

Generating and counting Hamilton cycles in random regular graphs

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1 Introduction

This paper deals with computational problems involving Hamilton cycles in random regular graphs. Thus let $\mathcal{G} = \mathcal{G}(r, n)$ denote the set of r -regular (simple) graphs with vertex set $[n] = \{1, 2, \dots, n\}$. While it is NP-Complete to tell whether or not a cubic ($r = 3$) graph has a Hamilton cycle, it has been known for some time that for r fixed but sufficiently large, G chosen at random from $\mathcal{G}(r, n)$ is Hamiltonian **whp**¹, see Bollobás [2], Fenner and

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¹An event \mathcal{E}_n is said to occur **whp** (with high probability) if $\Pr(\mathcal{E}_n) = 1 - o(1)$ as $n \rightarrow \infty$.

Frieze [5]. These results were non-constructive and Frieze [6] described an $O(n^3 \log n)$ time algorithm that found a Hamilton cycle **whp**, provided $r \geq 85$. Thus until quite recently it was not known whether or not a random cubic graph was Hamiltonian **whp**. (Experiments with the algorithm of [6] strongly suggested that it was.)

In two recent papers Robinson and Wormald [12], [13] used a second moment approach and showed that random r -regular graphs are Hamiltonian **whp** for $r \geq 3$. Their proof is non-constructive and the purpose of this paper is to provide corresponding algorithmic results. We abandon the rotation-extension approach of [6] in favour of an approach based on rapidly mixing Markov chains. We prove

Theorem 1 *Let $r \geq 3$ be fixed and let G be chosen uniformly at random from $\mathcal{G}(r, n)$. There is a polynomial time algorithm *FIND* which constructs a Hamilton cycle in G **whp**.*

For a graph G let $\text{HAM}(G)$ denote the set of Hamilton cycles of G . Assuming $\text{HAM}(G) \neq \emptyset$, a *near uniform generator* for $\text{HAM}(G)$ is a randomised algorithm which on input $\epsilon > 0$ outputs a cycle $H \in \text{HAM}(G)$ such that for any fixed $H_1 \in \text{HAM}(G)$

$$\left| \Pr(H = H_1) - \frac{1}{|\text{HAM}(G)|} \right| \leq \frac{\epsilon}{|\text{HAM}(G)|}. \quad (1)$$

The probabilities here are with respect to the algorithm's random choices, as G is considered fixed in (1). The algorithm is polynomial if it runs in time polynomial in n and $1/\epsilon$.

Theorem 2 *Let $r \geq 3$ be fixed. There is a procedure *GENERATE* such that if G is chosen uniformly at random from $\mathcal{G}(r, n)$ then **whp** *GENERATE* is a polynomial time generator for $\text{HAM}(G)$.*

Given a polynomial time generator for a set X one can usually estimate its size. This notion is made precise in Jerrum, Valiant and Vazirani [10]. The results there are based on the notion of self-reducibility (Schnorr [14]), which we do not have here. On the other hand, our method of proof does lead to an FPRAS (Fully Polynomial Randomised Approximation Scheme) for almost every $G \in \mathcal{G}(r, n)$.

An *FPRAS* for $\text{HAM}(G)$ is a randomised algorithm which on input $\epsilon, \delta > 0$ produces an estimate Z such that

$$\Pr \left(\left| \frac{Z}{|\text{HAM}(G)|} - 1 \right| \geq \epsilon \right) \leq \delta. \quad (2)$$

Again, the probabilities in (2) are with respect to the algorithm's choices. The running time of the algorithm is polynomial in $n, 1/\epsilon$ and $\log(1/\delta)$.

Theorem 3 *Let $r \geq 3$ be fixed. There is a procedure *COUNT* such that if G is chosen uniformly at random from $\mathcal{G}(r, n)$ then **whp** *COUNT* is an *FPRAS* for $\text{HAM}(G)$.*

These results can be extended to random regular digraphs, see Frieze, Molloy and Cooper [7], Janson [8] for the non-constructive counterparts.

Our final result concerns a problem left open by Broder, Frieze and Shamir [4]. A graph G with vertex set $[n]$ is obtained by adding a random perfect matching M to a random Hamilton cycle H . The problem is to find a Hamilton

cycle in G without knowing H . One motivation for this problem is in the design of authentication protocols. Our positive result on finding a Hamilton cycle can be viewed as a negative result for such a protocol.

Theorem 4 *Let n be even and let G be obtained as the union of a random perfect matching M and a random (disjoint) Hamilton cycle H . Applying *FIND* to G will lead to the construction of a Hamilton cycle **whp**.*

The next section outlines the proof of these results and the remaining sections fill in the missing details.

2 Outline proofs of Theorems

2.1 Configurations

Initially we will not work directly with $\mathcal{G}(r, n)$. Instead we will use the configuration model as developed by Bender and Canfield [1] and Bollobás [3]. Thus let $W = [n] \times [r]$ ($W_v = v \times [r]$ represents r half edges incident with vertex $v \in [n]$.) The elements of W are called *points* and a 2-element subset of W is called a *pairing*. A *configuration* F is a partition of W into $rn/2$ pairings. We associate with F a multigraph $\mu(F) = ([n], E(F))$ where, as a multi-set,

$$E(F) = \{(v, w) : \{(v, i), (w, j)\} \in F \text{ for some } 1 \leq i, j \leq r\}.$$

(Note that $v = w$ is possible here.)

Let Ω denote the set of possible configurations. Thus

$$|\Omega| = P(rn)$$

where

$$P(2m) = \frac{(2m)!}{m!2^m}.$$

We say that F is *simple* if the multigraph $\mu(F)$ has no loops or multiple edges. Let Ω_0 denote the set of simple configurations.

We turn Ω into a probability space by giving each element the same probability. The main properties that we need of this model are

P1 Each $G \in \mathcal{G}(n, r)$ is the image (under μ) of exactly $(r!)^n$ simple configurations.

P2 $\Pr(F \in \Omega_0) \approx e^{-(r^2-1)/4}$.

(Here $\alpha \approx \beta$ means that $\alpha/\beta \rightarrow 1$ as $n \rightarrow \infty$.)

Suppose now that \mathcal{A}^* is a property of configurations and \mathcal{A} is a property of graphs such that when $F \in \Omega_0$, $\mu(F) \in \mathcal{A}$ implies $F \in \mathcal{A}^*$. Then P1 and P2 imply

$$\Pr(G \in \mathcal{A}) \leq (1 + o(1))e^{(r^2-1)/4}\Pr(F \in \mathcal{A}^*)$$

where G is chosen randomly from \mathcal{G} and F is chosen randomly from Ω . We will *generally* use this to prove

$$\Pr(F \in \mathcal{A}^*) = o(1) \text{ implies } \Pr(G \in \mathcal{A}) = o(1). \quad (3)$$

2.2 Generating and counting

We now begin the proof proper. For $F \in \Omega$ let

$$Z_H = Z_H(F) = |\text{HAM}(\mu(F))|.$$

Then

$$\mathbf{E}(Z_H) = \frac{H(n, r)P((r-2)n)}{P(rn)}, \quad (4)$$

where

$$H(n, r) = \frac{(n-1)!}{2} (r(r-1))^n.$$

Explanation: $H(n, r)$ is the number of sets of n pairings which would be projected by μ to a Hamilton cycle (an H -configuration). $P((r-2)n)/P(rn)$ is the probability that a given H -configuration appears in F .

Note that Stirling's approximation gives

$$\mathbf{E}(Z_H) \approx \sqrt{\frac{\pi}{2n}} \left((r-1) \left(\frac{r-2}{r} \right)^{(r-2)/2} \right)^n.$$

which grows exponentially with n for $r \geq 3$.

Using the method of Robinson and Wormald we prove (Section 5) that

$$Z_H \geq n^{-1} \mathbf{E}(Z_H) \quad \mathbf{whp}, \quad (5)$$

which by (3) implies

$$|\mathbf{HAM}(G)| \geq \frac{1}{n} \mathbf{E}(Z_H) \quad \mathbf{whp}.^2 \quad (6)$$

A 2-factor of a graph G is a set of vertex disjoint cycles which contain all vertices. Let $2\mathbf{FACTOR}(G)$ denote the set of 2-factors of G . Then

$$\mathbf{HAM}(G) \subseteq 2\mathbf{FACTOR}(G).$$

²Robinson and Wormald prove this for $r = 3$ but decline to do it for $r \geq 4$. They proceed indirectly. This has advantages and disadvantages. The advantage is that they show that a random $r+1$ -regular graph is *close* to a random r -regular graph plus a random matching ($r \geq 2$). But for our purposes, (6) is what is needed.

For $F \in \Omega$ let

$$Z_f = Z_f(F) = |\text{2FACTOR}(\mu(F))|.$$

Now

$$\mathbf{E}(Z_f) \leq \frac{\binom{r}{2}^n P(2n)P((r-2)n)}{P(rn)}. \quad (7)$$

Explanation: there are $\binom{r}{2}^n$ ways of choosing two points from each W_v . There are then $P(2n)$ ways of pairing these points. If the set X of n pairings contains no loops or multiple edges then μ projects X to a 2-factor. The remaining terms give the probability that X exists in F . We have inequality in (7) as some sets X do not yield 2-factors and some yield the same. On the other hand all 2-factors of G arise in this way. By the Markov inequality

$$Z_f \leq n\mathbf{E}(Z_f) \quad \mathbf{whp},$$

which by (3) implies

$$|\text{2FACTOR}(G)| \leq n\mathbf{E}(Z_f) \quad \mathbf{whp}. \quad (8)$$

Now by (4) and (7)

$$\begin{aligned} \frac{\mathbf{E}(Z_f)}{\mathbf{E}(Z_H)} &\leq \frac{(2n)!}{2^{2n-1}n!(n-1)!} \\ &\leq 2n^{1/2}. \end{aligned} \quad (9)$$

Combining (6) and (8) we obtain

$$\frac{|\text{HAM}(G)|}{|\text{2FACTOR}(G)|} \geq \frac{1}{2n^{5/2}} \quad \mathbf{whp}. \quad (10)$$

We will show in Section 3 that **whp** there is a polynomial time generator and an FPRAS for $\text{2FACTOR}(G)$. This and (10) easily verifies Theorems 1,2 and 3. Indeed we estimate $|\text{2FACTOR}(G)|$ and the ratio $|\text{HAM}(G)|/|\text{2FACTOR}(G)|$.

The former is estimated by the assumed FPRAS and the latter by generating $O(n^{5/2}/\epsilon^2)$ 2-factors and computing the proportion that are Hamilton cycles (ϵ is the required relative accuracy).

2.3 Hidden Hamilton cycles

Let $X = \{(H, M) : H \text{ is a Hamilton cycle, } M \text{ is a perfect matching of } K_n \text{ and } H \cap M = \emptyset\}$. Consider X to be a probability space in which each element is equally likely. Let \mathbf{Pr}_1 refer to probabilities in this space and $\mathbf{Pr}_0, \mathbf{E}_0$ refer to probability and expectation with respect to F chosen randomly from Ω_0 .

Let $A = \{F \in \Omega_0 : \text{GENERATE is not a polynomial time generator for } \text{HAM}(\mu(F))\}$ and $\hat{A} = \{(H, M) \in X : \text{GENERATE is not a polynomial time generator for } \text{HAM}(H \cup M)\}$ be the corresponding subset of X . Now for each $(H, M) \in X$ there are 6^n configurations F for which $\mu(F) = H \cup M$ and for each $F \in \Omega_0$ there are $Z_H(F)$ corresponding pairs (H, M) in X . Thus, where 1_A is the indicator function of the set A ,³

$$\begin{aligned}
\mathbf{Pr}_1(\hat{A}) &= \sum_{(H,M) \in \hat{A}} \frac{1}{|X|} \\
&= \sum_{F \in A} \frac{Z_H(F)}{6^n |X|} \\
&= \frac{1}{\mathbf{E}_0(Z_H)} \mathbf{E}_0(1_A Z_H) && \text{since } 6^n |X| = |\Omega_0| \mathbf{E}_0(Z_H) \\
&\leq \frac{1}{\mathbf{E}_0(Z_H)} \sqrt{\mathbf{E}_0(1_A^2) \mathbf{E}_0(Z_H^2)}.
\end{aligned}$$

³This elegant use of the Cauchy-Schwarz inequality was pointed out to us by Svante Janson.

Robinson and Wormald proved [11] that

$$\mathbf{E}_0(Z_H^2) \approx \frac{3}{e} \mathbf{E}_0(Z_H)^2.$$

Hence

$$\begin{aligned} \mathbf{Pr}_1(\hat{A}) &\leq (1 + o(1))\sqrt{3/e} \mathbf{Pr}_0(A) \\ &= o(1), \end{aligned} \tag{11}$$

by Theorem 1.

3 Generating and counting 2-factors

For any graph $G = (V, E)$, a construction of Tutte [15] gives a graph $G' = (V', E')$ such that the perfect matchings in G' correspond in a natural fashion to the 2-factors of G . Specifically, (assuming G is r -regular) for each vertex $v \in V$ we have a complete bipartite graph $H_v \cong K_{r,r-2}$ with bipartition

$$U_v = \{u_{v,w} : \{v, w\} \in E\}, \quad W_v = \{w_{v,i} : 1 \leq i \leq r - 2\}.$$

Now $V' = \bigcup_{v \in V} (U_v \cup W_v)$ and E' contains the edge set of H_v for each $v \in V$. Additionally, for each edge $\{v, w\} \in E$ we have a unique edge $\{u_{v,w}, u_{w,v}\} \in E'$. We will call these the G -edges of G' and the remainder the H -edges.

In any perfect matching in G' exactly two vertices in U_v will not be matched by H -edges. They must therefore be matched by two G -edges incident with H_v . Thus the n G -edges in the matching correspond to a 2-factor K in G . For each such choice of edges, the remaining $(r - 2)n$ H -edges can be chosen in $(r - 2)!^n$ ways. Therefore each 2-factor in G corresponds to $(r - 2)!^n$ perfect matchings in G' . In particular, by generating a near uniform perfect

matching G' we can generate a near uniform 2-factor of G . Similarly, by approximately counting perfect matchings in G' , we can approximately count 2-factors in G .

The problem of generating near uniform perfect matchings in a graph Γ was studied by Jerrum and Sinclair [9]. They describe an algorithm which runs in time polynomial in $|V(\Gamma)|$, $1/\epsilon$ and $\rho = \rho(\Gamma)$, where ρ is the ratio of the number of near perfect to perfect matchings of Γ (a near perfect matching covers all but two vertices). In light of this we have only to show that **whp** $\rho(G')$ is bounded by a polynomial in n .

Let ν_p and ν_{np} denote the number of perfect and near perfect matchings in G' , assuming G is chosen at random from $\mathcal{G}(r, n)$. Then, from (6), we have

$$\nu_p \geq \frac{(r-2)!^n}{n} \mathbf{E}(Z_H) \quad \mathbf{whp}.$$

To estimate ν_{np} we consider the G -edges of some near perfect matching M' of G . Let M denote the corresponding set of edges in G itself. It is straightforward to verify that the subgraph $G(M)$ induced by M has

- (i) $n - 2$ vertices of degree 2, and
- (ii) 2 vertices with degrees $d_1, d_2 \in \{0, 1, 2, 3, 4\}$ where $d_1 + d_2 = 2, 4$ or 6.

Let Z_{nf} denote the number of subgraphs of G satisfying (i) and (ii). Clearly

$$\nu_{np} \leq (r-2)!^{n-2} (r!)^2 Z_{nf} \tag{12}$$

and a (crude) argument similar to that for (7) yields

$$\mathbf{E}(Z_{nf}) \leq \frac{\binom{n}{2} (r^4)^2 \binom{r}{2}^{n-2} \sum_{k=-1}^1 P(2(n+k)) P((r-2)n-2k)}{P(rn)}.$$

Applying (12) and the Markov inequality we see that

$$\nu_{np} \leq n(r-2)!^{n-2}(r!)^2 \mathbf{E}(Z_{nf}) \quad \mathbf{whp}$$

and so **whp**

$$\begin{aligned} \rho(G') &\leq \frac{n^2 \binom{n}{2} (r-2)!^{n-2} r!^2 r^8 \binom{r}{2}^{n-2} \sum_{k=-1}^1 P(2(n+k)) P((r-2)n-2k)}{(r-2)!^n \mathbf{E}(Z_H) P(rn)} \\ &= O(n^{9/2}), \end{aligned}$$

as required.

4 The Variance of Z_H

The method of Robinson and Wormald is an analysis of variance. We will partition the probability space Ω into *groups* according to the number of cycles of each size. We will then show that $\mathbf{Var}(Z_H)$ can be “explained” almost entirely by the variance between groups. Thus, within most groups Z_H is concentrated around its mean, which in most groups is “close” to $\mathbf{E}(Z_H)$. In this section we compute the variance of Z_H .

We will from now on assume that $r \geq 4$. The case $r = 3$ has been dealt with in [12]. The calculations there are done directly on $\mathcal{G}(3, n)$.

We will count the number of potential pairs of Hamilton cycles by counting the number of pairs (H, H') of H -configurations whose intersection is a set of a paths containing a total of k edges, and summing over all feasible a, k . If H, H' coincide, then we have $k = n$ and we take $a = 0$. Thus:

$$\mathbf{E}(Z_H^2) = \mathbf{E}(Z_H) \sum_{k,a} N(k, a) P((r-4)n+2k) / P((r-2)n), \quad (13)$$

where $N(k, a)$ is the number of ways of selecting H' given H , k and a . Note that this quantity is independent of H .

Explanation: for each fixed H, k, a the number of possible H' is independent of H . Taking out the factor $\mathbf{E}(Z_H)N(k, a)$ leaves us with $\mathbf{Pr}(H' | H)$ which comprises the last two factors.

Claim 1: The number of ways of selecting k edges from H consisting of a paths is

$$\frac{an}{k(n-k)} \binom{k}{a} \binom{n-k}{a}.$$

provided we interpret $an/k(n-k)$ as 1 when $a = 0$ (equivalently $k = 0$ or $k = n$).

Proof We will assume $a > 0$ and $k < n$. Fix an orientation of H . Remove any edge of H and insist that it is not one of the k edges. We now have a path of length $n - 1$ from which we must choose k edges forming a paths. There are $\binom{k-1}{a-1}$ ways to choose the lengths of the paths, and $\binom{n-k}{a}$ ways to pick their initial vertices. There were n ways to choose the edge that was removed, and each choice of paths had $n - k$ eligible choices for this edge. Therefore, the number of ways of selecting the paths is $\frac{n}{n-k} \binom{k-1}{a-1} \binom{n-k}{a} = \frac{an}{k(n-k)} \binom{k}{a} \binom{n-k}{a}$. \square

Claim 2: Given our choice of the k edges of H , the number of ways to complete H' is:

$$\left(\frac{2(r-2)}{r-3} \right)^a H(n-k, r-2).$$

Proof Imagine that each of these a paths is contracted to a single vertex. The selection of a Hamilton cycle H' extending the chosen fragments of H can be divided into two steps: (i) select a Hamilton cycle on $n - k$ vertices

which joins up all the (contracted) fragments, and then (ii) select a way of splicing in the (expanded) fragments to obtain a full n -edge H-configuration H' . The number of choices in (i) is simply $H(n - k, r - 2)$, where we must interpret $H(0, r - 2)$ as 1, while for (ii) it is $(2(r - 2)/(r - 3))^a$. (For each fragment we may choose a direction of traversal; then, on expanding each fragment from a point to a path, the number of ways of connecting H' to the endpoints of a fragment is increased from $(r - 2)(r - 3)$ — as counted by the formula for $H(n - k, r - 2)$ — to $(r - 2)^2$.) \square

Substituting for $N(k, a)$ in (13), and applying (4) and Stirling's formula, we have

$$\begin{aligned}
\mathbf{E}(Z_H^2) &= \mathbf{E}(Z_H) \sum_Q \frac{an}{k(n-k)} \binom{k}{a} \binom{n-k}{a} \left(\frac{2(r-2)}{r-3} \right)^a \\
&\quad \times \frac{H(n-k, r-2) P((r-4)n + 2k)}{P((r-2)n)}, \\
&\approx \frac{\sqrt{\pi n}}{4} \left(\frac{n}{e} \right)^{-(r-2)n/2} \left(\frac{(r-1)(r-2)(r-3)}{2^{(r-4)/2} r^{(r-2)/2}} \right)^n \\
&\quad \times \sum_Q \frac{a}{k(n-k)^2} T_{a,k}, \tag{14}
\end{aligned}$$

where

$$T_{a,k} = \frac{k!(n-k)!^2((r-4)n + 2k)!(r-2)^{a-k} 2^{a-k}}{a!^2(k-a)!(n-k-a)!((r-4)n/2 + k)!(r-3)^{a+k}},$$

and

$$Q = \{(k, a) \mid a, k - a, n - k - a \geq 0\}.$$

It is straightforward to check that we can ignore all terms on the border of Q , as they each contribute $o(n^{-2}\mathbf{E}(Z_H)^2)$, and so we define

$$Q' = \{(k, a) \mid a, k - a, n - k - a > 0\}.$$

Now set $\kappa = \frac{k}{n}, \alpha = \frac{a}{n}$. Using Stirling's approximation, we have:

$$\mathbf{E}(Z_H^2) \approx \frac{1}{4n^2} \left(\frac{2^{(r-4)/2}(r-1)(r-2)(r-3)}{r^{(r-2)/2}} \right)^n \times \sum_{Q'} F^n \lambda(1 + \epsilon), \quad (15)$$

where $F = F_r(\kappa, \alpha)$ is defined by

$$F = \frac{2^{\kappa+\alpha}(r-2)^{\alpha-\kappa}g(\kappa)g(1-\kappa)^2g(\frac{r}{2}-2+\kappa)}{(r-3)^{\kappa+\alpha}g(\alpha)^2g(\kappa-\alpha)g(1-\kappa-\alpha)},$$

with $g(x) = x^x$,

$$\lambda = (\kappa(\kappa-\alpha)(1-\kappa-\alpha)(1-\kappa)^2)^{-\frac{1}{2}},$$

and

$$\epsilon = O\left(\frac{1}{a} + \frac{1}{k-a} + \frac{1}{n-k-a}\right).$$

We extend the domain of F_r to

$$R = \{(\alpha, \kappa) \mid \alpha, \kappa - \alpha, 1 - \kappa - \alpha \geq 0\},$$

by defining $g(0) = 1$. It is straightforward to verify that F_r is continuous over R . We now wish to find its maximum, so we will look for the critical points of F_r in the interior of R . We set the partial derivatives of $\ln F_r$ with respect to κ and α equal to 0, yielding the two equations:

$$\kappa(1-\kappa-\alpha)(r-4+2\kappa) - (r-2)(r-3)(1-\kappa)^2(\kappa-\alpha) = 0, \quad (16)$$

and

$$(r-3)\alpha^2 - 2(r-2)(\kappa-\alpha)(1-\kappa-\alpha) = 0. \quad (17)$$

It is easily verified that $\kappa = \kappa_0 = 2/r$, $\alpha = \alpha_0 = 2(r-2)/r(r-1)$ is a solution of the simultaneous equations (16) and (17). As we now show, this solution is the only one in the interior of R .

Solving equation (16) for α , noting that the equation is linear in α , we obtain

$$\alpha = \frac{\kappa(\kappa - 1)[(r^2 - 5r + 8)\kappa - (r^2 - 6r + 10)]}{Q_r(\kappa)}, \quad (18)$$

where

$$Q_r(\kappa) = (r^2 - 5r + 4)\kappa^2 - (2r^2 - 9r + 8)\kappa + (r^2 - 5r + 6).$$

Substituting this expression for α in equation (17) yields an equation of the form $P_r(\kappa)/Q_r(\kappa)^2 = 0$, where

$$P_r(\kappa) = (r - 3)\kappa(\kappa - 1)^2(r\kappa - 2)P'_r(\kappa),$$

and

$$\begin{aligned} P'_r(\kappa) = & (r^3 - 10r^2 + 25r - 16)\kappa^2 + (-2r^3 + 16r^2 - 36r + 22)\kappa \\ & + (r^3 - 8r^2 + 20r - 16). \end{aligned} \quad (19)$$

Clearly, any solution to (16) and (17) will also be a solution to $P_r(\kappa) = 0$.

When $\kappa = \kappa_0 = 2/r$, the solution $\alpha = \alpha_0$ is unique, except in the case $r = 4$, when equation (16) holds for all α and equation (17) allows the additional solution $\alpha = 1$ which is not in the interior of R . Clearly the roots $\kappa = 0$ and $\kappa = 1$ do not lead to solutions in the interior of R .

We have considered all roots of $P_r(\kappa)$, except those given by the quadratic (19). Our aim is to show that all such roots κ lead to solution pairs (α, κ) that do not lie in the interior of R . In analysing the quadratic $P'_r(\kappa)$, it is convenient

to assume $r \geq 7$, and leave $r = 4, 5, 6$ as special cases to be treated later. We first establish a lower bound on roots κ of equation (19), by recasting (19) in the form

$$P'_r(\kappa) = (r^3 - 10r^2 + 25r - 16)(\kappa - 1)^2 - (4r^2 - 14r + 10)\kappa + (2r^2 - 5r).$$

Under the assumption $r \geq 7$, the factor $r^3 - 10r^2 + 25r - 16$ is strictly positive, and hence any root κ of $P'_r(\kappa) = 0$ must satisfy

$$\kappa \geq \frac{2r^2 - 5r}{4r^2 - 14r + 10} > \frac{1}{2}.$$

Now, from equation (18),

$$1 - \kappa - \alpha = \frac{-(r-2)(r-3)(2\kappa-1)(\kappa-1)^2}{Q_r(\kappa)}. \quad (20)$$

In the light of our lower bound on κ , we see immediately that the numerator of (20) is negative. We show that, for $r \geq 7$, the denominator of (20) is positive, from which it follows that $1 - \kappa - \alpha$ is negative, and the point (α, κ) cannot lie in the interior of R .

By direct calculation,

$$\begin{aligned} & (2r-5)Q_r(\kappa) - 2P'_r(\kappa) \\ &= (5r^2 - 17r + 12)\kappa^2 - (4r^2 - 11r + 4)\kappa + (r^2 - 3r + 2). \end{aligned} \quad (21)$$

The discriminant of quadratic (21) is $-(2r-5)(r-4)(2r^2-7r+4)$, which is negative for all $r > 4$; furthermore, the leading coefficient of (21) is positive under the same condition on r . It follows that $(2r-5)Q_r(\kappa) - 2P'_r(\kappa)$ is positive for all $r > 4$ and all κ , and hence that $Q_r(\kappa)$ is positive for all $r > 4$

and all κ satisfying $P'_r(\kappa) = 0$. This verifies the claim that the denominator of (20) is positive, and completes the analysis of the case $r \geq 7$.

The case $r = 5$ may be eliminated by noting that, of the two roots $\kappa = (-1 \pm \sqrt{10})/4$ of (19), one is negative, and the other yields a corresponding value for α that is greater than 1. A similar argument eliminates the case $r = 6$. When $r = 4$, the two roots of (19) are $\kappa = 0$ and $\kappa = 1/2$; the former leads to a solution not in the interior R , while the latter is just a repeat of the root $\kappa = \kappa_0 = 2/r$ that we have already considered.

Now that we have established (κ_0, α_0) as the only critical point of F_r in R , other than $(0, 0)$, we will see that it is a local maximum, and it will follow that we can ignore all (κ, α) not nearby (κ_0, α_0) . Set

$$\delta_k = \frac{k - \kappa_0 n}{\sqrt{n}}, \delta_a = \frac{a - \alpha_0 n}{\sqrt{n}},$$

and perform a Taylor expansion of $\ln(F_r(\kappa, \alpha))$ around (κ_0, α_0) , yielding:

$$F^n = F_r(\kappa_0, \alpha_0)^n \exp(-(A\delta_k^2 + B\delta_k\delta_a + C\delta_a^2) + \text{cubic terms and greater}),$$

where

$$\begin{aligned} A &= \frac{1}{2(\kappa_0 - \alpha_0)} + \frac{1}{2(1 - \kappa_0 - \alpha_0)} - \frac{1}{2\kappa_0} - \frac{1}{1 - \kappa_0} - \frac{1}{r - 4 + 2\kappa_0}, \\ B &= \frac{1}{1 - \kappa_0 - \alpha_0} - \frac{1}{\kappa_0 - \alpha_0}, \\ C &= \frac{1}{\alpha_0} + \frac{1}{2(\kappa_0 - \alpha_0)} + \frac{1}{2(1 - \kappa_0 - \alpha_0)}. \end{aligned}$$

Substituting $\kappa_0 = 2/r, \alpha_0 = 2(r - 2)/r(r - 1)$ we get:

$$A = \frac{r(r^4 - 9r^3 + 28r^2 - 34r + 16)}{4(r - 2)^2(r - 3)},$$

$$\begin{aligned}
B &= -\frac{r(r-1)^2(r-4)}{2(r-2)(r-3)}, \\
C &= \frac{r(r-1)^2}{4(r-3)}.
\end{aligned}$$

The determinant $D = 4AC - B^2$ of the Hessian of $A\delta_k^2 + B\delta_k\delta_a + C\delta_a^2$ is

$$D = \frac{r^3(r-1)^2}{4(r-2)(r-3)}.$$

Now it is easily checked that $A > 0$ for $r \geq 4$ and since $D > 0$ we have that F is strictly concave, and (κ_0, α_0) is a local maximum. It follows that we can ignore all terms of (15) outside of

$$X = \{(k, a) \mid |k - \kappa_0 n|, |a - \alpha_0 n| \leq \sqrt{n} \log n\}.$$

Now,

$$F_r(\kappa_0, \alpha_0) = \frac{(r-1)(r-2)^{r-3}}{2^{(r-4)/2}(r-3)r^{(r-2)/2}},$$

and so by (15),

$$\begin{aligned}
\mathbf{E}(Z_H^2) &\approx \frac{1}{4n^2} \left(\frac{2^{(r-4)/2}(r-1)(r-2)(r-3)}{r^{(r-2)/2}} \right)^n \sum_X F(\kappa, \alpha)^n \lambda \\
&\approx \frac{1}{4n^2} \left(\frac{2^{(r-4)/2}(r-1)(r-2)(r-3)}{r^{(r-2)/2}} \right)^n \\
&\quad \times \left(\frac{r^{5/2}(r-1)}{2(r-2)^{3/2}(r-3)^{1/2}} \right) F(\kappa_0, \alpha_0)^n \\
&\quad \times n \int_X \exp\{-(A\delta_k^2 + B\delta_k\delta_a + C\delta_a^2)\} d\delta_k d\delta_a \\
&\approx \frac{1}{4n} \left(\frac{r^{5/2}(r-1)}{2(r-2)^{3/2}(r-3)^{1/2}} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\left(\frac{r-2}{r} \right)^{r-2} (r-1)^2 \right)^n \frac{2\pi}{\sqrt{D}} \\
& = \frac{\pi r}{2(r-2)n} \left(\left(\frac{r-2}{r} \right)^{r-2} (r-1)^2 \right)^n,
\end{aligned}$$

and comparing with (4), we have

$$\frac{\mathbf{E}(Z_H^2)}{\mathbf{E}(Z_H)^2} \approx \frac{r}{r-2}. \tag{22}$$

5 Bounding Z_H whp

In the following, b, x are considered to be arbitrary large *fixed* positive integers. Let C_ℓ denote the number of ℓ -cycles of $\mu(F)$ for $\ell \geq 1$. We will be concerned mainly with C_ℓ where ℓ is odd. For $\mathbf{c} = (c_1, c_2, \dots, c_b) \in N^b$, where $N = \{0, 1, 2, \dots\}$, let group $\Omega_{\mathbf{c}} = \{F \in \Omega : C_{2k-1} = c_k, 1 \leq k \leq b\}$. Let

$$\lambda_k = \frac{(r-1)^{2k-1}}{2(2k-1)}.$$

It is straightforward to show that the $C_\ell, \ell \geq 1$, are asymptotically independent Poisson variables with mean $(r-1)^\ell/2\ell$; thus if \mathbf{c} is fixed, then

$$\pi_{\mathbf{c}} = \mathbf{Pr}(F \in \Omega_{\mathbf{c}}) \approx \prod_{k=1}^b \frac{\lambda_k^{c_k} e^{-\lambda_k}}{c_k!}.$$

Now let

$$S(x) = \{\mathbf{c} \in N^b : c_k \leq \lambda_k + x\lambda_k^{2/3}, 1 \leq k \leq b\},$$

and

$$\bar{\Omega} = \bigcup_{\mathbf{c} \notin S(x)} \Omega_{\mathbf{c}}.$$

Let

$$\bar{\pi} = \mathbf{Pr}(F \in \bar{\Omega}).$$

For $\mathbf{c} \in N^b$ let

$$E_{\mathbf{c}} = \mathbf{E}(Z_H \mid F \in \Omega_{\mathbf{c}})$$

and

$$V_{\mathbf{c}} = \mathbf{Var}(Z_H \mid F \in \Omega_{\mathbf{c}}).$$

Then we have

$$\mathbf{E}(Z_H^2) = \sum_{\mathbf{c} \in N^b} \pi_{\mathbf{c}} V_{\mathbf{c}} + \sum_{\mathbf{c} \in N^b} \pi_{\mathbf{c}} E_{\mathbf{c}}^2. \quad (23)$$

The following two lemmas contain the most important observations. Lemma 1 shows that for most groups, the group mean is large and Lemma 2 shows that most of the variance can be explained by the *variance between groups*.

Lemma 1 *For all sufficiently large x*

(a) $\bar{\pi} \leq e^{-\alpha x}$ for some absolute constant $\alpha > 0$.

(b) $\mathbf{c} \in S(x)$ implies $E_{\mathbf{c}} \geq e^{-(\beta+\gamma x)} \mathbf{E}(Z_H)$, for some absolute constants $\beta, \gamma > 0$.

Lemma 2 *If x is sufficiently large then*

$$\sum_{\mathbf{c} \in S(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 \geq (1 - be^{-3\gamma x}) \left(1 - \left(\frac{2}{r-1}\right)^{2b}\right) \left(\frac{r}{r-2}\right) \mathbf{E}(Z_H)^2.$$

where γ is as in Lemma 1

Hence we have from (22) and (23) and Lemma 2,

$$\sum_{\mathbf{c} \in N^b} \pi_{\mathbf{c}} V_{\mathbf{c}} \leq \delta \mathbf{E}(Z_H)^2, \quad (24)$$

where $\delta = \left(b e^{-3\gamma x} + \left(\frac{2}{r-1} \right)^{2b} \right) \frac{r}{r-2}$. The rest is an application of the Chebycheff inequality. Define the random variable \hat{Z}_H by

$$\hat{Z}_H = E_{\mathbf{c}}, \text{ if } F \in \Omega_{\mathbf{c}}.$$

Then for any $t > 0$

$$\begin{aligned} \Pr(|Z_H - \hat{Z}_H| \geq t) &\leq \mathbf{E}((Z_H - \hat{Z}_H)^2/t^2) \\ &= \sum_{\mathbf{c} \in N^b} \pi_{\mathbf{c}} V_{\mathbf{c}}/t^2 \\ &\leq \delta \mathbf{E}(Z_H)^2/t^2 \end{aligned}$$

where the last inequality follows from (24).

Put $t = e^{-(\beta+\gamma x)} \mathbf{E}(Z_H)/2$ where β, γ are from Lemma 1. Applying Lemma 1 we obtain that for n large,

$$\begin{aligned} \Pr\left(Z_H \geq \frac{\mathbf{E}(Z_H)}{n}\right) &\geq \Pr(Z_H \geq e^{-(\beta+\gamma x)} \mathbf{E}(Z_H)/2) \\ &\geq \Pr(|Z_H - \hat{Z}_H| \leq t \wedge (F \notin \bar{\Omega})) \\ &\geq 1 - 4\delta e^{2(\beta+\gamma x)} - \bar{\pi} \\ &\geq 1 - 4\delta e^{2(\beta+\gamma x)} - e^{-\alpha x}. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \Pr\left(Z_H \geq \frac{\mathbf{E}(Z_H)}{n}\right) \geq 1 - \left(4b e^{2\beta - \gamma x} + 4 \left(\frac{2}{r-1} \right)^{2b} e^{2(\beta+\gamma x)} \right) \frac{r}{r-2} e^{-\alpha x}. \quad (25)$$

This is true for all b, x and so the left hand side limit of (25) must in fact be one, proving (5), (putting $b = x^2$ and x arbitrarily large makes the right-hand side of (25) arbitrarily close to 1).

All that remains are the proofs of Lemmas 1 and 2

Proof of Lemma 2:

Let H_0 be some fixed Hamilton cycle.

$$\begin{aligned}
E_{\mathbf{c}} &= \sum_{F \in \Omega_{\mathbf{c}}} \frac{1}{|\Omega_{\mathbf{c}}|} \sum_{H \subseteq F} 1 \\
&= \sum_H \sum_{\substack{F \supseteq H \\ F \in \Omega_{\mathbf{c}}}} \frac{1}{|\Omega_{\mathbf{c}}|} \frac{|\Omega|}{|\Omega|} \\
&= \frac{|\Omega|}{|\Omega_{\mathbf{c}}|} \sum_H \Pr(F \supseteq H \text{ and } F \in \Omega_{\mathbf{c}}) \\
&= \frac{\Pr(F \supseteq H_0)}{\Pr(\Omega_{\mathbf{c}})} \sum_H \Pr(F \in \Omega_{\mathbf{c}} \mid F \supseteq H) \\
&= \frac{\mathbf{E}(Z_H) \Pr(F \in \Omega_{\mathbf{c}} \mid F \supseteq H_0)}{\Pr(\Omega_{\mathbf{c}})}. \tag{26}
\end{aligned}$$

So we will now compute $\Pr(F \in \Omega_{\mathbf{c}} \mid F \supseteq H_0)$, by first computing the expected number of cycles of length l , conditional on F containing H_0 . Here l can be considered fixed as $n \rightarrow \infty$.

To choose a cycle C of length l , we will first fix s , the number of edges in $C \cap H_0$ (hereafter called H -edges), and t , the number of H -paths, i.e. the paths formed by the H -edges.

First we will count the number of ways to choose the edges of C which will form the H -paths. Fix a starting vertex of C , and an orientation. We will

insist that the last edge of this orientation does *not* lie in an H -path. This will have the effect of multiplying the number of choices by $2(l - s)$. Now we will consider the generating function in which x, y, z mark the number of edges, H -edges, and H -paths respectively.

We go around the cycle and at each point we decide whether the next edge lies outside of H_0 , an option we represent by x , or if it is the first edge of an H -path, an option which we represent by $x^{i+1}y^i z$ where i is the length of the H -path. Note that the first edge following the H -path must of course lie outside of H_0 , explaining the exponent of x . Thus we find that the number of choices of H -edges in C is (where as usual $[x^l y^s z^t]$ stands for “coefficient of $x^l y^s z^t$ ”):

$$\begin{aligned} & [x^l y^s z^t] \frac{1}{2(l-s)} \left(x + \sum_{i \geq 1} x^{i+1} y^i z \right)^{l-s} \\ = & [x^l y^s z^t] \frac{1}{2(l-s)} \left(x + \frac{x^2 y z}{1 - xy} \right)^{l-s}. \end{aligned}$$

Given such a choice, we now compute the number of ways to finish the cycle. The number of ways to choose the sequence of vertices in the cycle is $\approx n^{l-s} 2^t$. The number of choices for copies of those vertices is $(r-2)^{l-s+t} (r-3)^{l-s-t}$. Also, the number of configurations containing $H_0 \cup C$ is $P((r-2)n - 2(l-s))$, so we multiply by:

$$\begin{aligned} & \approx n^{l-s} 2^t (r-2)^{l-s+t} (r-3)^{l-s-t} \\ & \quad \times P((r-2)n - 2(l-s)) / P((r-2)n) \end{aligned}$$

$$\approx \left(\frac{2(r-2)}{r-3} \right)^t (r-3)^{-s} (r-3)^l,$$

to get

$$\begin{aligned} & [x^l y^s z^t] \frac{1}{2(l-s)} \left((r-3)x + \frac{2(r-2)x^2 yz}{1-xy} \right)^{l-s} \\ &= -\frac{1}{2} [x^l y^s z^t] \ln \left(1 - (r-3)x - \frac{2(r-2)x^2 yz}{1-xy} \right). \end{aligned}$$

(Observe that $((r-3)x + 2(r-2)x^2 yz/(1-xy))^k$ only contributes to terms of the form $x^i (x^2 yz)^{k-i} (xy)^j$ for some i, j . So only i, j, k such that $l = 2k - i + j$, $s = k - i + j$ and $t = k - i$ affect our expression. But this implies that $k = l - s$.)

Summing over all s, t (or equivalently putting $y = z = 1$), we get:

$$\begin{aligned} & -\frac{1}{2} [x^l] \ln \left(1 - (r-3)x - \frac{2(r-2)x^2}{1-x} \right) \\ &= -\frac{1}{2} [x^l] \ln \left(\frac{1 - (r-2)x - (r-1)x^2}{1-x} \right) \\ &= -\frac{1}{2} [x^l] (\ln(1+x) + \ln(1 - (r-1)x) - \ln(1-x)) \\ &= \frac{(r-1)^l + (-1)^l - 1}{2l}. \end{aligned}$$

Note that for l even, this is equal to the unconditional expected number of l -cycles in F , explaining why we are concentrating on l odd. Let

$$\mu_k = \frac{(r-1)^{2k-1} + (-1)^{2k-1} - 1}{2(2k-1)} = \lambda_k - \frac{1}{2k-1},$$

the expected number of cycles of length $2k-1$ in F conditional on $F \supseteq H_0$.

The next step is to compute

$$\mathbf{E} \left([C_3]_{i_3} [C_5]_{i_5} \cdots [C_{2k-1}]_{i_{2k-1}} \mid F \supseteq H_0 \right)$$

for any fixed $i_3, i_5, \dots, i_{2k-1}$. This is done by counting the expected number of sets of i_3 distinct 3-cycles, i_5 distinct 5-cycles, ..., and i_{2k-1} distinct $(2k-1)$ -cycles in F , conditional on $F \supseteq H_0$. It follows from a straightforward first moment argument that F a.s. has no two intersecting cycles of length at most k . It follows that the cycles appear almost independently, and we get:

$$\mathbf{E} \left([C_3]_{i_3} [C_5]_{i_5} \cdots [C_{2k-1}]_{i_{2k-1}} \mid F \supseteq H_0 \right) \approx \prod_{j=1}^k \mu_j^{i_j}. \quad (27)$$

Therefore, conditional on $F \supseteq H_0$, the C_k are asymptotically independent Poisson variables with means μ_k . Hence, from (26),

$$E_{\mathbf{c}} \approx \mathbf{E}(Z_H) \prod_{k=1}^b \left(\frac{\mu_k}{\lambda_k} \right)^{c_k} e^{\lambda_k - \mu_k}. \quad (28)$$

So,

$$\begin{aligned} \sum_{\mathbf{c} \in \mathcal{S}(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 &\approx \mathbf{E}(Z_H)^2 \sum_{\mathbf{c} \in \mathcal{S}(x)} \prod_{k=1}^b \left(\frac{\mu_k^2}{\lambda_k} \right)^{c_k} \frac{e^{-(2\mu_k - \lambda_k)}}{c_k!} \\ &= \mathbf{E}(Z_H)^2 \prod_{k=1}^b \sum_{c_k=0}^{\lambda_k + x \lambda_k^{2/3}} \left(\frac{\mu_k^2}{\lambda_k} \right)^{c_k} \frac{e^{-(2\mu_k - \lambda_k)}}{c_k!} \\ &= \mathbf{E}(Z_H)^2 \prod_{k=1}^b (1 - Z_k) e^{\frac{(\mu_k - \lambda_k)^2}{\lambda_k}} \end{aligned}$$

where

$$Z_k = \sum_{c_k = \lambda_k + x \lambda_k^{2/3}}^{\infty} \left(\frac{\mu_k^2}{\lambda_k} \right)^{c_k} \frac{e^{-(\mu_k^2/\lambda_k)}}{c_k!} \quad (29)$$

The following lemma appears in [13]

Lemma 3 *Let η_1, η_2, \dots be given. Suppose that $\eta_1 > 0$ and that for some $c > 1, \eta_{i+1}/\eta_i > c$ for all $i > 1$. Then uniformly over $x \geq 1$,*

$$R(x) = \sum_{i=1}^{\infty} \sum_{t=\eta_i(1+y_i)}^{\infty} \frac{\eta_i^t}{t!e^{\eta_i}} = O(e^{-c_0x})$$

where $y_i = x\eta_i^{-1/3}$ and $c_0 = \min\{\eta_1^{1/3}, \eta_1^{2/3}\}/4$.

Applying this lemma with $\eta_i = \mu_i^2/\lambda_i$ and observing that $\eta_1 = (r-3)^2/(2(r-1)) \geq 6$ we see that

$$\sum_{k \geq 3} Z_k \leq O(e^{-x/20})$$

Hence, for x sufficiently large,

$$\sum_{\mathbf{c} \in \mathcal{S}(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 \geq \mathbf{E}(Z_H)^2 (1 - be^{-x/20}) \prod_{k=1}^b \exp \left\{ \frac{(\mu_k - \lambda_k)^2}{\lambda_k} \right\}. \quad (30)$$

Now,

$$\begin{aligned} \prod_{k=b+1}^{\infty} \exp \left\{ \frac{(\mu_k - \lambda_k)^2}{\lambda_k} \right\} &= \exp \left\{ \sum_{k=b+1}^{\infty} \frac{2}{(2k-1)(r-1)^{2k-1}} \right\} \\ &\leq \exp \left\{ \frac{2}{(r-1)^{2b}} \right\} \\ &\leq \left(1 - \frac{2}{(r-1)^{2b}} \right)^{-1}. \end{aligned}$$

Thus, from (30), with

$$1 - \theta = (1 - be^{-x/20}) \left(1 - \frac{2}{(r-1)^{2b}} \right),$$

$$\begin{aligned}
\sum_{\mathbf{c} \in S(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 &\geq (1 - \theta) \mathbf{E}(Z_H)^2 \prod_{k=1}^{\infty} \exp \left\{ \frac{(\mu_k - \lambda_k)^2}{\lambda_k} \right\} \\
&= (1 - \theta) \mathbf{E}(Z_H)^2 \exp \left\{ \sum_{k=1}^{\infty} \frac{2}{(2k-1)(r-1)^{2k-1}} \right\} \\
&= (1 - \theta) \mathbf{E}(Z_H)^2 \left(\frac{r}{r-2} \right).
\end{aligned}$$

□

Proof of Lemma 1

(a) Putting $\eta_i = \lambda_i$ satisfies the conditions of Lemma 3 with $c = 4/3$. Now

$$\begin{aligned}
\bar{\pi} &\leq \sum_{k=3}^b \sum_{c \geq \lambda_k(1+y_k)} \mathbf{Pr}(C_k = c) \\
&\approx \sum_{k=1}^b \sum_{c \geq \lambda_k(1+y_k)} \frac{\lambda_k^c e^{-\lambda_k}}{c!} \\
&= O(e^{-\alpha x}),
\end{aligned}$$

for some constant α , independent of x .

(b) Applying (28) we obtain

$$\begin{aligned}
E_{\mathbf{c}} &\approx \mathbf{E}(Z_H) \prod_{k=1}^b \left(1 - \frac{2}{(r-1)^{2k-1}} \right)^{c_k} \exp \left\{ \frac{1}{2k-1} \right\} \\
&\geq AB^x,
\end{aligned}$$

where

$$A = \prod_{k=1}^b \left(1 - \frac{2}{(r-1)^{2k-1}} \right)^{\lambda_k} \exp \left\{ \frac{1}{2k-1} \right\}$$

and

$$B = \prod_{k=1}^b \left(1 - \frac{2}{(r-1)^{2k-1}} \right)^{\lambda_k^{2/3}}.$$

Now

$$\begin{aligned}
A &= \prod_{k=1}^b \exp \left\{ \frac{1}{2k-1} - \left(\frac{2\lambda_k}{(r-1)^{2k-1}} + \frac{4\lambda_k}{2(r-1)^{2(2k-1)}} + \dots \right) \right\} \\
&\geq \prod_{k=1}^{\infty} \exp \left\{ -\frac{2\lambda_k}{(r-1)^{2(2k-1)}} \right\} \\
&= \exp \left\{ -\sum_{k=1}^{\infty} \frac{1}{(2k-1)(r-1)^{2k-1}} \right\}.
\end{aligned}$$

The sum in the exponential term is convergent and so A is bounded below by a positive absolute constant.

Also

$$\begin{aligned}
B &\geq \prod_{k=1}^{\infty} \left(1 - \frac{2}{(r-1)^{2k-1}} \right)^{\lambda_k^{2/3}} \\
&\geq \exp \left\{ -\sum_{k=1}^{\infty} \frac{2}{(2k-1)^{\frac{2}{3}}(r-1)^{\frac{2k-1}{3}}} \right\}.
\end{aligned}$$

Again, the sum in the exponential term is convergent and so B is bounded below by a positive absolute constant, completing the proof. \square

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