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LONG CYCLES IN SPARSE RANDOM GRAPHS

B. Bollobás, T. I. Fenner and A. M. Frieze

ABSTRACT It is proved that if c is a sufficiently large constant then almost every graph of order n and size $\frac{1}{2}cn$ contains a cycle of length at least $(1-c^6e^{-c})n$.

This note is a continuation of [**4**]. As in [**4**], we shall study the maximal length of a path or cycle of a random graph $G_{c/n}$. As is customary, we write G_p for a random element of the probability space $\mathcal{G}(n, p(\text{edge})=p)$ of all graphs with a fixed set of n labelled vertices, in which the edges are chosen independently and with probability p. Furthermore, we say that *almost every* (a.e.) G_p has a property Q if the probability that a graph $G \in \mathcal{G}(n, P(\text{edge})=p)$ has Q tends to 1 as $n \to \infty$. For $c > 0$ set

$$1-\beta(c) = \sup\{\beta \geq 0\text{: a.e. } G_{c/n} \text{ contains a cycle of length at least } \beta n\}.$$

It was proved by Ajtai, Komlós and Szemerédi [**1**] and by de la Vega [**9**] that a.e. $G_{c/n}$ contains long cycles and $\beta(c) \leq c_0/c$ for some absolute constant c_0. On the other hand, results of Erdös and Rényi [**5**] imply that $\beta(c) \geq (c+1)e^{-c}$. In [**4**] it was shown that $\beta(c)$ decays exponentially: $\beta(c) < c^{24}e^{-c/2}$. As a consequence of our main result we find that, in fact, $-\log \beta(c) \sim c$ as $c \to \infty$.

THEOREM *There is a polynomial P of degree at most* 6 *such that a.e.* $G_{n,c/n}$ *contains a cycle of length at least* $(1-P(c)e^{-c})n$. *In particular,*

$$\beta(c) \leq P(c)e^{-c}.$$

PROOF We shall show that if c is sufficiently large then $P(c) = c^6$ will do. Our proof is based on the method used by Fenner and Frieze [**6**] to solve a related problem and on the model of random graphs with a fixed degree sequence introduced by Bollobás [**2**]. In fact, the present proof is rather close to the proof of the theorem, due to Bollobás [**3**] and Fenner and Frieze [**7**], that if k is a sufficiently large constant then a.e. k-regular graph is Hamiltonian. Because of this similarity we shall not give all the details of the proof. We shall start with a lemma enabling us to locate an

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appropriate large subgraph of a.e. $G_{c/n}$. Then we shall show that this subgraph is Hamiltonian for a.e. $G_{c/n}$.

As is customary, we take $V=\{1,2,\ldots,n\}$. Consider a graph $G\in \mathscr{G}(n, P(\text{edge})=c/n))$ and define

$$U_0=\{x\in V\colon d(x)\leqslant 6 \text{ or } d(x)\geqslant 4c\}.$$

Suppose we have constructed a sequence of sets $U_0, U_1, \ldots, U_j$. Set

$$U'_{j+1}=\{x\in V-U_j\colon |\Gamma(x)\cap U_j|\geqslant 2\}.$$

If $U'_{j+1}=\varnothing$, stop the sequence. Otherwise let x_{j+1} be the minimal element of U'_{j+1} and put $U_{j+1}=U_j\cup\{x_{j+1}\}$. Suppose the sequence stops with $U_s\neq V$. Let H be the subgraph spanned by $V-U_s$ and write h for the order of H. Then every vertex of H has degree at least 6 and at most $4c$, since every vertex $x\in V-U_s$ has degree at least 7 in G and is joined to at most one vertex of U_s.

LEMMA 1 *Let $\varepsilon>0$. If c is sufficiently large then a.e. $G_{c/n}$ is such that*

(i) *Any $t\leqslant n/(6c^3)$ vertices of $G_{c/n}$ span at most $3t/2$ edges.*
(ii) *Any $t\leqslant n/3$ vertices of $G_{c/n}$ span at most $ct/5$ edges.*
(iii) *$n-h<c^6\mathrm{e}^{-c}n$.*
(iv) *The set $W=\{x\colon (1-\varepsilon)c<d_H(x)<(1+\varepsilon)c\}$ has at least $(1-c^{-4})h$ elements, and spans at least $(1-\varepsilon)ch/2$ edges.*
(v) *H has at most $(1+\varepsilon)ch/2$ edges.*

PROOF The proofs of all the assertions are rather straightforward so we shall not give all the details. For $t\leqslant n/c$ the expected number of t-sets of vertices spanning at least $3t/2$ edges is at most

$$\begin{aligned}\sum_{u\geqslant 3t/2}\binom{n}{t}\binom{\binom{t}{2}}{u}\left(\frac{c}{n}\right)^u\left(1-\frac{c}{n}\right)^{\binom{t}{u}-u} &\leqslant c_i\sum_{u\geqslant 3t/2}^{n}\left(\frac{\mathrm{e}n}{t}\right)^t\left(\frac{\mathrm{e}ct^2}{2un}\right)^u \mathrm{e}^{-ct^2/2n}\\ &\leqslant c_2\left(\frac{\mathrm{e}n}{t}\right)^t\left(\frac{\mathrm{e}ct^2}{3tn}\right)^{3t/2}\mathrm{e}^{-ct^2/2n}\\ &= c_2\left(\left(\frac{c^3\mathrm{e}^5t}{27n}\right)\mathrm{e}^{-ct/n}\right)^{t/2}.\end{aligned}$$

Since

$$\sum_{t=1}^{\lceil n/6c^3\rceil}\left(\left(\frac{\mathrm{e}^5c^3t}{27n}\right)\mathrm{e}^{-ct/n}\right)^{t/2}=O(n^{-1/2}),$$

assertion (i) follows. The proof of (ii) is similar.

To prove (iii) note first that the expectation of $|U_0|$ is

$$n\sum_{k=0}^{6}\binom{n-1}{k}\left(\frac{c}{n}\right)^k\left(1-\frac{c}{n}\right)^{n-1-k}+n\sum_{k\geq 4c}\binom{n-1}{k}\left(\frac{c}{n}\right)^k\left(1-\frac{c}{n}\right)^{n-1-k}$$
$$<\left(\frac{c^6}{700}\right)e^{-c}n+2\left(\frac{enc}{4cn}\right)^{4c}e^{-c}n<\left(\frac{c^6}{600}\right)e^{-c}n,$$

provided that c is sufficiently large. It is easily checked that the variance of $|U_0|$ is $O(n)$, so by Chebyshev's inequality $|U_0|<(c^6/500)e^{-c}n$ for a.e. $G_{c/n}$. Now if this last inequality holds and the graph satisfies (i) then $n-h\geq c^6e^{-c}n$ does not hold since otherwise for some j we have $|U_j|=\lfloor c^6e^{-c}n\rfloor<n/6c^3$ and this set U_j spans at least $2j\geq\frac{3}{2}|U_j|$ edges. Hence (iii) follows.

Assertion (iv) follows from the fact that if c is sufficiently large and (iii) holds, then the degree sequence of H is close to the degree sequence of $G_{c/n}$: $d_H(X)\geq d(X)-1$ for every $X\in V(H)$.

Assertion (v) is an immediate consequence of (iii). □

Let us assume that the graph $H=H(G_{c/n})$ in Lemma 1 has vertex set $V(H)=\{1,2,\ldots,h\}$ and degree sequence $6\leq d_1\leq d_2\leq\ldots\leq d_h\leq 4c$. Let $\mathcal{H}=\mathcal{H}(G_{c/n})$ be the set of all graphs with vertex set $\{1,2,\ldots,h\}$ and degree sequence $(d_i)_1^h$. Turn $\mathcal{H}$ into a probability space by giving all members of $\mathcal{H}$ the same probability. Note that all members of $\mathcal{H}$ occur as $H=H(G_{c/n})$ with the same probability. Hence a.e. $G_{c/n}$ is such that a.e. element H of $\mathcal{H}(G_{c/n})$ satisfies the conclusions of Lemma 1 with $G_{c/n}$ replaced by H in (i) and (ii).

Consider a graph $G_{c/n}$ which satisfies these conditions. In view of the remarks above, our theorem will follow if we prove that for some $\varepsilon>0$ and large enough c almost every graph in $\mathcal{H}$ is Hamiltonian.

The graphs in $\mathcal{H}$ are fairly close to being regular, so this assertion resembles the theorem, proved in **[3]** and **[7]**, that a.e. regular graph is Hamiltonian. Hence it is no surprise that we can adapt the proofs in **[3]** and **[7]** to the present case.

In order to study $\mathcal{H}$, we consider the model defined in **[2]**. Let $D_1,D_2,\ldots,D_h$ be disjoint sets with $|D_i|=d_i$ and set

$$D=\bigcup_1^h D_i,\qquad 2m=|D|=\sum_1^h d_i.$$

A *configuration* C is a partition of D into m pairs, the *edges* of C. Let Φ be the set of all $N(m)=(2m-1)!!=(2m)!2^{-m}/m!$ configurations. Turn Φ into a probability space by giving all members of Φ the same probability. For $C\in\Phi$ let $\phi(C)$ be the graph with vertex set $\{1,2,\ldots,h\}$ in which i is joined to j $(i \neq j)$ if and only if C has an edge with one end-vertex in D_i and the other in D_j. Then clearly $\mathscr{H} \subset \phi(\Phi)$ and

$$|\phi^{-1}(H)| = \prod_1^h d_i!$$

for every $H \in \mathscr{H}$.

Let Q be a property of the graphs in $\mathscr{H}$ and let Q^* be a property of the configurations in Φ. Suppose that these properties are such that for $H \in \mathscr{H}$ and $C \in \phi^{-1}(H)$ the configuration C has Q^* if and only if H has Q. All we shall need from **[3]** is that in this case, if almost no C has Q^*, then almost no H has Q.

LEMMA 2 *A.e.* $H \in \mathscr{H}$ *is connected.*

PROOF Let us say that $H \in \mathscr{H}$ has property Q if H is disconnected and satisfies the conclusions of Lemma 1. Let Q^* be such that $C \in \Phi$ has Q^* if and only if $\phi(C) \in \mathscr{H}$ and $\phi(C)$ has Q. We shall show that almost no $C \in \Phi$ has Q^*.

Note that if $C \in \Phi$ has Q^* then there is a set $U \subset \{1, 2, \ldots, h\}$, $1 \leqslant u = |U| \leqslant h/2$, such that C is the union of a partition of

$$X = \bigcup_{i \in U} D_i$$

and a partition of $Y = D - X$. If X is odd, this cannot happen. Suppose $|X| = 2x$ and $|Y| = 2y$. Then $6u \leqslant 2x \leqslant 4cu$ and $6(h-u) \leqslant 2y \leqslant 4c(h-u)$. The probability that C is the union of two such partitions is

$$\frac{(2x-1)!!\,(2y-1)!!}{(2x+2y-1)!!} \leqslant \left(\frac{(2x)!}{(2(x+y))_{2x}}\right)^{1/2} = \binom{2(x+y)}{2x}^{-1/2} \leqslant \binom{6h}{6u}^{-1/2}.$$

Hence the probability that C has Q^* is at most

$$\sum_{u=1}^{\lfloor h/2 \rfloor} \binom{h}{u}\binom{6h}{6u}^{-1/2} = O(h^{-2}) = o(1). \quad \square$$

REMARK The simple proof above implies that if Δ is fixed and $6 \leqslant d_i = d_i(n) \leqslant \Delta$, $i = 1, 2, \ldots, n$, then a.e. graph with vertex set $\{1, 2, \ldots, n\}$ and degree sequence $(d_i)_1^n$ is connected.

Let us continue the proof of the theorem. Put $\mathscr{H}_0 = \{H \in \mathscr{H}: H$ satisfies the conclusions of Lemmas 1 and 2$\}$ and let Q be the property that $H \in \mathscr{H}_0$ and H is not Hamiltonian. Let Q^* be such that $C \in \Phi$ has Q^* if and only if $\phi(C)$ has Q and let $\Phi_0 = \{C: C$ has $Q^*\}$. To complete the proof of our theorem we shall show that almost no C has Q^*. This will be done by the colouring method introduced in [**6**] and used in [**3**] and [**7**].

Suppose that $C \in \Phi_0$. Let P_H be a longest path of $H = \phi(C)$, say of length l. Since H is connected and not Hamiltonian, it does not contain a cycle of length l. Consider all red–blue colourings of the edges of C in which there are exactly $3h$ red edges, the red edges join vertices of

$$E = \bigcup_{i \in W} D_i$$

and every edge mapped into an edge of P_H is blue. Colour each element of W with the colour of the edge incident with it. Denote by C^b the subconfiguration of C formed by the blue edges and denote by H^b the corresponding subgraph of H.

By making use of the properties guaranteed by Lemma 1, one can show that there are many colourings of C for which $B = H^b$ is such that

$$|U \cup \Gamma_B(U)| \geqslant 3\,|U|$$

whenever $U \subset \{1, 2, \ldots, h\}$ and $|U| \leqslant h/9$. Combining this with the lemma of Pósa [**8**], which is often used in the search for Hamilton cycles, one can prove the following crucial lemma. The proof, which is an exact analogue of the proofs in [**3**] and [7], is omitted.

LEMMA 3 *There is an absolute constant $c_1 > 0$ such that $c \in \Phi_0$ has at least*

$$(1-3\varepsilon)^{3h}\binom{m}{3h}$$

colourings with the following properties. Set $u = \lfloor c_1 h \rfloor$. There are distinct red elements $y_1, y_2, \ldots, y_u$ and not necessarily distinct sets $Y_1, Y_2, \ldots, Y_u$ such that each Y_i consists of red elements, $|Y_i| = u$ and the red edge incident with y_i does not join y_i to Y_i. □

Now we are ready to complete the proof of the theorem. Let Ψ_0 be the collection of all coloured configurations such that the configuration belongs to Φ_0 and the colouring satisfies the conclusions of Lemma 3. Then

$$|\Psi_0| \geqslant |\Phi_0|\,(1-3\varepsilon)^{3h}\binom{m}{3h}. \tag{1}$$

In order to show that $|\Phi_0|/|\Phi|$ is small, we shall give a suitable upper bound for $|\Psi_0|$. Suppose that $B_0 = C_0^b$ for some $C_0 \in \Psi_0$. At most how many $C \in \Psi_0$ satisfy $C^b = B_0$? When extending B_0 to a configuration $C \in \Psi_0$, we have at most $6h - u$ choices for the red edge, incident with y_i. Hence we have at most $(6h-u)^{\lceil u/2 \rceil}$ choices for the first $w = \lceil u/2 \rceil$ edges incident with $y_1, y_2, \ldots, y_u$. The remaining $3h-w$ edges can be chosen in at most $N(3h-w)$ ways. Hence at most

$$(6h-u)^w N(3h-w) < N(3h)e^{-c_2 h}$$

configurations $C \in \Psi_0$ satisfy $C^b = B_0$, where c_2 is a positive absolute constant.

As clearly

$$|\{C^b \colon C \in \Psi_0\}| \leq N(m)\binom{m}{3h}\Big/ N(3h),$$

we have

$$|\Psi_0| \leq \left\{N(m)\binom{m}{3h}\Big/ N(3h)\right\} N(3h)e^{-c_2 h}. \tag{2}$$

Inequalities (1) and (2) imply

$$|\Phi_0|/N(m) \leq |\Psi_0| \Big/ \left\{N(m)\binom{m}{3h}(1-3\varepsilon)^{3h}\right\} = o(1),$$

provided that ε is small enough. This completes our proof. □

With a little more work one can prove that a polynomial of degree 4 will do for P. However, as it is very likely that P can be chosen to be linear, the additional complications are hardly worthwhile.

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