How Many Random Edges Make a Dense Graph Hamiltonian?

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ABSTRACT: This paper investigates the number of random edges required to add to an arbitrary dense graph in order to make the resulting graph hamiltonian with high probability. Adding $\Theta(n)$ random edges is both necessary and sufficient to ensure this for all such dense graphs. If, however, the original graph contains no large independent set, then many fewer random edges are required. We prove a similar result for directed graphs. © 2002 Wiley Periodicals, Inc. Random Struct. Alg., 22: 33–42, 2003

1. INTRODUCTION

In the classical model of a random graph (Erdős and Rényi [3]) we add random edges to an empty graph, all at once or one at a time, and then ask for the probability that certain structures occur. At the present time, this model and its variants, have generated a vast number of research papers and at least two excellent books, Bollobás [1] and Janson, Łuczak, and Ruciński [5]. It is also of interest to consider random graphs generated in other ways. For example, there is a well-established theory of considering random subgraphs of special graphs, such as the n-cube. In this paper we take a slightly different line. We start with a graph H chosen *arbitrarily* from some class of graphs and then

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consider adding a random set of edges R. We then ask if the random graph $G = H + R$ has a certain property. This for example would model graphs which were basically deterministically produced, but for which there is some uncertainty about the complete structure. In any case, we feel that there is the opportunity here for asking interesting and natural questions.

As an example we consider the following scenario: Let $0 < d < 1$ be a fixed positive constant. We let $\mathcal{G}(n, d)$ denote the set of graphs with vertex set $[n]$, which have minimum degree $\delta \geq dn$. We choose H arbitrarily from $\mathcal{G}(n, d)$ and add a random set of m edges R to create the random graph G. We prove two theorems about the number of edges needed to have G Hamiltonian **whp**.[1] Since $d \geq 1/2$ implies that H itself is Hamiltonian (Dirac's Theorem), this could be considered to be a probabilistic generalisation of this theorem to the case where $d < 1/2$. We henceforth assume $d < 1/2$. Also, let

$$\theta = \ln d^{-1} \geq .69.$$

Theorem 1. *Suppose* $0 < d \leq 1/2$ *is constant,* $H \in \mathcal{G}(n, d)$. *Let* $G = H + R$, *where* $|R| = m$ *is chosen randomly from* $\bar{E} = [n]^{(2)} \setminus E(H)$.

- **(a)** *If* $m \geq (30\theta + 13)n$, *then* G *is Hamiltonian* **whp**.
- **(b)** *For* $d \leq 1/10$ *there exist graphs* $H \in \mathcal{G}(n, d)$ *such that if* $m < \theta n/3$, *then* **whp** G *is not Hamiltonian.*

So it seems that we have to add $\Theta(n)$ random edges in order to make G Hamiltonian **whp**. Since a random member of $\mathcal{G}(n, d)$ is already likely to be Hamiltonian, this is a little disappointing. Why should we need so many edges in the worst case? It turns out that this is due to the existence of a large independent set. Let $\alpha = \alpha(H)$ be the independence number of H.

Theorem 2. *Suppose* $H \in \mathcal{G}(n, d)$ *and* $1 \leq \alpha < d^2n/2$ *and so* $d > n^{-1/2}$ (*d need not be constant in this theorem*). *Let* $G = H + R$, *where* $|R| = m$ *is chosen randomly from* $\bar{E}$. *If*

$$\frac{md^3}{\ln d^{-1}} \to \infty, \tag{1}$$

then G *is Hamiltonian* **whp**.

Note that if d is constant then Theorem 2 implies that $m \to \infty$ is sufficient.

We have also considered a similar problem in relation to adding random arcs to a dense digraph. For a digraph D we denote its arc-set by $A(D)$. We denote its minimum out-degree by δ^+ and its minimum in-degree by δ^-, and then we let $\delta = \min\{\delta^+, \delta^-\}$. Let $0 < d < 1$ be a fixed positive constant. We let $\mathcal{D}(n, d)$ denote the set of digraphs with vertex set $[n]$ which have $\delta \geq dn$.

Theorem 3. *Suppose* $0 < d < 1/2$ *is constant and* $H \in \mathcal{D}(n, d)$. *Let* $D = H + R$, *where* $|R| = m$ *is chosen randomly from* $\bar{A} = [n]^2 \setminus E(H)$.

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[1]A sequence of events $\mathcal{E}_n$ is said to occur "with high probability" (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

(a) *If* $m \geq (d^{-1}(15 + 6\theta) + 5d^{-2})n$*, then D is Hamiltonian* **whp**.
(b) *For* $d \leq 1/10$ *there exist digraphs* $H \in \mathfrak{D}(d)$ *such that if* $m < \theta n/3$*, then* **whp** *D is not Hamiltonian.*

If $\delta \geq n/2$, then H itself is Hamiltonian (Ghouila-Houri [4]).

Theorem 1 is proven in the next section, Theorem 2 is proven in Section 3, and Theorem 3 is proved in Section 4.

2. THE WORST-CASE FOR GRAPHS

We will assume from now on that m is exactly $\lceil 30\theta n \rceil + 13n$. We let $R = R_1 \cup R_2$, where $|R_1| = \lceil 30\theta n \rceil$. Then let $G_1 = H + R_1$.

We first show that

Lemma 1. G_1 *is connected* **whp**.

Proof. Let $N = \binom{n}{2}$. If $u, v \in [n]$ then either they are at distance one or two in H or

$$\mathbf{Pr}(dist_{G_1}(u, v) > 3) \leq \left(1 - \frac{|R_1|}{N}\right)^{d^2n^2} \leq e^{-60\theta d^2 n}.$$

Hence,

$$\mathbf{Pr}(diam(G) > 3) \leq n^2 e^{-60\theta d^2 n} = o(1). \qquad \blacksquare$$

Given a longest path P in a graph Γ with end-vertices x_0, y and an edge $\{y, v\}$, where v is an internal vertex of P, we obtain a new longest path $P' = x_0 \ldots vy \ldots w$, where w is the neighbor of v on P between v and y. We say that P' is obtained from P by a *rotation* with x_0 *fixed.*

Let $END_\Gamma(x_0, P)$ be the set of end-vertices of longest paths of Γ which can be obtained from P by a sequence of rotations keeping x_0 as a fixed end-vertex. Let $END_\Gamma(P) = \{x_0\} \cup END_\Gamma(x_0, P)$. Note that if Γ is connected and non-Hamiltonian, then there is no edge $\{x_0, y\}$ where $y \in END_\Gamma(x_0, P)$.

It follows from Pósa [6] that

$$|N_\Gamma(END_\Gamma(P))| < 2|END_\Gamma(P)|, \tag{2}$$

where for a graph Γ and a set $S \subseteq V(\Gamma)$

$$N_\Gamma(S) = \{w \notin S : \exists\, v \in S \text{ such that } vw \in E(\Gamma)\}.$$

Lemma 2. Whp

$$|N_{G_1}(S)| \geq 3|S| \tag{3}$$

for all $S \subseteq [n]$, $|S| \leq n/5$.

Proof. Now $|N_H(S)| \geq 3|S|$ for all $S \subseteq [n]$, $|S| \leq dn/3$. So,

$$\begin{aligned}\mathbf{Pr}(\exists\,|S| \leq n/5 : |N_{G_1}(S)| < 3|S|) &\leq \sum_{k=dn/3}^{n/5} \binom{n}{k}\binom{n}{3k}\left(1-\frac{m}{N}\right)^{k(n-4k)} \\ &\leq \sum_{k=dn/3}^{n/5}\left(\frac{n^4e^4}{27k^4}\,e^{-12\theta}\right)^k \\ &= o(1).\end{aligned}$$

∎

It follows from Lemma 2 that, for any longest path P in a graph Γ that contains G_1 as a subgraph, we have $n/5 \leq |END_\Gamma(P)| \leq |P|$.

Now let R_2 be obtained from a random sequence $e_1, e_2, \ldots$ of edges chosen from $\bar{E}$ *with replacement.*

Let P_0 be a longest path in G_1 of length $\lambda_0 \geq dn$. Now consider the following process: At a general stage we will have a path P_i of length at least $\lambda_0 + i$. We will have used a set $S_i \subseteq R_2$ of size Y_i to go from P_{i-1} to P_i, for $i \geq 1$. Here $S_1 = \{e_1, e_2, \ldots, e_{Y_1}\}$, $S_2 = \{e_{Y_1}, e_{Y_1+1}, \ldots, e_{Y_1+Y_2}\}$, and so on. Let $Z_i = Y_1 + Y_2 + \cdots + Y_i$, and let $\Gamma_i = G_1 + \{e_1, e_2, \ldots, e_{Z_i}\}$.

In order to see how the Y_i are determined, let P_i be a longest path in Γ_i and let $END_{\Gamma_i}(P_i)$ be as defined above and note that by Lemma 2, we can assume that $|END_{\Gamma_i}(P_i)| \geq n/5$. We now add edges $e_{Z_i+1}, e_{Z_i+2}, \ldots$ in turn until we find an edge $e_{Z_i+k} = \{a, b\}$, where $a \in END_{\Gamma_i}(P_i)$ and $b \in END_{\Gamma_i}(a, P_i)$. Since Γ_i is connected the addition of $\{a, b\}$ will increase the length of the longest path or close a Hamilton cycle. We let $Y_{i+1} = k$ in this case. Finally, once we have formed a Hamilton cycle, at stage r say, we let $Y_{r+1} = \cdots = Y_n = 0$.

Now the random variables $Y_1, Y_2, \ldots, Y_n$ are independent random variables which are either geometric with probability of success at least $\frac{2}{25}$ or are zero valued. Thus

$$\mathbf{E}(Z_n) \leq \frac{25n}{2}.$$

Since the variance of Z_n is $O(n)$ it is easy to show by an application of Chebychev's inequality that $Z_n \leq 13n$ **whp** and this completes the proof of (a).

Remark 1. The calculations above go through quite happily for $\delta(H) \geq n^{3/4}$, say. For this degree bound the number of additional edges required in the worst-case is $\Omega(n \ln n)$. Since only $\frac{1}{2}n \ln n$ random edges are required for Hamiltonicity when we start with the empty graph, there is no point in considering smaller values of d, unless we can improve the constant factor.

(b) Let $m = cn$ for some constant c and let H be the complete bipartite graph $K_{A,B}$, where $|A| = dn$ and $|B| = (1 - d)n$. Let I be the set of vertices of B which are not incident with an edge in R. If $|I| > |A|$, then G is not Hamiltonian. Instead of choosing m random edges for R, we choose each possible edge independently with probability $p = \frac{2m}{(d^2 + (1-d)^2)n^2}$. (We can use monotonicity, see, for example, Bollobás [1, II.1] to justify this simplification). Then

$$\mathbf{E}(|I|) = (1-d)n(1-p)^{(1-d)n-1} \sim (1-d)\exp\left\{-\frac{2(1-d)m}{(d^2+(1-d)^2)n}\right\}n.$$

It follows from the Chebychev inequality that $|I|$ is concentrated around its mean and so G will be non-Hamiltonian **whp** if c satisfies

$$c < \frac{1}{2(1-d)}(d^2 + (1-d)^2)\ln(d^{-1} - 1).$$

This verifies (b). ∎

3. GRAPHS WITH SMALL INDEPENDENCE NUMBER

Proof of Theorem 2. We will first show that we can decompose H into a few large cycles.

Lemma 3. *Suppose that G has minimum degree dn, where $d \le 1/2$ and that $\alpha(G) < d^2n/2$. Let $k_0 = \lfloor \frac{2}{d} \rfloor$. Then the vertices of G can be partitioned into $\le k_0$ vertex disjoint cycles.*

Proof. Let C_1 be the largest cycle in H. $|C_1| \ge dn + 1$, and we now show that the graph $H \backslash C_1$ has minimum degree $\ge dn - \alpha$.

To see this, let $C_1 = v_1, \ldots, v_c, v_{c+1} = v_1$. Let $w \in V(H \backslash C_1)$. Because C_1 is maximum sized, no such w is adjacent to both v_i and v_{i+1}. Also, if $w \sim v_i$ and $w \sim v_j$ with $i < j$ and $v_{i-1} \sim v_{j-1}$, then

$$w,\ v_j,\ \ldots,\ v_c,\ v_1,\ \ldots,\ v_{i-1},\ v_{j-1},\ \ldots,\ v_i,\ w$$

is a larger cycle. So the predecessors of $N(w)$ in C_1 must form an independent set and $|N(w) \cap C_1| \le \alpha$. Similar arguments are to be found in [2].

We can repeat the above argument to create disjoint cycles $C_1, \ldots, C_r$, where $|C_1| \ge |C_2| \ge \cdots \ge |C_r|$ and C_j is a maximum sized cycle in the graph $H_{j-1} = H \backslash (C_1 \cup \cdots \cup C_{j-1})$ for $j = 1, 2, \ldots, r$. Now H_k has minimum degree at least $dn - k\alpha$ and at most $n - dn - 1 - (dn - \alpha + 1) - \cdots - (dn - (k-1)\alpha + 1) = n - k(dn + 1 - (k-1)\alpha/2)$ vertices. Since $d^2n > 2\alpha$, H_{k_0}, if it existed, would have minimum degree at least 2 and a negative number of vertices. ∎

Let $C_1, \ldots, C_r$ be the cycles given by Lemma 3.

In order to simplify the analysis, we assume the edges of R are chosen from $\bar{E}$ by including each $e \in \bar{E}$ independently with probability $\frac{m}{|\bar{E}|}$. Because Hamiltonicity is a monotone property, showing that G is Hamiltonian **whp** in this model implies the theorem. We get a further simplification in the analysis if we choose these random edges in rounds: Set $R = R_1 \cup R_2 \cup \cdots \cup R_r$ where each edge set R_i is independently chosen by including $e \in \bar{E}$ with probability p, where $1 - (1-p)^r = \frac{m}{|\bar{E}|}$. Each R_i will be used to either extend a path or close a cycle and will only be used for one such attempt. In this way each such attempt is independent of the previous. To this end, let $G_t = H \cup \bigcup_{s=1}^{t} R_t$ for $t = 0, 1, \ldots, r$. Thus $G_0 = H$ and $G_r = G$.

Let $e = \{x, y\}$ be an edge of C_r and let Q be the path $C_r - e$. In each phase of our procedure, we have a current path P with endpoints x, y together with a collection of vertex disjoint cycles $A_1, A_2, \ldots, A_s$ which cover V. Initially $P = Q$, $s = r - 1$, and $A_i = C_i$, $i = 1, 2, \ldots, r - 1$.

Suppose a path P and collection of edge disjoint cycles have been constructed in G_{t-1} (initially $t = 1$). Consider the set $Z = END_{G_{t-1}}(x, P)$ created from rotations with x as a fixed endpoint, as in Section 2. We identify the following possibilities:

Case 1. There exists $z_1 \in Z$, $z_2 \notin P$ such that $f = (z_1, z_2)$ is an edge of H. Let Q be the corresponding path with endpoints x, z_1 which goes through $V(P)$. Now suppose that $z_2 \in C_i$ and let $f' = (z_2, z_3)$ be an edge of C_i incident with z_2. Now replace P by the path Q, f, Q', where $Q' = C_i - f$. This construction reduces the number of cycles by one.

Case 2. $|V(P)| \leq n/2$ and $z \in Z$ implies that $N_{G_{t-1}}(z) \subseteq V(P)$. It follows from (2) that $|Z| \geq dn/3$. Now add the next set R_t of random edges. Since $|V(P)| \leq n/2$, the probability that no edge in R_t joins $z_1 \in Z$ to $z_2 \in V \backslash V(P)$ is at most $(1-p)^{(dn/3)(n/2)}$. If there is no such edge, we fail; otherwise we can use (z_1, z_2) to proceed as in Case 1. We also replace t by $t + 1$.

Case 3. $|V(P)| > n/2$ and $z \in Z$ implies that $N_{G_{t-1}}(z) \subseteq V(P)$. Now we close P to a cycle. For each $z \in Z$ let $A_z = END_{G_{t-1}}(z, Q_z)$, where Q_z is as defined in Case 1. Each A_z is of size at least $dn/3$. Add in the next set R_t of random edges. The probability that R_t contains no edge of the form (z, z'), where $z \in Z$ and $z' \in A_z$ is at most $(1 - p)^{d^2n^2/10}$. If there is no such edge, we fail. Otherwise, we have constructed a cycle C through the set $V(P)$ in the graph G_t. If C is Hamiltonian, we stop. Otherwise, we choose a remaining cycle C', distinct from C, and replace P by $C' - e$, where e is any edge of C'. Now $|V(P)| < n/2$ and we can proceed to Case 1 or Case 2.

After at most r executions of each of the above three cases, we either fail or produce a Hamilton cycle. The probability of failure is bounded by

$$\begin{aligned} k_0((1-p)^{(dn/3)(n/2)} + (1-p)^{d^2n^2/10}) &\leq 2d^{-1}\left(\left(1 - \frac{m}{|\bar{E}|}\right)^{dn^2/6r} + \left(1 - \frac{m}{|\bar{E}|}\right)^{d^2n^2/10r}\right) \\ &\leq 4d^{-1}e^{-md^3/10} \\ &= o(1) \end{aligned}$$

provided that (1) holds. ■

An observation: We do not actually need the condition that $\alpha(H) \le d^2n/2$ to complete this proof. The weaker condition that $d^2n/2$ bounds the independence number of the neighborhood of each vertex is enough.

4. DIRECTED GRAPHS

For a digraph $D = ([n], A)$ we let B_D be the bipartite graph $([1, n], [n + 1, 2n], E)$ such that B_D contains an edge (x, y) for every arc $(x, y - n) \in A$. Perfect matchings of B_D correspond to *cycle covers* of D, i.e., sets of vertex disjoint directed cycles which contain all vertices of D.

We divide our arcs R into two subsets: $R = R_1 \cup R_2$, where each R_i is independently randomly chosen from $[n]^2 \backslash A(H)$. Here

$$|R_i| = \rho_i n, \qquad \text{where } \rho_1 = d^{-1}(15 + 6\theta) \qquad \text{and} \qquad \rho_2 = 5d^{-2}.$$

Lemma 4. **Whp** $H_1 = H + R_1$ *has a cycle cover* Σ_1.

Proof. We apply Hall's theorem to show that B_{H_1} has a perfect matching **whp**. If B_{H_1} does not have a perfect matching, then there exists a *witness* $K \subseteq [1, n]$, $|K| \le n/2$ (or $L \subseteq [n + 1, 2n]$, $|L| \le n/2$) such that its neighbor set $N(K)$ in B_{H_1} satisfies $|N(K)| \le |K| - 1$ (resp. $|N(L)| \le |L| - 1$). Clearly any such witness must be of size at least dn.

Since having a perfect matching is a monotone increasing property, we can assume that the arcs of R_1 are chosen independently with $p_1 = \frac{\rho_1}{n}$.

The probability that B_{H_1} does not contain a perfect matching is therefore bounded by

$$2 \sum_{k=dn}^{n/2} \binom{n}{k} \binom{n}{k-1} (1 - p_1)^{k(n-k)} \le 2 \sum_{k=dn}^{n/2} \left(\frac{n^2e^2}{k^2} \cdot e^{-\rho_1/2} \right)^k = o(1). \qquad \blacksquare$$

We also need to know that there are many arcs joining large sets. For $S \subseteq [n]$ let $N^+(S) = \{t \notin S : \exists\ s \in S$ such that (s, t) is an arc of $H_1\}$. Define $N^-(S)$ similarly.

Lemma 5. **Whp**, *for all disjoint* $S, T \subseteq [n]$ *with* $|S|, |T| \ge dn/2$, $|N^-(S) \cap T|, |N^+(S) \cap T| \ge |T|/2$.

Proof. Let $\mathscr{E}$ denote the event $\{\exists$ disjoint $S, T \subseteq [n] : |S|, |T| \ge dn/2$ and $|N^+(S) \cap T| < |T|/2\}$. Now fix S, T with $|S| = s$, $|T| = t$. If $|N^+(S) \cap T| < |T|/2$, then there exists $T' \subseteq T$ of size $|T|/2$ such that there are no arcs from S to T' in H_1. The probability of this is at most

$$2^t(1 - p_1)^{ts/2} \le (2e^{-\rho_1 s/(2n)})^t.$$

Thus

$$\begin{aligned}
\mathbf{Pr}(\mathscr{E}) &\leq \sum_{s=dn/2}^{(1-d/2)n} \sum_{t=dn/2}^{n-s} \binom{n}{s}\binom{n}{t}(2e^{-\rho_1 s/(2n)})^t \\
&\leq \sum_{s=dn/2}^{(1-d/2)n} \sum_{t=dn/2}^{n-s} \left(\frac{ne}{s}\right)^s \left(\frac{2ne^{1-\rho_1 s/(2n)}}{t}\right)^t \\
&\leq \sum_{s=dn/2}^{(1-d/2)n} \sum_{t=dn/2}^{n-s} \left(\frac{ne}{s}\right)^s e^{-\rho_1 st/(3n)} \\
&\leq n^2 e^{-\rho_1 dn^2/24} \\
&= o(1).
\end{aligned}$$

The proof for $N^-(S) \cap T$ is identical. ∎

Corollary 4. *H_1 is strongly connected* **whp**.

Proof. If H_1 is not strongly connected, then there exists $S \subseteq [n]$, $|S| \leq n/2$ such that either $N^+(S) = \emptyset$ or $N^-(S) = \emptyset$. But this would contradict Lemma 5 with $T = \bar{S}$. ∎

Assume that H_1 is strongly connected and satisfies the condition of Lemma 5.

We now describe a procedure for converting the cycle cover Σ_1 to a Hamilton cycle. We start with an arbitrary cycle C for which there is an arc (y, z) with $y \in C$, $z \in C' \neq C$. Such an arc must exist because H_1 is strongly connected. Let (y, x) be the arc of C leaving y and (y', z) be the arc of C' entering z. Now delete arcs (y, x), (y', z) from Σ_1 and add the arc (y, z). This yields a path P from x to y' plus a collection of disjoint cycles which covers $[n]$. Call this a *near cycle cover* (NCC).

Given an NCC we first try to perform an *out path extension.* We can do this if there is an arc e joining the terminal endpoint of the path P to a vertex v of one of the cycles, C' say. By adding the arc e and deleting the arc of C' entering v, we create an NCC with one fewer cycle. Note that this construction is the same as that of the previous paragraph, except that we do not invoke strong connectivity. We also try to perform an analogous *in path extension* by checking if there is an arc (w, s), where s is the start vertex of P and $w \notin P$. If such an arc exists, we may extend the path P by adding a path section at its beginning.

We continue with these path extensions until the NCC Σ_2 that we have no longer admits one. Let $Q = (u_0, u_1, \ldots, u_k)$ be the path of Σ_2 and define the successor function σ by $\sigma(u_i) = u_{i+1}$ for $i < k$. Now $k \geq dn$ since $\delta \geq dn$ and there are no out path extensions available. All of u_k's out neighbors are in Q. Let

$$S = \{u_{i-1} : i \leq k - dn/2,\ (u_k, u_i) \text{ is an arc of } H_1\},\ T = \{u_{k-dn/2}, \ldots, u_k\}.$$

Clearly $|S| \geq dn/2$. Let $T' = N^+(S) \cap T$ so that $|T'| \geq dn/2$ by Lemma 5. For $v \in T'$ choose $\lambda(v) \in S$ such that $(\lambda(v), v)$, $(u_k, \sigma\lambda(v))$ are both arcs of H_1. For each such $v \in T'$ consider the path

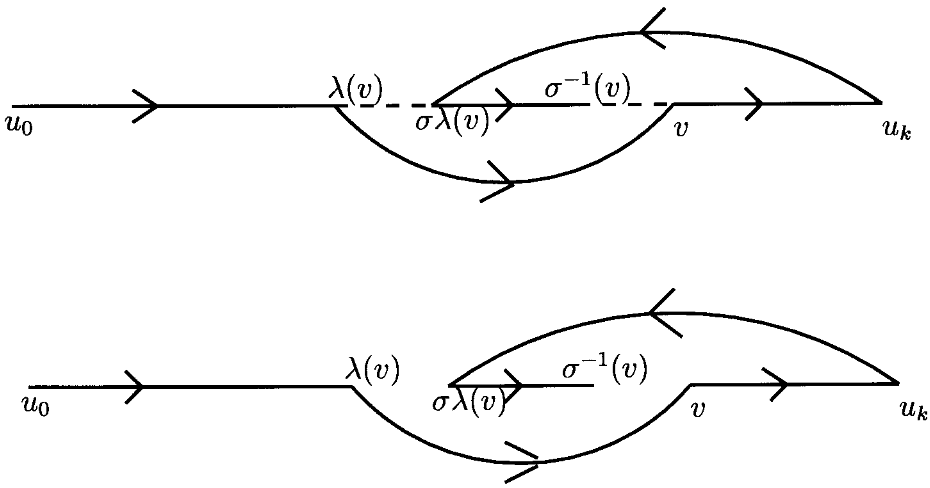

Fig. 1.

$$Q_v = Q + (u_k, \sigma\lambda(v)) - (\lambda(v), \sigma\lambda(v)) + (\lambda(v), v) - (\sigma^{-1}(v), v).$$

Note that Q_v has the same vertex set as Q and has endpoints u_0, $\sigma^{-1}(v)$—see Figure 1.

For each $v \in T'$ we see if there is an out path extension available for Q_v. Suppose no such out extension exists. By an analogous procedure to the creation of Q_v, we can, for each $v \in T'$, construct a set $\mathcal{Q}_v$ of $\geq dn/2$ paths each with a distinct start vertex and all with the same end vertex $\sigma^{-1}(v)$, and all covering the vertices of Q, the start vertices are distinct within $\mathcal{Q}_v$ that is. (There are no in path extensions available into u_0 and we just look at the arcs that enter u_0.)

If there is an in path extension available for a $v \in T'$, $Q' \in \mathcal{Q}_v$, then we carry it out.

Now suppose that we fail in all of these attempts at path extension. We generate a sequence of random arcs $e_1, e_2, \ldots$, part of R_2. Each e_i is chosen uniformly from the arcs not in H_1, with replacement. We continue until we find an arc which closes a path in some $\mathcal{Q}_v$, $v \in T'$ to a cycle C^* say. Note that each e_i has probability at least $d^2/4$ of achieving this.

Now note that the sequence, starting with a cycle cover, replacing two cycles by a path, doing path extensions, using random arcs to close a path to C^*, produces a new cycle cover with one less cycle. Thus eventually a Hamilton cycle is produced.

The number of random edges required can be bounded by the sum $Z = Z_1 + Z_2 + \cdots + Z_n$, where the Z_i are independent geometric random variables with probability of success $d^2/4$. Thus $\mathbf{E}(Z) = 4d^{-2}n$ and **whp** $Z < 5d^{-2}n$. We could use the Chebychev inequality for example to prove the latter claim. Thus, if we add $5d^{-2}n$ random edges to H_1, then we will create a Hamilton cycle **whp**. This completes the proof of part (a) of Theorem 3.

For part (b) we can start with $K_{A,B}$ of Theorem 1(b) and then replace each edge by an arc in both directions to create $H = \vec{K}_{A,B}$. Once again, we let I be the set of vertices of B which are not incident with an arc in R. If $|I| > |A|$, then D is not Hamiltonian. Instead of choosing $m = cn$ random arcs for R, we choose each possible arc independently with probability $p = \frac{m}{(d^2 + (1-d)^2)n^2}$. Then

$$\mathbf{E}(|I|) = (1-d)n(1-p)^{2(1-d)n-2} \sim (1-d)\exp\left\{-\frac{2(1-d)m}{(d^2+(1-d)^2)n}\right\}n.$$

It follows from the Chebychev inequality that $|I|$ is concentrated around its mean and so G will be non-Hamiltonian **whp** if c satisfies

$$c < \frac{1}{2(1-d)}(d^2+(1-d)^2)\ln(d^{-1}-1).$$

This verifies (b).

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