + \frac{n}{\Delta}) < 1,$$

and Lemma 7 follows. ∎

We now prove Lemma 9. We use the Markov inequality for $t, k > 0$ and any event $\mathcal{E}$, to obtain

$$\mathbf{Pr}(Z_y > t \mid \mathcal{E}) \leq \frac{\mathbf{E}\left(\binom{Z_y}{k} \mid \mathcal{E}\right)}{\binom{t}{k}}. \tag{4}$$

But

$$\mathbf{E}\left(\binom{Z_y}{k} \mid \mathcal{E}\right) = \sum_{|S|=k} \mathbf{Pr}(g(v) = y, v \in S \mid \mathcal{E}). \tag{5}$$

Now fix $S = \{v_1, v_2, \ldots, v_k\}$ in (5). For $v \in S$ let $N_S(v) = N(v) \setminus S$, and let $\mu(v) = |N_S(v)|$. Note that $d_v - \mu(v) \leq k - 1$. For $v \in S$ let $\xi_1 < \xi_2 < \cdots < \xi_{d_v}$ be the values of $X_u, u \in N(v)$, sorted in increasing order and let $\eta_1 < \eta_2 < \cdots < \eta_{\mu(v)}$ be the values of $X_u, u \in N_S(v)$, also sorted in increasing order.

Note that, in general, if $\xi_1 < \xi_2 < \cdots < \xi_s$ is the sequence of order statistic from the uniform distribution over $[0,1]$, then ξ_i has the same distribution as $(Y_1 + Y_2 + \cdots + Y_i)/(Y_1 + Y_2 + \cdots + Y_{s+1})$ where $Y_1, Y_2, \ldots, Y_{s+1}$ is a sequence of independent random variables, each having exponential distribution with mean one, see for example Ross, Theorem 2.3.1 [9].

To prove the lemma we need to show the following general statement.

Lemma 11 *Let $Y_1, Y_2, \ldots, Y_s$ be a sequence of independent random variables, each having exponential distribution with mean one. Then for any real $a > 0,\ 0 < b < 1$ we have*

$$\begin{aligned}
\mathbf{Pr}(Y_1 + \ldots + Y_s \geq (1+a)s)) &\leq ((1+a)e^{-a})^s \\
\mathbf{Pr}(Y_1 + \ldots + Y_s \leq (1-b)s)) &\leq ((1-b)e^{b})^s.
\end{aligned}$$

Proof:

$$\begin{aligned}
\mathbf{Pr}(Y_1 + \ldots + Y_s \geq t) &\leq \mathbf{Pr}(e^{\lambda(Y_1 + \cdots + Y_s - t)} \geq 1) \\
&\leq e^{-\lambda t}\mathbf{E}(e^{\lambda(Y_1 + \cdots + Y_s)}) \\
&= \frac{e^{-\lambda t}}{(1-\lambda)^s},
\end{aligned}$$

provided $\lambda \in (0,1)$.

So putting $t = (1+a)s$, we see that

$$\mathbf{Pr}(Y_1 + \ldots + Y_s \geq (1+a)s) \leq \left(\frac{e^{-\lambda(1+a)}}{1-\lambda}\right)^s = ((1+a)e^{-a})^s$$

on putting $\lambda = a/(1+a)$.

A similar argument shows that

$$\mathbf{Pr}(Y_1 + \ldots + Y_s \leq (1-b)s) \leq ((1-b)e^{b})^s,$$

completing the proof of Lemma 11. ∎

Let $k = \lfloor \log n \rfloor$ and

$$\mathcal{E} = (\Theta < (16 \log n)/\delta),$$

where

$$\Theta = \max_{v \in V} \Theta_v, \quad \text{and} \quad \Theta_v = \max_{0 \leq i \leq d_v - 2k + 1} \xi_{i+2k} - \xi_i \ .$$

Here, by default, we take $\xi_0 = 0$ and $\xi_{d_v+1} = 1$.

Now, observe that $g(v) = y$ implies

$$1 - X_v \in [\xi_{2d_v - y}, \xi_{2d_v - y + 1}] \subset [\eta_{2d_v - y - k + 1}, \eta_{2d_v - y + 1}] \subseteq [\xi_{2d_v - y - k + 1}, \xi_{2d_v - y + k}].$$

In the above formula, we take $\xi_j = \eta_j = 0$ for $j \le 0$, and $\xi_{d_v + j} = \eta_{\mu(v) + j} = 1$ for $j \ge 1$.

Applying Lemma 11 to the order statistics defining Θ, we see that

$$\begin{aligned}
\mathbf{Pr}(\neg\mathcal{E}) &= \mathbf{Pr}\left(\exists v \in V : \Theta_v \ge \frac{16 \log n}{\delta}\right) \\
&\le n\mathbf{Pr}\left(\exists 0 \le i \le \Delta - 2k + 1 : \frac{Y_i + \cdots + Y_{i+2k-1}}{Y_1 + \cdots + Y_{\delta+1}} \ge \frac{16 \log n}{\delta}\right) \\
&\le n\mathbf{Pr}(Y_1 + \cdots + Y_{\delta+1} \le \delta/2) + n^2 \mathbf{Pr}(Y_1 + \cdots + Y_{2k} \ge 8k) \\
&\le n(e^{1/2}/2)^{\delta+1} + n^2(4e^{-3})^{2k} \\
&\le 1/10. \qquad (6)
\end{aligned}$$

Further,

$$\mathbf{Pr}(g(v) = y, v \in S \mid \mathcal{E}) \le \mathbf{Pr}(1 - X_{v_i} \in [\eta_{2d_{v_i} - y - k + 1}, \eta_{2d_{v_i} - y + 1}],\ i = 1, 2, \ldots, k \mid \mathcal{E})$$

$$\begin{aligned}
&\le 2\mathbf{Pr}(1 - X_{v_i} \in [\eta_{2d_{v_i} - y - k + 1}, \eta_{2d_{v_i} - y - k + 1} + \tfrac{16 \log n}{\delta}],\ i = 1, 2, \ldots, k) \\
&\le 2\left(\frac{16 \log n}{\delta}\right)^k.
\end{aligned}$$

From (4) and (5) we obtain

$$\mathbf{Pr}(\exists y :\ Z_y > t \mid \mathcal{E}) \le 2n \binom{t}{k}^{-1} \binom{n}{k} \left(\frac{16 \log n}{\delta}\right)^k.$$

Putting $t = 48(\log n) n \delta^{-1}$ together with (6) establishes

$$\mathbf{Pr}(\exists y :\ Z_y > t) \le \mathbf{Pr}(\exists y :\ Z_y > t \mid \mathcal{E}) + \mathbf{Pr}(\neg\mathcal{E}) < 1,$$

proving Lemma 9.

∎

Proof of Lemma 8. For every vertex v independently assign a number W_v from $\{0, \ldots, d_v\}$ uniformly at random. Now pick a random subset $N \subseteq N(v)$ of size W_v, and for every $u \in N$, set $v_u = 1$, and for every $u \in N(v) \setminus N$, set $v_u = 0$.

Let $g : E \to [3]$ as follows: For $uv \in E$, let $g(uv) = 1 + v_u + u_v$. For a vertex v, let $g(v) = \sum_{u \in N(v)} g(uv)$. For some integer y with $\delta \leq y \leq 3\Delta$, let $Z_y = |\{v \in V : g(v) = y\}|$. Then

$$\mathbf{E}(Z_y) \leq \frac{n}{\delta}, \tag{7}$$

since

$$\mathbf{Pr}(g(v) = y) = \mathbf{Pr}(W_v = y - d - \sum_{u \in N(v)} u_v) \leq \frac{1}{d_v + 1}.$$

By the symmetry of the construction we know that $\forall x \in V, v, u \in N(x)$:

$$\begin{aligned}
&\mathbf{Pr}(x_v = 1) = 1/2, \\
&\mathbf{Pr}(x_v = x_u = 1) = \mathbf{Pr}(x_v = x_u = 0) = 1/3, \\
&\mathbf{Pr}(x_v = 1, x_u = 0) = \mathbf{Pr}(x_v = 0, x_u = 1) = 1/6.
\end{aligned} \tag{8}$$

To use Chebyshev's inequality, we have to bound the variance of Z_y:

$$\mathbf{Var}(Z_y) = \sum_{v \in V} \sum_{u \in V} (\mathbf{Pr}(g(v) = g(u) = y) - \mathbf{Pr}(g(v) = y)\mathbf{Pr}(g(u) = y)).$$

Fix a $v \in V$, and consider

$$S_v = \sum_{u \in V} (\mathbf{Pr}(g(v) = g(u) = y) - \mathbf{Pr}(g(v) = y)\mathbf{Pr}(g(u) = y)).$$

Divide V into three classes V_1, V_2, V_3, and consider the partial sums

$$S_i = \sum_{u \in V_i} (\mathbf{Pr}(g(v) = g(u) = y) - \mathbf{Pr}(g(v) = y)\mathbf{Pr}(g(u) = y)).$$

Class 1: $V_1 = \{v\}$.

$$S_1 \leq \mathbf{Pr}(g(v) = y) \leq \frac{1}{d_v} \leq \frac{\Delta}{\delta^2}. \tag{9}$$

Class 2: $V_2 = N(v)$.

$$\begin{aligned}
S_2 &\leq d_v \mathbf{Pr}(g(v) = g(u) = y) \\
&\leq d_v \mathbf{Pr}\left(W_v = y - d_v - \sum_{x \in N(v)} x_v \mid g(u) = y\right) \mathbf{Pr}\left(W_u = y - d_u - \sum_{x \in N(u)} x_u\right) \\
&\leq d_v \frac{2}{(d_v+1)} \frac{1}{(d_u+1)} < \frac{2}{d_u} \leq \frac{2\Delta}{\delta^2}.
\end{aligned} \tag{10}$$

Class 3: $V_3 = V \setminus (\{v\} \cup N(v))$.

Let $u \in V_3$, and let $c = |N(v) \cap N(u)|$. For the sake of the analysis, pick a random subset $\mathcal{A}$ from $\{x \in N(u) \cap N(v) : x_u = x_v\}$, by choosing each vertex with probability $1/2$. So, using (8), for every vertex $x \in N(u) \cap N(v)$,

$$\begin{aligned}
&\mathbf{Pr}(x_u = x_v = 1 \wedge x \in \mathcal{A}) = \mathbf{Pr}(x_u = x_v = 1 \wedge x \notin \mathcal{A}) = \\
&\quad \mathbf{Pr}(x_u = x_v = 0 \wedge x \in \mathcal{A}) = \mathbf{Pr}(x_u = x_v = 0 \wedge x \notin \mathcal{A}) = \\
&\qquad \mathbf{Pr}(x_u = 0 \wedge x_v = 1) = \mathbf{Pr}(x_u = 1 \wedge x_v = 0) = 1/6,
\end{aligned}$$

and

$$\mathbf{Pr}(x \in \mathcal{A}) = 1/3.$$

Let $A \subseteq N(u) \cap N(v)$, and let $a = |A|$.
Then, for every vertex $x \in N(u) \cap N(v)$,

$$\begin{aligned}
\mathbf{Pr}(x_u = x_v = 1 \mid \mathcal{A} = A \wedge x \notin A) = \frac{\mathbf{Pr}(x_u = x_v = 1 \wedge \mathcal{A} = A \mid x \notin A)}{\mathbf{Pr}(\mathcal{A} = A \mid x \notin A)} &= \\
\frac{(1/6)(1/3)^a(2/3)^{c-a-1}}{(1/3)^a(2/3)^{c-a}} &= \frac{1}{4}.
\end{aligned}$$

By symmetry, we get

$$\begin{aligned}
&\mathbf{Pr}(x_u = x_v = 0 \mid \mathcal{A} = A) = \mathbf{Pr}(x_u = 0, x_v = 1 \mid \mathcal{A} = A) = \\
&\qquad \mathbf{Pr}(x_u = 1, x_v = 0 \mid \mathcal{A} = A) = 1/4.
\end{aligned}$$

Thus, given $x \notin A$ and $\mathcal{A} = A$, the events $(x_v = 1)$ and $(x_u = 1)$ are independent. For $x \in A$, we get

$$\mathbf{Pr}(x_u = x_v = 1 \mid \mathcal{A} = A \wedge x \in A) = \mathbf{Pr}(x_u = x_v = 0 \mid \mathcal{A} = A \wedge x \in A) = 1/2.$$

We introduce the following notation:

$$\begin{aligned} P_A &= \mathbf{Pr}(g(v) = g(w) = y \mid \mathcal{A} = A) - \mathbf{Pr}(g(v) = y \mid \mathcal{A} = A)\mathbf{Pr}(g(w) = y \mid \mathcal{A} = A) \\ &= \mathbf{Pr}(g(v) = g(w) = y \mid \mathcal{A} = A) - \mathbf{Pr}(g(v) = y)\mathbf{Pr}(g(w) = y), \end{aligned}$$

since $\mathbf{Pr}(g(v) = y)$ is independent from the choice of $\mathcal{A}$. In particular,

$$P_\emptyset = \mathbf{Pr}(g(v) = g(w) = y \mid \mathcal{A} = \emptyset) - \mathbf{Pr}(g(v) = y)\mathbf{Pr}(g(w) = y) = 0. \tag{11}$$

For $A \neq \emptyset$, pick any $x \in A$. We want to bound the difference $P_A - P_{A\setminus x}$. Let

$$b_v = d_v + \sum_{z \in N(v)\setminus x} z_v \ , \ b_u = d_u + \sum_{z \in N(u)\setminus x} z_u.$$

Now consider the difference between P_A and $P_{A\setminus x}$, given that $b_v = l$ and $b_u = r$, and denote it by

$$\begin{aligned} P_A^{l,r} \ - \ & P_{A\setminus x}^{l,r} = \\ = \quad & \mathbf{Pr}(g(v) = g(w) = y \mid \mathcal{A} = A \wedge b_v = l \wedge b_u = r) \\ - \quad & \mathbf{Pr}(g(v) = g(w) = y \mid \mathcal{A} = A \setminus x \wedge b_v = l \wedge b_u = r) \\ = \quad & [\mathbf{Pr}(x_u = x_v = 1 \mid \mathcal{A} = A) - \mathbf{Pr}(x_u = x_v = 1 \mid \mathcal{A} = A \setminus x)] \\ & \times \mathbf{Pr}(W_v = y - l - 1)\mathbf{Pr}(W_u = y - r - 1) \\ + \quad & [\mathbf{Pr}(x_u = x_v = 0 \mid \mathcal{A} = A) - \mathbf{Pr}(x_u = x_v = 0 \mid \mathcal{A} = A \setminus x)] \\ & \times \mathbf{Pr}(W_v = y - l)\mathbf{Pr}(W_u = y - r) \\ + \quad & [\mathbf{Pr}(x_u = 1 \wedge x_v = 0 \mid \mathcal{A} = A) - \mathbf{Pr}(x_u = 1 \wedge x_v = 0 \mid \mathcal{A} = A \setminus x)] \\ & \times \mathbf{Pr}(W_v = y - l)\mathbf{Pr}(W_u = y - r - 1) \\ + \quad & [\mathbf{Pr}(x_u = 0 \wedge x_v = 1 \mid \mathcal{A} = A) - \mathbf{Pr}(x_u = 0 \wedge x_v = 1 \mid \mathcal{A} = A \setminus x)] \\ & \times \mathbf{Pr}(W_v = y - l - 1)\mathbf{Pr}(W_u = y - r) \\ = \quad \tfrac{1}{4} \quad & [\mathbf{Pr}(W_v = y - l - 1)\mathbf{Pr}(W_u = y - r - 1) + \mathbf{Pr}(W_v = y - l)\mathbf{Pr}(W_u = y - r) \\ & - \mathbf{Pr}(W_v = y - l)\mathbf{Pr}(W_u = y - r - 1) - \mathbf{Pr}(W_v = y - l - 1)\mathbf{Pr}(W_u = y - r)]. \end{aligned}$$

Therefore,

$$P_A^{l,r} - P_{A\setminus x}^{l,r} = \begin{cases} 1/[4(d_v+1)(d_u+1)] & \text{if } (r = y - d_u - 1 \wedge l = y - d_v - 1) \\ & \text{or } (r = y \wedge l = y), \\ -1/[4(d_v+1)(d_u+1)] & \text{if } (r = y - d_u - 1 \wedge l = y) \\ & \text{or } (r = y \wedge l = y - d_v - 1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, summing over all possible values of l, r and $t = |\{z \in \mathcal{A} \setminus x : z_u = z_v = 1\}|$,

$$\begin{aligned}
P_A \quad - \quad & P_{A\setminus x} \leq \\
\leq \quad & 1/[4(d_v+1)(d_u+1)] \\
& \times[\mathbf{Pr}(b_u = y - d_u - 1 \wedge b_v = y - d_v - 1) + \mathbf{Pr}(b_u = y \wedge b_v = y)] \\
\leq \quad & 1/[4(d_v+1)(d_u+1)] \\
& \times \left[\sum_{t=0}^{a-1} \binom{a-1}{t} 2^{-a+1} \binom{d_u - a}{y - 2d_u - 1 - t} \binom{d_v - a}{y - 2d_v - 1 - t} 2^{-d_u - d_v + 2a} \right. \\
& \left. + \sum_{t=0}^{a-1} \binom{a-1}{t} 2^{-a+1} \binom{d_u - a}{y - d_u - t} \binom{d_v - a}{y - d_v - t} 2^{-d_u - d_v + 2a} \right] \\
\leq \quad & \frac{1}{(d_v+1)(d_u+1)} \binom{d_u - a}{(d_u - a)/2} \binom{d_v - a}{(d_v - a)/2} 2^{-d_u - d_v + a} \sum_{t=0}^{a-1} \binom{a-1}{t}.
\end{aligned}$$

Suppose first that $1 \leq a \leq \delta/3$. Then,

$$\begin{aligned}
P_A - P_{A\setminus x} \leq \frac{2^{-d_v - d_u + 2a - 1}}{(d_v+1)(d_u+1)} \left(\frac{2^{d_v - a + 1}}{(d_v - a)^{1/2}}\right) \left(\frac{2^{d_u - a + 1}}{(d_u - a)^{1/2}}\right) = \\
\frac{2}{(d_v+1)(d_u+1)(d_v - a)^{1/2}(d_u - a)^{1/2}} \leq \frac{3}{d_v \delta^2}.
\end{aligned}$$

Hence,

$$P_A \leq \frac{3a}{d_v \delta^2} \leq \frac{3c}{d_v \delta^2}. \tag{12}$$

Note that for all A,

$$\mathbf{Pr}(g(v) = g(u) = y \mid \mathcal{A} = A) \leq \frac{1}{(d_v+1)(d_u+1)},$$

hence, for $a > \delta/3$,

$$P_A \leq \mathbf{Pr}(g(v) = g(u) = y \mid \mathcal{A} = A) \leq \frac{3a}{d_v \delta^2} \leq \frac{3c}{d_v \delta^2}. \tag{13}$$

Therefore, combining (11), (12) and (13),

$$\begin{aligned}
\mathbf{Pr}(g(v) = g(u) = y) - \mathbf{Pr}(g(v) = y)\mathbf{Pr}(g(u) = y) \quad & \leq \\
\sum_{A \subseteq N(u) \cap N(v)} (3c/d_v \delta^2)\mathbf{Pr}(\mathcal{A} = A) \quad & = \quad \frac{3|N(v) \cap N(u)|}{d_v \delta^2}.
\end{aligned}$$

Now notice that $\sum_{u \in V} |N(v) \cap N(u)|$ counts the number of walks of length two starting in v, thus $\sum_{u \in V} |N(v) \cap N(u)| \leq d_v \Delta$, and therefore

$$S_3 \leq \sum_{u \in V_3} \frac{3|N(v) \cap N(u)|}{d_v \delta^2} \leq \frac{3\Delta}{\delta^2}. \tag{14}$$

Altogether, we get from (9), (10) and (14),

$$S_v = S_1 + S_2 + S_3 \leq \frac{6\Delta}{\delta^2},$$

and thus,

$$\mathbf{Var}(Z_y) = \sum_{v \in V} S_v \leq \frac{6n\Delta}{\delta^2}.$$

By Chebyshev's inequality and (7) we get

$$\mathbf{Pr}(Z_y > 6n/\delta) \leq \frac{\mathbf{Var}(Z_y)}{(5n/\delta)^2} < \frac{1}{3\Delta},$$

and thus,

$$\mathbf{Pr}(\exists y : Z_y > 6n/\delta) < 1,$$

finishing the proof. ∎

Proof of Lemma 10.
Choose g randomly from $\{1,2\}^E$. Observe that $g(v) - d_v$ has the binomial distribution $Bi(d_v, 1/2)$. For a non-negative integer y let

$$V_y = \{v : \ |y - \frac{3}{2}d_v| \leq (2d_v \log n)^{1/2}\}.$$

The Chernoff bounds for the tails of the binomial (see for example [6]) imply that for any $t > 0$,

$$\mathbf{Pr}(|g(v) - \frac{3}{2}d_v| \geq t) \leq e^{-2t^2/d_v}.$$

Hence,

$$\mathbf{Pr}(g(v) = y) \leq \frac{1}{n^4} \qquad \text{if } v \notin V_y. \tag{15}$$

Now consider $v \in V_y$. Clearly,

$$\mathbf{Pr}(g(v) = y) = 0 \qquad \text{if } d_v < y/2. \tag{16}$$

Case 1: $y \geq n^{1/4}$

If $d_v \geq y/2 \geq n^{1/4}/2$ then we can use Stirling's inequality or apply Feller [5], Chapter VII (2.7) to get

$$\mathbf{Pr}(g(v) = y) = \frac{1}{2^{d_v}} \binom{d_v}{y - d_v} \approx \sqrt{\frac{2}{\pi d_v}} e^{-z^2/2}, \tag{17}$$

where $z = 2(y - \frac{3}{2} d_v)/d_v^{1/2}$.

Let $Z_y = |\{v : \ g(v) = y\}|$. It follows from (15), (16) and (17) that

$$\mathbf{E}(Z_y) \leq \frac{|V_y|}{\delta^{1/2}}. \tag{18}$$

Let

$$Z_y^1 = |\{v \in V_y : \ g(v) = y\}| \text{ and } Z_y^2 = |\{v \notin V_y : \ g(v) = y\}|.$$

It follows from (15) that

$$\mathbf{Pr}(Z_y^2 \neq 0) \leq \frac{1}{n^3}. \tag{19}$$

Note also that $v \in V_y$ implies that

$$y = \frac{3}{2} d_v + O\left((d_v \log n)^{1/2}\right). \tag{20}$$

Now for $t > 0$ and $k = (\log n)^2$ we use the Markov inequality to obtain

$$\mathbf{Pr}(Z_y^1 > t) \leq \frac{\mathbf{E}\left(\binom{Z_y^1}{k}\right)}{\binom{t}{k}}. \tag{21}$$

But

$$\begin{aligned}\mathbf{E}\left(\binom{Z_y^1}{k}\right) &= \sum_{S \subseteq V_y, |S| = k} \mathbf{Pr}(g(v) = y, v \in S) \\ &= \sum_{S \subseteq V_y, |S| = k} \sum_{\xi \in \{1,2\}^{E_S}} \mathbf{Pr}(g(v) = y, v \in S \mid g(E_S) = \xi) \mathbf{Pr}(g(E_S) = \xi) \end{aligned} \tag{22}$$

where $E_S = \{e \in E :\ e \subseteq S\}$.

Now fix S in (22). For $v \in S$ let

$$A_v = \{e = uv \in E :\ u \notin S\} \text{ and } B_v = \{e = uv \in E :\ u \in S\}.$$

Then, if $|g(B_v)|$ denotes $\sum_{u \in B_v} g(u)$,

$$\begin{aligned} \mathbf{Pr}(g(v) = y \mid g(E_S) = \xi) &= \mathbf{Pr}(|g(A_v)| = y - |g(B_v)|) \qquad (23) \\ &= 2^{-|A_v|} \binom{|A_v|}{y - |g(B_v)| - |A_v|}. \end{aligned}$$

Therefore,

$$\frac{\mathbf{Pr}(|g(A_v)| = y - |g(B_v)|)}{\mathbf{Pr}(g(v) = y)} = 2^{|B_v|} \frac{\binom{|A_v|}{y - |g(B_v)| - |A_v|}}{\binom{d_v}{y - d_v}}$$

$$= 2^{|B_v|} \frac{|A_v|(|A_v| - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)}{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)} \cdot \frac{1 \times 2 \times \cdots \times (y - d_v)}{d_v(d_v - 1) \cdots (2d_v - y + 1)}. \qquad (24)$$

Now we use

$$|A_v| + |B_v| = d_v \text{ and } |B_v| \leq |g(B_v)| \leq 2|B_v| \leq 2k$$

and (20) to verify that

$$\begin{aligned} &\frac{1 \times 2 \times \cdots \times (y - d_v)}{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)} = \\ &\quad (y - d_v)(y - d_v - 1) \cdots (y - |g(B_v)| - |A_v| + 1) = \\ &\quad \left(\frac{1}{2} d_v\right)^{|g(B_v)| - |B_v|} \left(1 + O\left(k \left(\frac{\log n}{d_v}\right)^{1/2}\right)\right) \qquad (25) \end{aligned}$$

and

$$\begin{aligned} &\frac{|A_v|(|A_v| - 1) \cdots (2|A_v| + |g(|B_v|) - y + 1)}{d_v(d_v - 1) \cdots (2d_v - y + 1)} = \\ &\quad \frac{(2d_v - y)(2d_v - y - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)}{d_v(d_v - 1) \cdots (|A_v| + 1)} = \\ &\quad d_v^{|B_v| - |g(B_v)|} \times 2^{|g(B_v)| - 2|B_v|} \left(1 + O\left(k \left(\frac{\log n}{d_v}\right)^{1/2}\right)\right). \qquad (26) \end{aligned}$$

Plugging (25) and (26) into (24) we see that

$$\frac{\mathbf{Pr}(|g(A_v)| = y - |g(B_v)|)}{\mathbf{Pr}(g(v) = y)} = 1 + O\left(k\left(\frac{\log n}{d_v}\right)^{1/2}\right).$$

So from (22) and (23) we see that

$$\mathbf{E}\left(\binom{Z_y^1}{k}\right) \leq$$

$$\sum_{S \subseteq V_y, |S|=k} \sum_{\xi \in \{1,2\}^{E_S}} \prod_{v \in S} \left(\left(1 + O\left(k\left(\frac{\log n}{d_v}\right)^{1/2}\right)\right) \mathbf{Pr}(g(v) = y)\right) \mathbf{Pr}(g(E_S) = \xi)$$

$$\leq \left(1 + O\left(k^2 \frac{(\log n)^{1/2}}{n^{1/8}}\right)\right) \sum_{S \subseteq V_y, |S|=k} \prod_{v \in S} \mathbf{Pr}(g(v) = y)$$

$$\leq (1 + o(1)) \frac{1}{k!} \left(\sum_{S \subseteq V_y, |S|=k} \mathbf{Pr}(g(v) = y)\right)^k$$

$$= (1 + o(1)) \frac{\mathbf{E}(Z_y^1)^k}{k!}.$$

So (18), (21) imply

$$\mathbf{Pr}\left(Z_y^1 > 2\frac{n}{\delta^{1/2}}\right) \leq (1 + o(1)) \frac{\mathbf{E}(Z_y^1)^k}{(2n/\delta^{1/2})^k} \leq (1 + o(1)) 2^{-k}$$

and then together with (19) we get

$$\mathbf{Pr}\left(\exists y :\ Z_y > 2\frac{n}{\delta^{1/2}}\right) \leq 2n((1 + o(1))2^{-k} + n^{-3}) = o(1). \tag{27}$$

Case 2: $y \leq n^{1/4}$.

Assume that $V_y \neq \emptyset$. We apply the Hoeffding-Azuma inequality. Changing the value of g on a single edge can only change the value of Z_y^1 by at most 2. Also, Z_y^1 is determined by the outcome of at most

$$\sum_{v \in V_y} d_v \leq |V_y|(y + (\log n)^2)$$

random choices. It follows that for $t > 0$,

$$\mathbf{Pr}(Z_y^1 \geq \mathbf{E}(Z_y^1) + t) \leq \exp\left\{-\frac{t^2}{2|V_y|(y + (\log n)^2)}\right\}. \tag{28}$$

Putting $t = n/\delta^{1/2}$ and observing that $V_y \neq \emptyset$ implies $\delta \leq n^{1/4}$ and $y\delta \leq n^{1/2}$, and applying (18), (19), (28), we see that

$$\mathbf{Pr}\left(Z_y^1 > 2\frac{n}{\delta^{1/2}}\right) \leq e^{-n^{1/2}/3}. \tag{29}$$

The lemma follows from (19), (27) and (29). ∎

4 Proofs of Theorems

We are now able to prove the Theorems.

Proof of Theorem 1. Let $\Delta \leq n^{1/2}$. By Lemma 8, there exists a weight assignment $g : E \to [w]$ with $m_g \leq 6n/\delta$ and $w = 3$. Now by Lemma 4, $s(G) \leq 3m_g w + m_g \leq 60n/\delta$, proving (b). Similar arguments, using Lemma 7 and Lemma 9 in place of Lemma 8, provide part (a) and (c). ∎

Proof of Theorem 2. The proof is similar to the proof of Theorem 1, just use Lemma 5 in place of Lemma 4. ∎

Proof of Theorem 3. The proof is similar to the proof of Theorem 1, just use Lemma 4 and Lemma 10. ∎

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