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ON THE INDEPENDENCE NUMBER OF RANDOM GRAPHS

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Let $\alpha(G_{n,p})$ denote the independence number of the random graph $G_{n,p}$. Let $d = np$. We show that if $\epsilon > 0$ is fixed then with probability going to 1 as $n \to \infty$

$$\left|\alpha(G_{n,p}) - \frac{2n}{d}(\log d - \log\log d - \log 2 + 1)\right| \leq \frac{\epsilon n}{d}$$

provided $d_\epsilon \leq d = o(n)$, where d_ϵ is some fixed constant.

This note is concerned with the independence number of random graphs. As usual $G_{n,p}$ denotes a random graph with vertex set $V_n = \{1, 2, \ldots, n\}$ in which each possible edge is independently included with probability $p = p(n)$. The independence number $\alpha(G_{n,p})$ is the size of the largest set of vertices not containing any edge. This has been studied by, inter alia, Matula [5], Grimmett and McDiarmid [4] and Bollobás and Erdős [3]. The aim of this paper is to prove the following

Theorem. *Let $d = np$ and $\epsilon > 0$ be fixed. Suppose $d_\epsilon \leq d = o(n)$ for some sufficiently large fixed constant d_ϵ. Then*

$$\left|\alpha(G_{n,p}) - \frac{2n}{d}(\log d - \log\log d - \log 2 + 1)\right| \leq \frac{\epsilon n}{d}$$

with probability going to 1 as $n \to \infty$.

(All logarithms are natural). The case p constant is well understood and the content of the theorem is already known for $d > n^{\frac{2}{3}}$ (see Bollobás [1, 2]). The upper bound of the theorem is already known and straightforward to prove (see [1] Lemma X1.21). The lower bound is close to what one might expect and our aim is to prove it and demonstrate what may turn out to be a useful approach for other problems.

Acknowledgment

This theorem represents a non-trivial strengthening of a result in a previous version. This comes about from an idea of Tomasz Luczak (see later). I am pleased to acknowledge his contribution.

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Let $m = \left[\frac{d}{(\log d)^2}\right]$, $n' = \lfloor n/m \rfloor$ and $P_i = \{(i-1)m+1, \ldots, im\}$, $i = 1, 2, \ldots, n'$ be a partition of $V_{mn'}$. Let a set $X \subseteq V_{mn'}$ be P-independent if it is independent and satisfies $|X \cap P_i| \leq 1$ for $i = 1, 2, \ldots, n'$. Let $\beta(G_{n,p}) \leq \alpha(G_{n,p})$ denote the size of the largest P-independent subset of $G_{n,p}$. (It was Luczak who suggested β in place of α. This leads to a strengthening of our original result).

Let X_l denote the (random) number of P-independent sets of size l in $G_{n,p}$. The theorem follows from the following

Lemma. (a) *Let* $\bar{\beta} = E(\beta(G_{n,p}))$. *Then*

$$\Pr(|\beta(G_{n,p}) - \bar{\beta}| \geq t) \leq 2\exp\left\{-\frac{t^2 d}{2(\log d)^2 n}\right\} \quad \textit{for } t > 0.$$

(b) *Let* $k = (2n/d)(\log d - \log\log d - \log 2 + 1 - (\epsilon/3))$. *Then*

$$\Pr(X_k > 0) \geq \exp\left\{-\frac{200(\log d)^{\frac{5}{2}}}{d^{\frac{3}{2}}} n\right\}.$$

Indeed, putting t equal to $t_0 = \epsilon n/6d$ in (a) and comparing with (b) we see that $\bar{\beta} \geq k - t_0$. We then apply (a) again with $t = t_0$ to obtain the lower bound of the theorem. (In the following, inequalities need only hold for sufficiently large d and sufficiently small values of p).

Proof of the lemma. (a) Using a martingale inequality of Azuma (see Stout [7]), Shamir and Spencer [6] have shown that

$$\Pr(|Z - E(Z)| \geq t) \leq 2e^{-t^2/2n} \quad \text{for } t > 0$$

for any random variable Z defined on $G_{n,p}$ satisfying

$$|Z(G) - Z(G')| \leq 1$$

whenever G' can be obtained from G by changing the edges incident with a single vertex. This is clearly true of the random variable $\alpha(G_{n,p})$.

The same proof yields

$$\Pr(|Z - E(Z)| \geq t) \leq 2e^{-t^2/2n'}$$

for any random variable Z defined on $G_{n,p}$ which satisfies

$$|Z(G) - Z(\hat{G})| \leq 1$$

whenever $\hat{G}$ can be obtained from G by changing some of the edges incident with the vertices in a single P_i. This is clearly true of $\beta(G_{n,p})$ and (a) follows. (See Bollobás [2] for a superlative use of a martingale inequality in the solution of the chromatic number problem for dense random graphs. Also Shamir and Spencer [6] prove a sharp concentration result for the chromatic number of sparse random graphs by using an "(a) type" inequality plus a "(b) type" inequality with an unknown k.)

(b) We use the inequality

$$\Pr(X_k > 0) \geqslant E(X_k)^2/E(X_k^2). \tag{1}$$

Now

$$E(X_k) = \binom{n'}{k} m^k (1-p)^{\binom{k}{2}}$$

and

$$E(X_k^2) \leqslant E(X_k) \sum_{l=0}^{k} \binom{k}{l}\binom{n'-l}{k-l} m^{k-l}(1-p)^{\binom{k}{2}-\binom{l}{2}}.$$

Thus

$$\frac{E(X_k)^2}{E(X_k)^2} \leqslant \sum_{l=0}^{k} \frac{\binom{k}{l}\binom{n'-l}{k-l}}{\binom{n'}{k} m^l (1-p)^{\binom{l}{2}}}$$

$$\leqslant \exp\{2(\log d)^2\} \sum_{l=0}^{k} u_l \tag{2}$$

where

$$u_l = \frac{\binom{k}{l}\binom{n'-l}{k-l}}{\binom{n'}{k} m^l} \exp\left\{\frac{l^2 d}{2n}\right\}.$$

Observe that $(A/l)^l$ is maximised at $l = A/\mathrm{e}$ and so

$$(A/l)^l \leqslant \mathrm{e}^{A/\mathrm{e}} \tag{3}$$

and

$$u_l \leqslant \left(\frac{k\mathrm{e}}{l} \cdot \frac{k}{n'm} \cdot \exp\left\{\frac{ld}{2n}\right\}\right)^l$$

$$\leqslant \left(\frac{k}{l} \cdot \frac{6 \log d}{d} \cdot \exp\left\{\frac{ld}{2n}\right\}\right)^l.$$

Case 1. $0 \leqslant l \leqslant k/2$.

Here $\exp\left\{\frac{ld}{2n}\right\} \leqslant \sqrt{d}$ and so, by (4)

$$u_l \leqslant \left(\frac{6k \log d}{l\sqrt{d}}\right)^l$$

$$\leqslant \exp\left\{\frac{6k \log d}{\mathrm{e}\sqrt{d}}\right\}, \quad \text{by (3)}$$

$$\leqslant \exp\left\{\frac{3(\log d)^2}{d^{\frac{3}{2}}} n\right\}. \tag{5}$$

Case 2. $k/2 < l \leqslant \frac{2n}{d}(\log d - \log\log d - 3)$.

By (4),

$$\begin{aligned} u_l &\leqslant \left(\frac{12\log d}{d}\exp\left\{\frac{ld}{2n}\right\}\right)^l \\ &\leqslant \left(\frac{12\log d}{d}\cdot\frac{d}{e^3\log d}\right)^l \\ &\leqslant 1. \end{aligned} \tag{6}$$

Case 3. $\frac{2n}{d}(\log d - \log\log d - 3) < l \leqslant k$.

Now

$$\begin{aligned} \frac{u_l}{u_{l+1}} &= \frac{m(l+1)(n'-l)}{(k-l)^2}\exp\left\{-\frac{(2l+1)d}{2n}\right\} \\ &\leqslant \frac{kn}{(k-l)^2}\frac{e^6(\log d)^4}{d^2}. \end{aligned}$$

Hence

$$\begin{aligned} u_l &\leqslant \frac{1}{((k-l)!)^2}\left(\frac{kne^6(\log d)^4}{d^2}\right)^{k-l}u_k \\ &\leqslant \left(\frac{kne^8(\log d)^4}{(k-l)^2d^2}\right)^{k-l}u_k. \end{aligned}$$

Now observe that $(A/l^2)^l$ is at most $\exp\{2A^{\frac{1}{2}}/e\}$ and so

$$\begin{aligned} u_l &\leqslant \exp\left\{2\left(\frac{kne^8(\log d)^4}{d^2}\right)^{\frac{1}{2}}\right\}u_k \\ &\leqslant \exp\left\{\frac{200(\log d)^{\frac{5}{2}}}{d^{\frac{3}{2}}}n\right\}u_k. \end{aligned} \tag{7}$$

Now

$$\begin{aligned} u_k^{-1} &= \binom{n'}{k}m^k\exp\left\{-\frac{k^2d}{2n}\right\} \\ &\geqslant \left(\frac{n'e}{k}\exp\left\{-\left(\frac{k}{2n'}+\left(\frac{k}{n'}\right)^2\right)\right\}m\exp\left\{-\frac{kd}{2n}\right\}\right)^k \\ &= ((1-\theta(d))e^{\epsilon/3})^k \quad \text{where } \lim_{d\to\infty}\theta(d)=0 \\ &\geqslant 1 \quad \text{for } d \text{ sufficiently large.} \end{aligned} \tag{8}$$

Part (b) follows from (2) and (5)–(8). □

Before the introduction of Azuma's inequality into the study of random graphs we would have to try something else if the variance "blew up". The proof of our theorem shows that in spite of this something can sometimes be gained.

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