HAMILTON CYCLES IN A CLASS OF RANDOM DIRECTED GRAPHS

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Abstract

We prove that almost every 3-in, 3-out digraph is Hamiltonian.

1 Introduction

The random, digraph $D_{k-in,\ell-out}$ is defined as follows: it has vertex set $[n] = \{1, 2, \ldots, n\}$ and each $v \in [n]$ chooses a set $in(v)$ of k random edges directed into v and a set $out(v)$ of ℓ random edges directed out of v. We call such a digraph a k-in,ℓ-out digraph. For our purposes it is not important if v chooses edges with or without replacement and we shall assume that they are chosen with replacement. We shall also allow v to choose loops. Thus $D_{k-in,\ell-out}$ usually has about $(k + \ell)n$ edges. The probability space for $D_{k-in,\ell-out}$ will be denoted by $\mathcal{D}_{k-in,\ell-out}$. This model was introduced by Fenner and Frieze [3] who discussed the strong connectivity of $D_{k-in,k-out}$ for $k \geq 2$. The remaining case, where $k = 1$ was discussed by Cooper and Frieze [1], and by McDiarmid and Reed [7].

If $\ell = 0$ we write D_{k-in} and if $k = 0$ then we write $D_{\ell-out}$. If we drop the orientation in D_{k-out} then we obtain the underlying *undirected* graph G_{k-out}. This has been the object of considerable study, and the main outstanding question, is how large should k be for G_{k-out} to have a Hamilton cycle **whp** (with high probability i.e. probability $1 - o(1)$ as $n \to \infty$).

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It is currently known that $k \geq 5$ is sufficient, (Frieze and Łuczak [4]), and it is conjectured that the correct lower bound for k is 3. This paper considers the directed version of this problem.

Theorem 1 $D_{3-in,3-out}$ *is Hamiltonian* **whp**.

$\square$

This result is unlikely to be best possible and we conjecture that $D_{2-in,2-out}$ is Hamiltonian **whp**.

To prove the theorem we will regard $D = D_{3-in,3-out}$ as the union of independent random digraphs $D_a \cup D_b$. Here $D_a \in \mathcal{D}_{2-in,2-out}$ and $D_b \in \mathcal{D}_{1-in,1-out}$.

To avoid confusion we refer to the functions selecting unique members of the in, out sets of D_b as in_b and out_b.

$B(n,p)$ denotes the Binomial random variable with parameters n, p. A *permutation digraph* is a set of vertex disjoint directed cycles that cover all n vertices. Its *size* is the number of cycles.

We will use a *three phase* method as outlined below.

Phase 1. We show that **whp** D_a contains a directed permutation digraph of size at most $2\log n$.

Phase 2. Using D_b we increase the minimum cycle length in the permutation digraph to at least $n_0 = \left\lceil \frac{100n}{\log n} \right\rceil$.

Phase 3. Using D_b we convert the *Phase 2* permutation digraph to a Hamilton cycle.

In what follows inequalities are only claimed to hold for n sufficiently large.

2 Phase 1. Making a permutation digraph with at most $2\log n$ cycles

With any digraph Δ on n vertices there is an associated bipartite graph G with $n+n$ vertices, which contains an edge (u,v) iff Δ contains the directed edge (u,v). It is well known that perfect matchings in G are in 1-1 correspondence with permutation digraphs of Δ.

We start with the random digraph D_a.

Lemma 2 **Whp** D_a *contains a permutation digraph with at most* $2\log n$ *cycles.*

Proof. Walkup [8] has shown that **whp** D_a's associated bipartite graph contains a perfect matching $\{(i,\phi(i)), i = 1,2,\dots,n\}$. We can argue by symmetry (as in [4]) that we can take ϕ to be a random permutation. It is well known (e.g. Feller [2]), that **whp** a random permutation contains at most $2\log n$ cycles, and thus the permutation digraph has size at most $2\log n$. □

Thus at the end of Phase 1, we can assume we have a permutation digraph Π_0 of size at most $2\log n$. The remaining unused edges of D_a have no further part to play.

3 Phase 2. Removing small cycles

We partition the cycles of the permutation digraph Π_0 into sets SMALL and LARGE, containing cycles C of length $|C| < n_0$ and $|C| \geq n_0$ respectively. We define a Near Permutation Digraph (NPD) to be a digraph obtained from a permutation digraph by removing one edge. Thus an NPD Γ consists of a path $P(\Gamma)$ plus a permutation digraph $PD(\Gamma)$ which covers $[n] \setminus V(P(\Gamma))$.

We now give an informal description of a process which removes a small cycle C from a *current* permutation digraph Π. We start by choosing an (arbitrary) edge (v_0, u_0) of C and delete it to obtain an NPD Γ_0 with $P_0 = P(\Gamma_0) \in \mathcal{P}(u_0, v_0)$, where $\mathcal{P}(x, y)$ denotes the set of paths from x to y in D. The aim of the process is to produce a *large* set S of NPD's such that for each $\Gamma \in S$, (i) $P(\Gamma)$ has a least n_0 edges and (ii) the small cycles of $PD(\Gamma)$ are a subset of the small cycles of Π. We will show that **whp** the endpoints of one of the $P(\Gamma)$'s can be joined by an edge to create a permutation digraph with (at least) one less small cycle.

The basic step in an *Out-Phase* of this process is to take an NPD Γ with $P(\Gamma) \in \mathcal{P}(u_0, v)$ and to examine the edges of D_b leaving v i.e. edges going *out* from the end of the path. Let w be the terminal vertex of such an edge and assume that Γ contains an edge (x, w). Then $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$ is also an NPD. Γ' is acceptable if (i) $P(\Gamma')$ contains at least n_0 edges and (ii) any new cycle created (i.e. in Γ' and not Γ) also has at least n_0 edges.

If Γ contains no edge (x, w) then $w = u_0$. We accept the edge if P has at least n_0 edges. This would (prematurely) end an iteration, although it is unlikely to occur.

We do not want to look at very many edges of D_b in this construction and we build a tree T_0 of NPD's in a natural breadth-first fashion where each non-leaf vertex Γ gives rise to NPD children Γ' as described above. The construction of T_0 ends when we first have $\nu = \left\lceil \sqrt{n \log n} \right\rceil$ leaves. The construction of T_0 constitutes an Out-Phase of our procedure to eliminate small cycles. Having constructed T_0 we need to do a further *In-Phase*, which is similar to a set of Out-Phases.

Then **whp** we close at least one of the paths $P(\Gamma)$ to a cycle of length at least n_0. If $|C| \geq 4$

and this process fails then we try again with a different independent edge of C in place of (u_0, v_0).

We now increase the the formality of our description. We start Phase 2 with a permutation digraph Π_0 and a general iteration of Phase 2 starts with a permutation digraph Π whose small cycles are a subset of those in Π_0. Iterations continue until there are no more small cycles. At the start of an iteration we choose some small cycle C of Π. There then follows an Out-Phase in which we construct a tree $T_0 = T_0(\Pi, C)$ of NPD's as follows: the root of T_0 is Γ_0 which is obtained by deleting an edge (v_0, u_0) of C.

We grow T_0 to a depth at most $\lceil 1.5 \log n \rceil$. The set of nodes at depth t is denoted by S_t. Let $\Gamma \in S_t$ and $P = P(\Gamma) \in \mathcal{P}(u_0, v)$. The *potential* children Γ' of Γ, at depth $t+1$ are defined as follows.

Let w be the terminal vertex of an edge directed from v in D_b.
Case 1. w is a vertex of a cycle $C' \in PD(\Gamma)$ with edge $(x, w) \in C'$. Let $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$.
Case 2. w is a vertex of $P(\Gamma)$. Either $w = u_0$, or (x, w) is an edge of P. In the former case $\Gamma \cup \{(v, w)\}$ is a permutation digraph Π' and in the latter case we let $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$.

In fact we only admit to S_{t+1} those Γ' which satisfy the following conditions.
C(i) The new cycle formed (Case 2 only) must have at least n_0 vertices, and the path formed (both cases) must either be empty or have at least n_0 vertices. When the path formed is empty we close the iteration and if necessary start the next with Π'.

We now define a set W of *used* vertices. Initially all vertices are *unused* i.e. $W = \emptyset$. Whenever we examine an edge (v, w), we add both v and w to W. So if $v \notin W$ then $out_b(v)$ is still unconditioned and $in_b(v)$ is a random member of a set $U \supseteq [n] \setminus W$. We do not allow $|W|$ to exceed $n^{3/4}$.

C(ii) $x, w \notin W$.

An edge (v, w) which satisfies the above conditions is described as *acceptable.*

Lemma 3 *Let $C \in SMALL$. Then, where $\nu = \left\lceil \sqrt{n \log n} \right\rceil$, (reminder)*

$$Pr(\exists t < \lceil \log_{1.9} \nu + 1000 \log \log n \rceil \textit{ such that } |S_t| \in [\nu, 3\nu]) = 1 - O((\log \log n)^3 / \log n).$$

Proof. We assume we stop an iteration, in mid-phase if necessary, when $|S_t| \in [\nu, 3\nu]$. Let us consider a generic construction in the growth of T_0. Thus suppose we are extending from Γ and $P(\Gamma) \in \mathcal{P}(u_0, v)$.

We consider S_{t+1} to be constructed in the following manner: we first examine $out_b(v), v \in S_t$ in the order that these vertices were placed in S_t to see if they produce acceptable edges. We then add in those vertices $x \notin W$ which arise from (x, w) with $v = in_b(w) \in S_t, w \notin W$

Let $Z(v)$ be the indicator random variable for $(v, out_b(v))$ being unacceptable and let $Z_t = \sum_{v \in S_t} Z(v)$. If $Z(v) = 1$ then either (i) $out_b(v)$ lies on $P(\Gamma)$ and is too close to an endpoint; this has probability bounded above by $201/\log n$, or (ii) the corresponding vertex x is in W; this has probability bounded above by $n^{-1/4}$, or (iii) $out_b(v)$ lies on a small cycle. Now in a random permutation the expected number of vertices on cycles of length at most n_0 is precisely n_0 ([6]). Thus, by the Markov inequality, **whp** Γ_0 contains at most $n \log\log n/(2\log n)$ vertices on small cycles. Condition on this event. Then $Pr(Z(v) = 1) \leq \log\log n/\log n$ regardless of the history of the process and so Z_t is stochastically dominated by $B(|S_t|, \log\log n/\log n)$.

Next let $X(v)$ denote the number of vertices w in $[n] \setminus W$ such that $in_b(w) = v$, $x \notin W$ where (v, w) is acceptable and $(x, w) \in \Gamma$ (if there is no such x then the iteration can end early.) Let $X_t = \sum_{v \in S_t} X(v)$. Now assuming $|W| \leq n^{3/4}$ we see that there are $n' = n - O(n \log\log n/\log n)$ vertices w which would produce an acceptable edge provided $in_b(w) \in S_t$. For these vertices $in_b(w)$ is a random choice from a set which contains S_t and so X_t stochastically dominates $B(n', |S_t|/n)$.

Summing $1 - Z(v) + X(v)$ over $v \in S_t$ might seem to overestimate $|S_{t+1}|$. In principle we should subtract off the number Y_t of vertices of S_{t+1} that are counted more than once in this sum. But these arise in two ways. First there are the pairs $v_1, v_2 \in S_t$ with $out_b(v_1) = out_b(v_2)$. Suppose we examine v_1 before v_2. Then when we examine v_2 we find that $out_b(v_2) \in W$ and so we do not get a contribution to S_{t+1}. Secondly there is the possibility of their being $v_1, v_2 \in S_t$ and w such that $w = out_b(v_1)$ and $v_2 = in_b(w)$. But in this case w will only be counted once as $w \in W$ when it is time for $in_b(w)$ to be examined. We can then write

$$|S_{t+1}| = |S_t| - Z_t + X_t.$$

Now let $t_0 = \lceil 1000 \log\log n \rceil$, $t_1 = 10t_0$, $t_2 = \lceil \log_{1.9} \nu + 1000 \log\log n \rceil$ and $s_0 = \lceil 1000 \log n \rceil$.

(a) $Pr(\exists t \leq t_0 : |S_t| \leq 500 \log\log n \text{ and } Z_t > 0) = O((\log\log n)^3/\log n)$

(b) $Pr(\sum_{t=1}^{t_0} X_t \leq 500 \log\log n \mid S_t \neq \emptyset) = O(1/\log n)$

(c) $Pr(\exists t \leq t_1 : 500 \log\log n \leq |S_t| \leq s_0 \text{ and } Z_t > X_t/100) = O(1/\log n)$.

(d) $Pr(\exists t \leq t_1 : X_t < |S_t|/2 \mid |S_t| \geq 500 \log\log n) = O(1/\log n)$.

(e) $Pr(\exists t \leq t_1 : |S_t| \leq s_0 \text{ and } X_t \geq 2s_0) = O(n^{-2})$.

(f) $Pr(\exists t \leq n : |S_t| \geq s_0 \text{ and } ||X_t - Z_t - |S_t|| \geq |S_t|/10) = O(n^{-2})$.

Explanations:- we use the following standard inequalities for the tails of the binomial distribution:

$$Pr(|B(n,p) - np| \geq \epsilon np) \leq 2e^{-\epsilon^2 np/3}, \qquad 0 \leq \epsilon \leq 1, \tag{1}$$

$$Pr(B(n,p) \geq anp) \leq (e/a)^{anp}. \tag{2}$$

(a) $Pr(Z_t > 0 \mid \; |S_t| \leq 500 \log\log n) = O((\log\log n)^2/\log n)$ by the Markov inequality.

(b) $\sum_{t=1}^{t_0} X_t$ dominates $B(t_0 n', 1/n)$ since given $S_t \neq \emptyset$, X_t dominates $B(n', 1/n)$.

(c) Condition on $|S_t| = s \geq 500 \log\log n$. Then $Z_t > X_t/100$ implies either that (i) $X_t \leq s/10$ or (ii) $Z_t > 10s$. Both of these events have probability $O(1/(\log n)^3)$.

(d) Immediate from (1).

(e) Immediate from (1) and (2).

(f) Similar to (c).

Let $\mathcal{E}_x, x \in \{a, b, \ldots, f\}$ be the low probability events described in (a)-(f) above. Assume the occurrence of $\bigcap_x \bar{\mathcal{E}}_x$. Then $\bar{\mathcal{E}}_a \cap \bar{\mathcal{E}}_b$ implies that $|S_t|$ reaches size at least $500 \log\log n$ before t reaches t_0. Once this happens, $\bar{\mathcal{E}}_c \cap \bar{\mathcal{E}}_d$ implies that $|S_t|$ then grows geometrically with t up to time t_1 at a rate of at least 1.49. Together with $\bar{\mathcal{E}}_e$ this proves that at some stage between 1 and t_1, $|S_t|$ reaches a size in the range $[s_0, 3s_0]$. $\bar{\mathcal{E}}_f$ then implies that $|S_t|$ increases at a rate $\lambda \in [1.9, 2.1]$ from then on. The lemma follows.

$\square$

The total number of vertices added to W in this way throughout the whole of Phase 2 is $O(\nu|SMALL|) = o(n^{3/4})$. (As we see later, we try this process once for $C \in SMALL, |C| \leq 3$ and once or twice for $C \in SMALL, |C| \geq 4$.)

Let t^* denote the value of t when we stop the growth of T_0. At this stage we have leaves Γ_i, for $i = 1, \ldots, \nu$, each with a path of length at least n_0, (unless we have already successfully made a cycle). We now execute an In-Phase. This involves the construction of trees $T_i, i = 1, 2, \ldots \nu$. Assume that $P(\Gamma_i) \in \mathcal{P}(u_0, v_i)$. We start with Γ_i and build T_i in a similar way to T_0 except that here all paths generated end with v_i. This is done as follows: if a current NPD Γ has $P(\Gamma) \in \mathcal{P}(u, v_i)$ then we consider adding an edge $(w, u) \in D_b$ and deleting an edge $(w, x) \in \Gamma$. Thus our trees are grown by considering edges directed into the start vertex of each $P(\Gamma)$ rather than directed out of the end vertex. Some technical changes are necessary however.

We consider the construction of our ν trees in two stages. First of all we grow the trees only enforcing condition C(ii) of success and thus allow the formation of small cycles and paths. We try to grow them to depth t_2. The growth of the ν trees can naturally be considered to occur simultaneously. Let $L_{i,\ell}$ denote the set of start vertices of the paths associated with the nodes at depth ℓ of the i'th tree, $i = 1, 2 \ldots, \nu, \ell = 0, 1, \ldots, t_2$. Thus $L_{i,0} = \{u_0\}$ for all i. We prove inductively that $L_{i,\ell} = L_{1,\ell}$ for all i, ℓ. In fact if $L_{i,\ell} = L_{1,\ell}$ then the acceptable D_b edges have the same set of initial vertices and since all of the deleted edges are D_a-edges (enforced by C(ii)) we have $L_{i,\ell+1} = L_{1,\ell+1}$.

The probability that we succeed in constructing trees $T_1, T_2, \ldots T_\nu$ is, by the analysis of Lemma 3, $1 - O((\log\log n)^3/\log n)$. Note that the number of nodes in each tree is $O(2.1^{t_2+1}) = O(n^{.74\ldots})$.

We now consider the fact that in some of the trees some of the leaves may have been constructed in violation of C(i). We imagine that we prune the trees $T_1, T_2, \ldots T_\nu$ by disallowing any node that was constructed in violation of C(i). Let a tree be BAD if after pruning it has less than ν leaves and GOOD otherwise. Now an individual pruned tree has been constructed in the same manner as the tree T_0 obtained in the Out-Phase. (We have chosen t_2 to obtain ν leaves even at the slowest growth rate of 1.9 per node.) Thus

$$Pr(T_1 \text{ is BAD}) = O\left(\frac{(\log\log n)^3}{\log n}\right)$$

and

$$E(\text{number of BAD trees}) = O\left(\frac{\nu(\log\log n)^3}{\log n}\right)$$

and

$$Pr(\exists \geq \nu/2 \text{ BAD trees}) = O\left(\frac{(\log\log n)^3}{\log n}\right).$$

Thus

$$\begin{array}{rl} & Pr(\exists < \nu/2 \text{ GOOD trees after pruning}) \\ \leq & Pr(\text{failure to construct } T_1, T_2, \ldots T_\nu) + Pr(\exists \geq \nu/2 \text{ BAD trees}) \\ = & O\left(\frac{(\log\log n)^3}{\log n}\right) \end{array}$$

Thus with probability 1-$O((\log\log n)^3/\log n)$ we end up with $\nu/2$ sets of ν paths, each of length at least $100n/\log n$ where the i'th set of paths all terminate in v_i. The $in_b(v_i)$ are still unconditioned and hence

$$\begin{array}{rcl} Pr(\text{no } D_b \text{ edge closes one of these paths}) & \leq & \left(1 - \frac{\nu}{n}\right)^{\nu/2} \\ & = & O(n^{-1/2}). \end{array}$$

Consequently the probability that we fail to eliminate a particular small cycle C after breaking an edge is $O((\log\log n)^3/\log n)$. If $|C| \geq 4$ then we try once or twice using independent edges of C and so the probability we fail to eliminate a given small cycle C is certainly $O(((\log\log n)^3/\log n)^2)$ for $|C| \geq 4$ (remember that we calculated all probabilities conditional on previous outcomes and assuming $|W| \leq n^{3/4}$.)

Now the number of cycles of length 1,2 or 3 in D_a is asymptotically Poisson with mean 11/6 and so there are fewer than $\log\log n$ **whp**. Hence, since **whp** $|C| = O(\log n)$,

Lemma 4 *The probability that Phase 2 fails to produce a permutation digraph with minimal cycle length at least* n_0 *is* $o(1)$.

$\square$

At this stage we have shown that a 3-in,3-out digraph almost always contains a permutation digraph Π^* in which the minimum cycle length is at least n_0.

We shall refer to Π^* as the *Phase 2* permutation digraph.

4 Phase 3. Patching the Phase 2 permutation digraph to a Hamilton cycle

Let $C_1, C_2, \dots, C_k$ be the cycles of Π^*, and let $c_i = |C_i \setminus W|$, $c_1 \le c_2 \le \dots \le c_k$, and $c_1 \ge n_0 - n^{3/4} \ge \frac{99 \log n}{n}$. If $k = 1$ we can skip this phase, otherwise let $a = \frac{n}{\log n}$. For each C_i we consider selecting a set of $m_i = 2\lfloor \frac{c_i}{a} \rfloor + 1$ vertices $v \in C_i \setminus W$, and deleting the edge (v, u) in Π^*. Let $m = \sum_{i=1}^{k} m_i$ and relabel (temporarily) the broken edges as $(v_i, u_i), i \in [m]$ as follows: in cycle C_i identify the lowest numbered vertex x_i which loses a cycle edge directed out of it. Put $v_1 = x_1$ and then go round C_1 defining $v_2, v_3, \dots v_{m_1}$ in order. Then let $v_{m_1+1} = x_2$ and so on. We thus have m path sections $P_j \in \mathcal{P}(u_{\phi(j)}, v_j)$ in Π^* for some permutation ϕ. We see that ϕ is an even permutation as all the cycles of ϕ are of odd length.

It is our intention to rejoin these path sections of Π^* to make a Hamilton cycle using D_b, if we can. Suppose we can. This defines a permutation ρ where $\rho(i) = j$ if P_i is joined to P_j by $(v_i, u_{\phi(j)})$, where $\rho \in H_m$ the set of cyclic permutations on $[m]$. We will use the second moment method to show that a suitable ρ exists **whp**. A technical problem forces a restriction on our choices for ρ. This will produce a variance reduction in a second moment calculation.

Given ρ define $\lambda = \phi\rho$. In our analysis we will restrict our attention to $\rho \in R_\phi = \{\rho \in H_m : \phi\rho \in H_m\}$. If $\rho \in R_\phi$ then we have not only constructed a Hamilton cycle in $\Pi^* \cup D_b$, but also in the *auxillary digraph* Λ, whose edges are $(i, \lambda(i))$.

Lemma 5 $(m-2)! \le |R_\phi| \le (m-1)!$

Proof. We grow a path $1, \lambda(1), \lambda^2(1), \dots, \lambda^r(1) \dots$ in Λ, maintaining feasibility in the way we join the path sections of Π^* at the same time.

We note that the edge $(i, \lambda(i))$ of Λ corresponds in D_b to the edge $(v_i, u_{\phi\rho(i)})$. In choosing

$\lambda(1)$ we must avoid not only 1 but also $\phi(1)$ since $\lambda(1) = 1$ implies $\rho(1) = 1$. Thus there are $m - 2$ choices for $\lambda(1)$ since $\phi(1) \neq 1$ from the definition of m_1.

In general, having chosen $\lambda(1), \lambda^2(1), \ldots, \lambda^r(1), 1 \leq r \leq m - 3$ our choice for $\lambda^{r+1}(1)$ is restricted to be different from these choices and also 1 and ℓ where u_ℓ is the initial vertex of the path terminating at $v_{\lambda^r(1)}$ made by joining path sections of Π^*. Thus there are either $m - (r + 1)$ or $m - (r + 2)$ choices for $\lambda^{r+1}(1)$ depending on whether or not $\ell = 1$.

Hence, when $r = m - 3$, there *may* be only one choice for $\lambda^{m-2}(1)$, the vertex h say. After adding this edge, let the remaining isolated vertex of Λ be w. We now need to show that we can complete λ, ρ so that $\lambda, \rho \in H_m$.

Which vertices are missing edges in Λ at this stage ? Vertices $1, w$ are missing in-edges, and h, w out-edges. Hence the path sections of Π^* are joined so that either

$$u_1 \longrightarrow v_h, \;\; u_w \longrightarrow v_w \quad \text{or} \quad u_1 \longrightarrow v_w, \;\; u_w \longrightarrow v_h.$$

The first case can be (uniquely) feasibly completed in both Λ and D by setting $\lambda(h) = w, \lambda(w) = 1$. Completing the second case to a cycle in Π^* means that

$$\lambda = (1, \lambda(1), \ldots, \lambda^{m-2}(1))(w) \tag{3}$$

and thus $\lambda \notin H_m$. We show this case cannot arise.

$\lambda = \phi\rho$ and ϕ is even implies that λ and ρ have the same parity. On the other hand $\rho \in H_m$ has a different parity to λ in (3) which is a contradiction.

Thus there is a (unique) completion of the path in Λ. $\square$

Let H stand for the union of the permutation digraph Π^* and D_b. We finish our proof by proving

Lemma 6 *Pr(H does not contain a Hamilton cycle)* $= o(1)$.

Proof. Let X be the number of Hamilton cycles in H obtainable by deleting edges as above, rearranging the path sections generated by ϕ according to those $\rho \in R_\phi$ and if possible reconnecting all the sections using edges of D_b. We will use the inequality

$$Pr(X > 0) \geq \frac{E(X)^2}{E(X^2)}. \tag{4}$$

Probabilities in (4) are thus with respect to the space of D_b choices for edges incident with vertices not in W.

Now the definition of the m_i yields that

$$\frac{2n}{a} - k \leq m \leq \frac{2n}{a} + k$$

and so

$$(1.99)\log n \le m \le (2.01)\log n.$$

Also

$$k \le \log n/100, m_i \ge 199 \text{ and } \frac{c_i}{m_i} \ge \frac{a}{2.01}, \qquad 1 \le i \le k.$$

Let Ω denote the set of possible cycle re-arrangements. $\omega \in \Omega$ is a *success* if D_b contains the edges needed for the associated Hamilton cycle. Thus,

$$\begin{aligned}
E(X) &= \sum_{\omega\in\Omega} Pr(\omega \text{ is a success}) \\
&= \sum_{\omega\in\Omega} \left(1-\left(1-\frac{1}{n}\right)^2\right)^m \\
&\ge (1-o(1))\left(\frac{2}{n}\right)^m (m-2)!\prod_{i=1}^k \binom{c_i}{m_i} \\
&\ge \frac{1-o(1)}{m\sqrt{m}}\left(\frac{2m}{en}\right)^m \prod_{i=1}^k \left(\left(\frac{c_i e^{1-1/12m_i}}{m_i^{1+(1/2m_i)}}\right)^{m_i}\left(\frac{1-2m_i^2/c_i}{\sqrt{2\pi}}\right)\right) \\
&\ge \frac{(1-o(1))(2\pi)^{-m/398}e^{-k/12}}{m\sqrt{m}}\left(\frac{2m}{en}\right)^m \prod_{i=1}^k \left(\frac{c_i e}{(1.02)m_i}\right)^{m_i} \\
&\ge \frac{(1-o(1))(2\pi)^{-m/398}}{n^{1/1200}m\sqrt{m}}\left(\frac{2m}{en}\right)^m \left(\frac{ea}{2.01\times 1.02}\right)^m \\
&\ge \frac{(1-o(1))(2\pi)^{-m/398}}{n^{1/1200}m\sqrt{m}}\left(\frac{3.98}{2.0502}\right)^m \\
&\ge n^{1.3}. \qquad (5)
\end{aligned}$$

Let M, M' be two sets of selected edges which have been deleted in Π^* and whose path sections have been rearranged into Hamilton cycles according to ρ, ρ' respectively. Let N, N' be the corresponding sets of edges which have been added to make the Hamilton cycles. What is the interaction between these two Hamilton cycles?

Let $s = |M \cap M'|$ and $t = |N \cap N'|$. Now $t \le s$ since if $(v, u) \in N \cap N'$ then there must be a unique $(\tilde{v}, u) \in M \cap M'$ which is the unique Π^*-edge into u. We claim that $t = s$ implies $t = s = m$ and $(M, \rho) = (M', \rho')$. (This is why we have restricted our attention to $\rho \in R_{\phi}$.) Suppose then that $t = s$ and $(v_i, u_i) \in M \cap M'$. Now the edge $(v_i, u_{\lambda(i)}) \in N$ and since $t = s$ this edge must also be in N'. But this implies that $(v_{\lambda(i)}, u_{\lambda(i)}) \in M'$ and hence in $M \cap M'$. Repeating the argument we see that $(v_{\lambda^k(i)}, u_{\lambda^k(i)}) \in M \cap M'$ for all $k \ge 0$. But λ is cyclic and so our claim follows.

We adopt the following notation. Let $< s, t >$ denote $|M \cap M'| = s$ and $|N \cap N'| = t$. So

$$E(X^2) \le E(X) + (1+o(1)) \sum_{M\in\Omega}\left(\frac{2}{n}\right)^m \sum_{\substack{\Omega \\ N'\cap N=\emptyset}} \left(\frac{2}{n}\right)^m$$

$$\begin{aligned} & +(1+o(1))\sum_{M\in\Omega}\left(\frac{2}{n}\right)^{m}\sum_{s=2}^{m}\sum_{t=1}^{s-1}\sum_{\substack{\Omega\\<s,t>}}\left(\frac{2}{n}\right)^{m-t} \\ = \quad & E(X)+E_1+E_2 \ \text{ say.} \end{aligned} \tag{6}$$

Clearly

$$E_1 \leq (1+o(1))E(X)^2. \tag{7}$$

For given ρ, how many ρ' satisfy the condition $< s,t >$? Previously $|R_{\phi}| \geq (m-2)!$ and now given $< s,t >$, $|R_{\phi}(s,t)| \leq (m-t-1)!$, (consider fixing t edges of Λ').
Thus

$$E_2 \leq E(X)^2 \sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\left[\sum_{\sigma_1+\cdots+\sigma_k=s}\prod_{i=1}^{k}\frac{\binom{m_i}{\sigma_i}\binom{c_i-m_i}{m_i-\sigma_i}}{\binom{c_i}{m_i}}\right]\frac{(m-t-1)!}{(m-2)!}\left(\frac{n}{2}\right)^{t}.$$

Now

$$\begin{aligned} \frac{\binom{c_i-m_i}{m_i-\sigma_i}}{\binom{c_i}{m_i}} & \leq \frac{\binom{c_i}{m_i-\sigma_i}}{\binom{c_i}{m_i}} \\ & \leq (1+o(1))\left(\frac{m_i}{c_i}\right)^{\sigma_i}\exp\left\{-\frac{\sigma_i(\sigma_i-1)}{2m_i}\right\} \\ & \leq (1+o(1))\left(\frac{2.01}{a}\right)^{\sigma_i}\exp\left\{-\frac{\sigma_i(\sigma_i-1)}{2m_i}\right\} \end{aligned}$$

where the $o(1)$ term is $O((\log n)^3/n)$. Also

$$\sum_{i=1}^{k}\frac{\sigma_i^2}{2m_i} \geq \frac{s^2}{2m} \qquad \text{for } \sigma_1+\cdots\sigma_k=s,$$

$$\sum_{i=1}^{k}\frac{\sigma_i}{2m_i} \leq \frac{k}{2},$$

and

$$\sum_{\sigma_1+\cdots+\sigma_k=s}\prod_{i=1}^{k}\binom{m_i}{\sigma_i} = \binom{m}{s}.$$

Hence

$$\begin{aligned} \frac{E_2}{E(X)^2} & \leq (1+o(1))e^{k/2}\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\exp\left\{-\frac{s^2}{2m}\right\}\left(\frac{2.01}{a}\right)^{s}\binom{m}{s}\frac{(m-t-1)!}{(m-2)!}\left(\frac{n}{2}\right)^{t} \\ & \leq (1+o(1))n^{.01}\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\exp\left\{-\frac{s^2}{2m}\right\}\left(\frac{2.01}{a}\right)^{s}\frac{m^{s-(t-1)}}{(s-1)!}\left(\frac{n}{2}\right)^{t} \end{aligned}$$

$$
\begin{aligned}
&= (1+o(1))n^{.01}\sum_{s=2}^{m}\left(\frac{2.01}{a}\right)^s\frac{m^s}{s!}\exp\left\{-\frac{s^2}{2m}\right\}m\sum_{t=1}^{s-1}\binom{s}{t}\left(\frac{n}{2m}\right)^t\\
&\leq (1+o(1))\left(\frac{2m^3}{n^{.99}}\right)\sum_{s=2}^{m}\left(\frac{(2.01)n\exp\{-s/2m\}}{2a}\right)^s\frac{1}{s!}\\
&= o(1) \qquad (8)
\end{aligned}
$$

To verify that the RHS of (8) is $o(1)$ we can split the summation into

$$S_1 = \sum_{s=2}^{\lfloor m/4\rfloor}\left(\frac{(2.01)n\exp\{-s/2m\}}{2a}\right)^s\frac{1}{s!}$$

and

$$S_2 = \sum_{s=\lfloor m/4\rfloor+1}^{m}\left(\frac{(2.01)n\exp\{-s/2m\}}{2a}\right)^s\frac{1}{s!}.$$

Ignoring the term $\exp\{-s/2m\}$ we see that

$$
\begin{aligned}
S_1 &\leq \sum_{s=2}^{\lfloor(.5025)\log n\rfloor}\frac{((1.005)\log n)^s}{s!}\\
&= o(n^{9/10})
\end{aligned}
$$

since this latter sum is dominated by its last term.

Finally, using $\exp\{-s/2m\} < e^{-1/8}$ for $s > m/4$ we see that

$$S_2 \leq n^{(1.005)e^{-1/8}} < n^{9/10}.$$

The result follows from (4) to (8). □

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