Edge disjoint Hamilton cycles in sparse random graphs of minimum degree at least k

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Abstract

Let $G_{n,m,k}$ denote the space of simple graphs with n vertices, m edges and minimum degree at least k, each graph G being equiprobable. Let G have property $\mathcal{A}_k$ if G contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. We prove that for $k \geq 3$, there is a constant C_k such that if $2m \geq C_k n$ then $\mathcal{A}_k$ occurs in $G_{n,m,k}$ with probability tending to 1 as $n \to \infty$.

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1 Introduction

Denote by $G_{n,m,k}$ the space of simple graphs with vertex set $[n] = \{1, 2, ..., n\}$, m edges and minimum degree at least k, each graph being equiprobable. We say a graph G has property $\mathcal{A}_k$, if it contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. We prove the following theorem.

Theorem 1 *Let $k \geq 3$. There exists a constant $C_k \leq 2(k+1)^3$ such that if $2m = cn$, $c \geq C_k$ then* **whp** [1] *$G \in G_{n,m,k}$ has property $\mathcal{A}_k$.*

In [BFF], Bollobás, Fenner and Frieze establish the following sharp threshold for a stronger property $\mathcal{A}_k^*$, in the case where $m/n \to \infty$. A graph G has property $\mathcal{A}_k^*$, if G contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is odd, a further edge disjoint matching of size $\lfloor n/2 \rfloor$.

Theorem 2 *Let $2m = n\left(\frac{\log n}{k+1} + k \log\log n + d_n\right)$. If $G \in G_{n,m,k}$, then*

$$\lim_{n\to\infty} \mathbf{Pr}(G \in \mathcal{A}_k^*) = \begin{cases} 0 & \text{if } d_n \to -\infty, \qquad \text{sufficiently slowly} \\ e^{-\theta_k(d)} & \text{if } d_n \to d, \\ 1 & \text{if } d_n \to +\infty, \end{cases}$$

were

$$\theta_k(d) = \frac{e^{-(k+1)d}}{(k+1)!\{(k-1)!\}^{k+1}(k+1)^{k(k+1)}}.$$

The primary obstruction to property $\mathcal{A}_k^*$ in $G_{n,m,k}$ is the presence of *k-spiders*. A k-spider is $k+1$ vertices of degree k having a common neighbour. In [BFF] it was shown that if m is as in Theorem 2 then

$$\lim_{n\to\infty} \mathbf{Pr}(G \in G_{n,m,k} \text{ has a } k\text{-spider }) = 1 - e^{-\theta_k(d)}.$$

When $2m = cn$, where c is constant, k-spiders occur **whp** in $G \in G_{n,m,k}$. Thus **whp** the property $\mathcal{A}_k^*$ does not occur, and property $\mathcal{A}_k$ is best possible.

The proof of Theorem 1 holds for $c \geq C_k$, in the case where c is constant or tends slowly to infinity, up to and including the value $c = \log n/(k+1) + k \log\log n + d_n$, given in Theorem 2, when the stronger property $\mathcal{A}_k^*$ applies.

For each proof in the paper, there is some proof specific constant c_k, such that the proof holds for $c \geq c_k$. It can be shown that if we choose $C_k = 2(k+1)^3$, then $C_k \geq c_k$ always.

[1] **whp** *with high probability.* With probability tending to 1 as $n \to \infty$.

The value selected for the constant C_k is clearly not optimal and could be improved by further work. We would not, however, expect to reach the obvious lower bound of $C_k = k$.

As $c \to k$ then $G_{n,m,k}$ tends to the space of k-regular graphs with the uniform distribution. Robinson and Wormald [RW] have proved that almost all k-regular graphs ($k \geq 3$) with an even number of vertices have a *Complete Decomposition* (a decomposition into a Hamilton cycle and a set of perfect matchings). They also conjecture a property similar to $\mathcal{A}_k^*$.

2 Models for the space $G_{n,m,k}$

For any graph G in $G_{n,m,k}$ there are $m!$ ways to order, and 2^m ways to orient the edges, to give $2^m m!$ sequences of vertex labels of length $2m$. The set $\mathcal{S}(n, 2m, k)$ of sequences arising in this manner has the uniform measure induced by $G_{n,m,k}$. Let $\mathcal{M}(n, 2m, k)$ be the space of equiprobable sequences $(a_i : a_i \in [n], i = 1, ..., 2m)$ specifying which of n labelled boxes contains each of $2m$ labelled balls; with the condition that the minimum occupancy of any box is at least k. Each element of $\mathcal{M}$ defines a multigraph with vertex set $[n]$. The set $\mathcal{S}$, is the subset of $\mathcal{M}$ whose underlying graphs are simple.

Denote by $\mathcal{O}(n, 2m, k)$ the space of sequences $(b_j : b_j \in [2m], j = 1, ..., n)$, giving the possible occupancies of the n boxes (the degrees of the vertices), arising from sequences in $\mathcal{M}(n, 2m, k)$, with the derived probability measure. A useful method of obtaining results about $\mathcal{O}(n, 2m, k)$ is to consider a larger space $\mathcal{P}(n, \lambda, k)$ in which each of the n boxes has independent occupancy given by a truncated Poisson random variable X with parameters λ and k. The space $\mathcal{O}(n, 2m, k)$ is obtained from $\mathcal{P}(n, \lambda, k)$ by conditioning on $\sum_{i=1}^n X_i = 2m$, as explained in [BFU]. Let

$$\mathbf{Pr}(X = j) = \frac{\lambda^j}{j!} \frac{e^{-\lambda}}{\beta(\lambda, k)} \qquad j \in \{k, k+1, ...\}$$

where

$$\beta(\lambda, k) = 1 - e^{-\lambda} \left(1 + \lambda + \cdots + \tfrac{\lambda^{k-1}}{k-1!}\right).$$

It is natural to choose λ so as to maximize the probability of the conditioning event $\{\sum_i X_i = 2m\}$. This can be approximately achieved by making $\mathbf{E}X$ equal to c, the average vertex degree. Explicitly, we choose

$$\lambda \frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = c, \tag{1}$$

so that $\mathbf{E}X = c$. The properties of (1) are given in Lemma A1 of the appendix.

For an event $\mathcal{E}$ in $G_{n,m,k}$ which specifies the degree sequence $\mathbf{d}(S)$ of some subset S of vertices, the set of multigraph sequences with $\mathbf{d}(S)$ in $\mathcal{M}(n, 2m, k)$ is well defined and corresponds to well defined events in $\mathcal{O}(n, 2m, k)$ and $\mathcal{P}(n, \lambda, k)$. For such an *occupancy event*, $\mathcal{E}$, we can regard $\mathcal{E}$ as being defined in each of the spaces under consideration, rather than just in $G_{n,m,k}$.

Lemma 3 *Let k be a fixed positive integer, and let $m = cn/2$, where $c \geq k$. Let $\mathcal{E}$ be an occupancy event in $G_{n,m,k}$. Then for sufficiently large n,*

(i)

$$\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)) \leq (1 + o(1)) \sqrt{2\pi cn}\, \mathbf{Pr}(\mathcal{E}; \mathcal{P}(n, \lambda, k))$$

(ii)

$$\mathbf{Pr}(\mathcal{E}; G_{n,m,k}) \leq e^{O(c^2)}\, \mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)).$$

We denote a generic expression of the form $\left[O(\sqrt{c})e^{O(c^2)}\right]$ by $A(c)$.

We will frequently use the model $\mathcal{P}(n, \lambda, k)$ to estimate the probability of an event $\mathcal{E}$, that a set S of vertices of a graph G in $G_{n,m,k}$ has degree sum T. In the model $\mathcal{P}(n, \lambda, k)$, the probability that a set S of s boxes (with occupancy $t_i : i = 1, ..., s$) has total occupancy T is

$$\begin{aligned} \mathbf{Pr}(t_1 + ... + t_s = T; \mathcal{P}(n, \lambda, k)) &= \sum_{\substack{t_1+...+t_s=T \\ t_i \geq k}} \left(\prod_{i=1}^{s} \frac{\lambda^{t_i}}{t_i!} \frac{e^{-\lambda}}{\beta(\lambda, k)} \right) \\ &= \frac{\lambda^T e^{-\lambda s}}{[\beta(\lambda, k)]^s} \frac{1}{T!} \sum_{\substack{t_1+...+t_s=T \\ t_i \geq k}} \binom{T}{t_1 ... t_s} \\ &\leq \frac{(\lambda s)^T}{T!} \frac{e^{-\lambda s}}{[\beta(\lambda, k)]^s}. \end{aligned} \tag{2}$$

Lemma 4 *Let k be a fixed positive integer, and let $2m = cn$ where $c \geq k$. For sufficiently large n,*

(i)

$$\frac{|G_{n,m,k}|}{|G_{n,m}|} \geq e^{-O(c^2)} \left[\left(\frac{c}{\lambda e} \right)^c e^{\lambda} \beta(\lambda, k) \right]^n.$$

(ii) Let $D = \frac{kc^{k-1}}{(k-1)!} e^{-(c-k)} e^{k^2/(c-k)}$, then

$$\mathbf{Pr}(\delta(G) \geq k; G_{n,m}) \geq e^{-nD}.$$

The proofs of Lemmas 3, 4 are provided in the appendix.

Although $|G_{n,m,k}|$ is bounded below by an exponentially small multiple of $|G_{n,m}|$, the constant D in the exponent also is small. For example, when $c = c_k,\ k = 3$, then $D = 1.4 \times 10^{-50}$. In particular, note that if $c \geq c_k$ then

$$D \leq e^{-8c/9}. \tag{3}$$

This allows us to prove results concerning $G_{n,m,k}$ directly in $G_{n,m}$ in Lemma 8 by using the general estimate

$$\mathbf{Pr}(\mathcal{E};\ G_{n,m,k}) = \mathbf{Pr}(\mathcal{E} \mid \delta(G) \geq k) \leq \frac{\mathbf{Pr}(\mathcal{E};\ G_{n,m})}{\mathbf{Pr}(\delta(G) \geq k;\ G_{n,m})}. \tag{4}$$

The following bound is also useful (where $p = m/\binom{n}{2}$), and is given in (eg) [Bo].

$$\mathbf{Pr}(\mathcal{E}; G_{n,m}) \leq O(\sqrt{n})\mathbf{Pr}(\mathcal{E}; G_{n,p}) \tag{5}$$

We will frequently use the following upper bound for the conditional probability that a sequence in $\mathcal{M}(n, 2m, k)$ corresponds to a multigraph in which there are at least q edges between A and B given that the sets of vertices A, B have degree sums a, b respectively, namely

$$\binom{m}{q}\left(\frac{a}{2m}\frac{b}{2m}\right)^q (2^q\ e)^{\delta(A,B)} \tag{6}$$

where $\delta(A, B) = 0$ if $A = B$ and 1 if A and B are disjoint.

Finally, we note that $\binom{m}{q} \leq (me/q)^q$ and that the unconstrained maximum of $(me/q)^q$ with respect to q occurs at $q = m$.

3 Relevant properties of $G_{n,m,k}$

Given y_0 let $P_{y_h} = y_0 y_1 ... y_h$ be a y_0 path (a longest path starting at y_0) in G. A *Pósa rotation* $P_{y_h} \to P_{y_{i+1}}$, [Po],[Bo] gives the path $P_{y_{i+1}} = y_0 y_1 ... y_i y_h y_{h-1} ... y_{i+1}$ formed from P_{y_h} by adding the edge $y_h y_i$ and erasing the edge $y_i y_{i+1}$. We call $y_h y_i,\ y_i y_{i+1}$ the *transformation edges* and $y_{i+1} y_{i+2}$ the *adjacent edge* of the rotation.

The Pósa rotations of a longest x_0 path $P_{x_h} = x_0 P x_h$ with x_0 fixed, define a *rotation subgraph* $\mathcal{R} = \mathcal{R}(x_0)$ of G, as follows. Initially $\mathcal{R} = x_{h-1} x_h$, where x_h is an *active* endpoint. Perform all possible rotations based on x_h due to edges $x_h x_i$, adding the transformation and adjacent edges of each rotation to $\mathcal{R}$. Each x_{i+1} is now an active endpoint, whereas x_h is now *passive*. Let x be an active endpoint with $P_x = x_0 ... yz ... x$ and where there is an edge xy in G. If z is a passive endpoint we add xy to $\mathcal{R}$ and consider this a transformation edge, else we add the transformation and adjacent edges corresponding to the rotation $P_x \to P_z$.

The final graph $\mathcal{R}^*$ is not necessarily unique, as its structure may depend on the order in which the endpoints are processed. It is however fully explored, in the sense that it has no active endpoints.

At any stage, we define a subgraph $\mathcal{T}$ of $\mathcal{R}$ which includes only transformation edges, where initially, $\mathcal{T} = x_h$.

Let U be the set of endpoints of $P_0 = x_0 P x_h$ obtained while constructing $\mathcal{R}^*(x_0)$, and let A be the vertices adjacent to vertices of U on P_0. Now, $N(U) = A \setminus U$, for if $N(U) \cap ([n] - V(P_0)) \neq \emptyset$ the condition 'longest path' would be contradicted. Thus we have the Pósa condition

$$|N(U)| \leq |A| = 2|U| - 1.$$

Here, $N(U) = N_G(U)$ is the set of neighbours of U in G that are not in U.

Lemma 5 *If G is a non-Hamiltonian graph of minimum degree at least 3 and P_0 is a longest path in G then the rotation subgraph $\mathcal{R}$ of P_0 contains at least two cycles.*

Proof We consider a final graph $\mathcal{R} = \mathcal{R}^*$, and the subgraph $\mathcal{T}$ of $\mathcal{R}$. As $\mathcal{T}$ has minimum degree at least 2, $\mathcal{T}$ is either a unique cycle or contains at least two cycles. If $\mathcal{T}$ is a unique cycle C, then by the definition of $\mathcal{T}$, $U \subseteq V(C)$ and every edge of C is incident with a vertex of U. There are two cases to consider.

Case I. The vertices of U alternate with vertices of $N(U)$ on C.
In this case the cycle C is of length $2|U|$. Each vertex of U in C has at least one further vertex $v \in N(U) \setminus C$ attached as a pendant leaf of C. This follows as the minimum degree in $\mathcal{R}$ of any vertex in U is at least 3, but there is only one cycle. Thus $|U \cup N(U)| \geq 3|U|$, contradicting the Pósa condition that $|N(U)| \leq 2|U| - 1$.

Case II. Two vertices of U are adjacent on C.
We claim that every vertex on the cycle is an endpoint. We proceed inductively.
Orient C, and let $(u, v) \in C$, $u, v \in U$. Consider P_u. If $P_u = x_0 ... vx ... u$, then $x \in U$. Hence $(v, x) \in C$ and $v, x \in U$. If $P_u = x_0 ... vu$ then $P_u = x_0 ... a_1 b_1 ... a_2 b_2 ... vu$ where there are edges $\{a_1, u\}$, $\{a_2, u\}$ and paths $u a_1 b_1$, $u a_2 b_2$ in $\mathcal{R}$. At best $C = ... b_1 a_1 uv ...$, and there is a chordal path $u a_2 b_2$ of C as $b_2 \in U$. This contradicts the unicyclicity assumption.

Suppose now that $U = V(C)$. Consider the initial longest path $P_0 = x_0 x_1 ... x_h$. Let $b \in U$ be the first occurrence of an endpoint vertex in P_0 beyond x_0. Thus either $P_0 = x_0 b ... x_h$ or $P_0 = x_0 Q ab ... x_h$. In either case there exists a sequence of transformations $P_{x_h}, ..., P_w$ where $P_w \to P_b$.

In the first case $x_0 w$ and $x_0 b$ are transformation edges of $\mathcal{R}$. However $x_0 \notin U$ so wx_0b is a chordal path of C in $\mathcal{R}$.

In the second case the transformation edges are wa, ab. As x_0Qa, the initial segment of P_0 is never broken, the vertex $a \notin U$. We conclude that $\mathcal{R}$ is not unicyclic as wab is a chordal path of C. $\square$

Lemma 6 *There exists a constant c_k such that if $c \geq c_k$ then $G \in G_{n,m,k}$ satisfies the following condition* **whp** .

Let $s_0 = n\,(3/c)^3 e^{-(2k+12)}$. No set of vertices S, of size $|S| < s_0$, induces at least $3|S|/2$ edges.

Proof Let $|S| = s$, fix the degree sum T of S. The expected number of vertex sets S inducing at least $3s/2$ edges is at most

$$\gamma(s) = A(c)\sqrt{n}\binom{n}{s}\sum_{T\geq ks}\frac{(\lambda s)^T}{T!}\frac{e^{-\lambda s}}{[\beta(\lambda,k)]^s}\binom{m}{\lfloor 3s/2\rfloor}\left(\frac{T}{2m}\right)^{3s}.$$

The right hand side of this expression follows from (2) and (6) and Lemma 3. Thus, as $\lambda \leq c$ from Lemma A1(i) we can apply Lemma A1 (iv) and (11) of Lemma A3 as follows:

$$\begin{aligned}\gamma(s) &\leq A(c)\sqrt{n}\left(\frac{ne}{s}\right)^s\frac{1}{[\beta(\lambda,k)]^s}\left(\frac{e}{3scn}\right)^{3s/2}\left[e^{-\lambda s}\sum_{T\geq ks}\frac{(\lambda s)^T}{T!}T^{3s}\right]\\ &\leq A(c)\sqrt{n}\frac{1}{[\beta(\lambda,k)]^s}\left(\frac{ne}{s}\right)^s\left(\frac{e}{3scn}\right)^{3s/2}(cse^{3/k})^{3s}\\ &\leq A(c)\sqrt{n}\left(\frac{s}{n}\frac{c^3}{3^3}e^{(5+18/k+2k^2/(c-k))}\right)^{s/2},\end{aligned}$$

where any constants have been absorbed into $A(c)$. Provided $k \geq 3$ and $c > 2k$,

$$\sum_{4\leq s\leq s_0}\gamma(s) = o\left(\frac{1}{n}\right)$$

$\square$

Lemma 7 *There exists a constant c_k such that if $c \geq c_k$ then $G \in G_{n,m,k}$ satisfies the following condition* **whp** .

No set of vertices S, $|S| = s$, $1 \leq s \leq n$ satisfies

(i) the degree sum T of S satisfies $ks \leq T \leq cs/4k$,

(ii) $|N(S)| \leq k|S|$,

(iii) the subgraph F consisting of the edges induced by S and by $S \times N(S)$, is connected and contains at least two cycles.

Proof Let the pair $(S, N(S))$ satisfy the conditions (i),(ii) and (iii). Let $T = 2q+p$ where there are q edges induced by S and p edges induced by $S \times N(S)$. Denote $|N(S)|$ by r. Let R be the degree sum of $N(S)$, where $R \geq \max\{p, kr\}$. Because F is connected and contains at least two cycles we have that

$$p + q \geq r + s + 1,$$

and thus

$$T \geq q + r + s + 1.$$

For fixed p, q, r, s the expected number of pairs $(S, N(S))$ satisfying conditions (i),(ii),(iii) is at most $\gamma(p, q, r, s)$ where from (2), (6),

$$\begin{aligned}\gamma(p,q,r,s) &= A(c)\sqrt{n}\binom{n}{s}\binom{n}{r}\frac{(\lambda s)^T}{T!}\frac{e^{-\lambda s}}{[\beta(\lambda,k)]^s}\left(\sum_{R\geq p}\frac{(\lambda r)^R}{R!}\frac{e^{-\lambda r}}{[\beta(\lambda,k)]^r}\right.\\ &\times \left.\binom{m}{p}\binom{m}{q}\left[\left(\frac{T}{2m}\right)\left(\frac{T}{2m}\right)\right]^q\left[\left(\frac{T}{2m}\right)\left(\frac{R}{2m}\right)\right]^p 2^p\right).\end{aligned}$$

Thus, as $c - k \leq \lambda \leq c$ an upper bound on $\gamma(p, q, r, s)$ is given by

$$\frac{A(c)\sqrt{n}}{[\beta(\lambda,k)]^s}\left(\frac{s}{n}\right)^{T-(r+s+q)} e^{(k+1)s-cs+T}\left(\frac{ces}{2q}\right)^q\left(\frac{e}{p}\right)^p\left(\frac{se}{\beta(\lambda,k)\ r}\right)^r\left[e^{-\lambda r}\sum_{R\geq p}\frac{(\lambda r)^R}{R!}R^p\right].$$

¿From (11) of Lemma A3, and (i) of Lemma A1,

$$e^{-\lambda r}\sum_{R\geq p}\frac{(\lambda r)^R}{R!}R^p \leq 2(cre)^p,$$

which is monotone increasing in r. The maximum of $(se/(\beta(\lambda,k)\ r))^r$ occurs at $r = s/\beta$, giving

$$\gamma(p,q,r,s) \leq \frac{A(c)}{[\beta(\lambda,k)]^s}\left(\frac{s}{\sqrt{n}}\right)\frac{e^{((k+1)+1/\beta)s}}{k^{ks}}e^{-cs}e^T\left(\frac{ces}{2q}\right)^q\left(\frac{kce^2s}{p}\right)^p.$$

Now, $p = T - 2q$ and $1 - x > \exp\{-x/(1-x)\}$ so, $p^{-p} \leq T^{-T}(eT)^{2q}$ and thus

$$\begin{aligned}e^T\left(\frac{ces}{2q}\right)^q\left(\frac{kce^2s}{p}\right)^p &\leq \left(\frac{kce^3s}{T}\right)^T\left(\frac{T^2}{2k^2ce^2s}\frac{e}{q}\right)^q\\ &\leq \exp\left\{\frac{T^2}{2k^2ce^2s}\right\}\left(kce^2s\frac{e}{T}\right)^T\\ &\leq \left[\exp\left\{\frac{1}{8k^3e^2}\right\}(4k^2e^3)\right]^{\frac{cs}{4k}}.\end{aligned}$$

Thus provided $k \geq 3$ and $c \geq \max\{k^2+k, 5(k+5)\}$ say, using Lemma A1 (iv)(b),

$$\sum_{p,q,r,s} \gamma(p,q,r,s) \leq \frac{A(c)}{\sqrt{n}} \sum_{s=1}^{n} s^4 \left[\frac{(5k^2e^3)^{c/4k}}{k^k} e^{-c+(k+1)+e^{k^2/(c-k)}+\frac{k^2}{c-k}} \right]^s$$

$$\leq \frac{A(c)}{\sqrt{n}} \sum_{s=1}^{n} s^4 e^{-cs/5}.$$

□

Lemma 8 *There exists a constant c_k such that if $c \geq c_k$ then $G \in G_{n,m,k}$ satisfies the following condition* **whp** .

Let $s_0 = n(3/c)^3 \exp\{-(2k+12)\}$, and let $s_1 = \alpha n$, where $\alpha = \frac{1}{\sqrt{2}(k+1)}$.

There is no subset of vertices S, $|S| = s$, $s_0/(k+1) \leq s \leq s_1$, such that $\Gamma_S = G[S \cup N(S)]$ is connected and $|N(S)| \leq k|S|$.

Proof Let $\mathcal{E}$ denote the event that there exists a set S, such that $|N(S)| \leq k|S|$.

We work in $G_{n,p}$, where $p = c/n$. There are $(k+1)s - 1$ edges in a spanning tree of Γ_S and suppose there are q other edges incident with S in Γ_S. The expected number $\gamma(q)$ of such Γ_S is at most,

$$\gamma(q) \leq \binom{n}{s}\binom{n-s}{ks}((k+1)s)^{(k+1)s-2}\binom{\binom{s}{2}+ks^2}{q}$$

$$\times \left(\frac{c}{n}\right)^{q+(k+1)s-1}\left(1-\frac{c}{n}\right)^{s(n-(k+1)s)+[\binom{s}{2}+ks^2-(q+(k+1)s-1)]}$$

$$\leq A(c)\frac{n}{s^2}\left(\frac{c^{k+1}e^{k+1}(k+1)^{k+1}e^{-c}}{k^k}\right)^s\left(\frac{s^2(k+\frac{1}{2})e^{1+c/n}c}{nq}\right)^q e^{(c-2k)s^2/(2n)}.$$

Now, $(xe/q)^q \leq e^x$, so

$$\sum_{q=0}^{\binom{s}{2}+ks^2} \gamma(q) \leq A(c)n\left(\frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k}e^{-c}e^{\frac{(k+1)cs}{n}}\right)^s$$

$$\leq A(c)n\exp\left\{-c\left(\left(1-\frac{1}{c}\log\frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k}\right)s-\frac{(k+1)s^2}{n}\right)\right\} \quad (7)$$

$$\leq A(c)ne^{-2Dn}.$$

Provided $\lambda < 1$ the function $f(s) = (1-\lambda)s-(k+1)s^2/n$ has an unconstrained maximum at $s = \frac{(1-\lambda)n}{2(k+1)}$. For the value $\lambda = \frac{1}{c}\log\frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k}$ given in (7) above, this is in the range (s_0, s_1). The minimum of $f(s)$ is at s_0. Thus (7) is maximized at s_0. The final inequality follows from (3). Thus from Lemma 4, and (4), (5)

$$\mathbf{Pr}(\mathcal{E}; G_{n,m,k}) \leq n^2 e^{-Dn}$$

□

Lemma 9 *For $c \geq c_k$, $G \in G_{n,m,k}$ satisfies the following conditions* **whp**.

(i) If $S \subset [n]$, $\alpha n \leq |S| \leq (1-\alpha)n$ then there are at least $4k|S|$ edges from S to $[n] - S$,

(ii) If $L(3c) = \{e \in E(G) : e$ is incident with a vertex of degree at least $3c\}$ then $|L(3c)| \leq ne^{-c/6}$

(iii) Let $L(k) = \{e \in E(G) : e$ is incident with a vertex of degree $k\}$ then $|L(k)| \leq ne^{-c/6}$.

Proof In all three cases we estimate the probability of failure in $G_{n,m}$ or $G_{n,p}, p = c/n$ and show it is at most e^{-2nD}. This estimate can then be inflated by $O(n^{1/2}e^{nD})$ as was done in Lemma 8.

(i) We assume without loss of generality that $s = |S| \leq n/2$. In $G_{n,p}$ the number of edges between S and $[n] - S$ has binomial distribution $B(s(n-s), p)$. This has mean $s(n-s)p$ and if $4ks = (1-\epsilon)s(n-s)p$ then $\epsilon \geq 2/3$ (using $c \geq 2(k+1)^3$ and $s \leq n/2$.) Applying the Chernoff-Hoeffding bound

$$\mathbf{Pr}(B(N,\theta) \leq (1-\epsilon)N\theta) \leq e^{-\epsilon^2 N\theta/3}$$

we obtain

$$\begin{aligned}
\mathbf{Pr}(\text{(i) fails in } G_{n,p}) &\leq \sum_{s=\alpha n}^{n/2} \binom{n}{s} e^{-sc/9} \\
&\leq \sum_{s=\alpha n}^{n/2} \left(\frac{ne}{s} e^{-c/9}\right)^s \\
&\leq n(e\sqrt{2}(k+1)e^{-2(k+1)^3/9})^{n/(\sqrt{2}(k+1))} \\
&\leq e^{-2nD}.
\end{aligned}$$

(ii) The probability that an edge $e \in G_{n,m}$ is incident with a vertex of degree at least $3c$ is at most 3^{-c}. Thus $\mathbf{E}(|L(3c)|) \leq 3^{-c}n$ in $G_{n,m}$. Changing one edge of $G_{n,m}$ changes $|L(3c)|$ by at most $12c$. Applying the Azuma-Hoeffding martingale tail inequality we get

$$\begin{aligned}
\mathbf{Pr}(\text{(ii) fails in } G_{n,m}) &\leq \exp\{-n(e^{-c/6} - 3^{-c})^2/(144c^2)\} \\
&\leq \exp -2nD.
\end{aligned}$$

(iii) The probability that edge $e \in G_{n,m}$ is incident with a vertex of degree k is at most $\exp -2c/3$. We proceed as in (ii). □

4 The proof of Theorem 1

We will write $k = 2l + 1$ if k is odd, and $k = 2l + 2$ otherwise. The property $\mathcal{A}_k$ requires the existence of l edge disjoint Hamilton cycles $H_i : i = 1, ..., l$, and, if k is even, a further edge disjoint (near) perfect matching H_0.

We prove the **whp** existence of these structures in a sequential manner. Initially the set of excluded edges $\mathcal{Q}(0)$ is empty. If k is even, we first prove **whp** the existence of a Hamilton cycle by the methods described below, and use the edges of this cycle to obtain a (near) perfect matching H_0. If k is odd, then $E(H_0) = \emptyset$. The edges of the matching H_0 are added to $\mathcal{Q}(1)$.

At the start of iteration $i = 1, ..., l$, the set $\mathcal{Q}(i)$ contains those edges to be excluded from the cycle H_i, by virtue of appearing in $H_0, ..., H_{i-1}$.Thus

$$\mathcal{Q}(i) = \cup_{j=0}^{i-1} E(H_j).$$

To prove the existence of H_i, we follow the method of Fenner and Frieze [FF]. A set $\mathcal{T}$ of edges of G is said to be *deletable* if

D(a) $\mathcal{T}$ is not incident with any vertex of degree k or degree at least $3c$,

D(b) $\mathcal{T}$ avoids a specified longest path $P_0 = x_0 P x_h$ in $G - \mathcal{Q}(i)$,

D(c) $\mathcal{T}$ avoids the specified set $\mathcal{Q}(i)$,

D(d) $\mathcal{T}$ is a matching.

Let $\mathcal{N}(G)$ be the set of edges of G which $\mathcal{T}$ must avoid in order to satisfy the conditions D(a),(b),(c) above and let $H = G - \mathcal{T} - \mathcal{Q}(i)$. Let $END(x_0; H)$ be a rotation endpoint set of the fixed longest path $x_0 P x_h$ in the subgraph H, and $\mathcal{R}(x_0; H)$ the associated final rotation subgraph.

Lemma 10 *Let $\mathcal{B}$ be the subset of graphs in $G_{n,m,k}$ which satisfy the conditions of Lemma's 6, 7, 8 and 9, then*

(i) $|\mathcal{B}| = (1 - o(1))|G_{n,m,k}|$,

(ii) If $G \in \mathcal{B}$ then $|END(x_0; H)| \geq \alpha n$,

(iii) $G \in \mathcal{B}$ implies H is connected.

Proof (i) This is a consequence of Lemmas 6-9.

(ii) Let $END(x_0; H) = S$. We assume $|S| < \alpha n$. $\mathcal{R}(x_0, H)$ is a connected subgraph induced by $S \cup N(S)$.

At the start of iteration $i = 0$, $\mathcal{Q}(0) = \emptyset$. At the start of iteration $i = 1, ..., l$ the degree in $\mathcal{Q}(i)$ of any vertex $v \in [n]$ is $2(i-1) + 1_{\{k=2l+2\}}$. We note that $|N_G(S)| < k|S|$, for if $|N_G(S)| \geq k|S|$ then $|N_{G-\mathcal{Q}(i)}(S)| \geq 3|S|$. We delete at most a matching from S in $G - \mathcal{Q}$, so this implies that $|N_H(S)| \geq 2|S|$ in contradiction of the Pósa condition. By Lemma 8, $|S| \leq s_0/(k+1)$.

As $\mathcal{R}(x_0)$ is a connected graph it satisfies condition (iii) of Lemma 7. Furthermore, $|N(S)| < ks$ satisfies condition (ii) of the same lemma. Thus we conclude that T, the degree of S, satisfies $T > cs/4k$. Let $|N(S)| = \theta s < ks$. By Lemma 6, the total number of edges in $S \cup N(S)$ satisfies

$$p + q \leq \frac{3}{2}(1+\theta)s,$$

and the total number of edges in S satisfies

$$q \leq \frac{3}{2}s.$$

A simple optimization shows that $T = 2q + p$ is maximized at $(p, q) = (3\theta s/2, 3s/2)$. Thus $T \leq 3s + \frac{3}{2}\theta s$ so that $T < \frac{3}{2}(k+2)s$. However, as $T > \frac{cs}{4k}$, this implies $c < 6k(k+2)$, and contradicts the assumption that $c \geq c_k$.

(iii) Starting with any vertex v, we see that both $\mathcal{R}(v)$ and P_0, are connected subgraphs of H, containing at least αn vertices by (ii). The connectivity of H follows from Lemma 9 (i). □

Let $\mathcal{E}$ be the subset of $G_{n,m,k}$ which does not have property $\mathcal{A}_k$. We will apply the edge colouring argument of Fenner and Frieze [FF] in an inductive manner to the set $\mathcal{E} \cap \mathcal{B}$ to prove that $\mathbf{Pr}(\mathcal{E}) \to 0$

Let $\mathcal{T}$ be a deletable set of edges of G of size $t = \lceil \log n \rceil$ avoiding the set $\mathcal{N}(G)$ of size s, and let αn be a lower bound on $|END(x_0)|$. By transforming P_0 in H using Pósa rotations there are at least $(\alpha n)^2/2$ longest paths aPb in H with distinct endpoint sets $\{a, b\}$. Thus if $G \in \mathcal{E}$ at least $(\alpha n)^2/2$ non-edges must be avoided when replacing a subgraph $\mathcal{T}'$, to form a graph $G' = G - \mathcal{T} + \mathcal{T}'$, $G' \in \mathcal{E}$. We will call such a replacement subgraph *addable.*

Let η be a lower bound on the number of ways of selecting a deletable $\mathcal{T}$ from G. Then, because of D(a),

$$\eta \geq \frac{1}{t!} \prod_{j=0}^{t-1} (m - s - 6cj)$$

$$\geq \frac{1}{t!} m^t \exp\left\{-\frac{t(s+6ct)}{m-(s+6ct)}\right\},$$

where, from Lemma 9,

$$s = |\mathcal{N}(G)| \leq \left(\frac{1}{2}(k-1) + 2e^{-c/6}\right) n.$$

Let μ be an upper bound on the number of ways of choosing an addable edge set $\mathcal{T}'$, then

$$\mu \leq \binom{\binom{n}{2} - \binom{\alpha n}{2} - (m-t)}{t} \leq \frac{1}{t!}\binom{n}{2}^t e^{-\alpha^2 t}.$$

The edge colouring argument of [FF], applied to $G_{n,m,k}$ shows that

$$\mathbf{Pr}(\mathcal{E} \cap \mathcal{B}) \leq (1+o(1)) \frac{\mu |G_{n,m-t,k}|}{\eta |G_{n,m,k}|}.$$

Now

$$\begin{aligned}
\frac{|G_{n,m-t,k}|}{|G_{n,m,k}|} &= \frac{\binom{\binom{n}{2}}{m-t}}{\binom{\binom{n}{2}}{m}} \frac{\mathbf{Pr}(\delta \geq k \text{ in } G_{n,m-t})}{\mathbf{Pr}(\delta \geq k \text{ in } G_{n,m})} \\
&\leq (1+o(1)) \left(\frac{m}{\binom{n}{2}}\right)^t
\end{aligned}$$

since the probability $\mathbf{Pr}(\delta \geq k \text{ in } G_{n,m-t})$ is non-decreasing as $t \to 0$.

We now find that

$$\mathbf{Pr}(\mathcal{E}) \leq O(1) \exp\left\{-O\left(t\left(\alpha^2 - \frac{s}{m-s}\right)\right)\right\}.$$

For $\mathbf{Pr}(\mathcal{E}) \to 0$, this requires that

$$\alpha = \frac{1}{\sqrt{2}(k+1)} > \sqrt{\frac{(k-1)+4e^{-c/6}}{c-((k-1)+4e^{-c/6})}},$$

which is satisfied when $c \geq c_k = 2(k+1)^3$. The value of α is the same as in Lemma 8.

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6 Appendix

Numerical calculations show that even for moderate values of c, the value of λ rapidly converges to c provided k is small. Similarly $\beta(\lambda, k)$ tends rapidly to 1. For example, $1 - 10^{-10} \leq \beta(30, 3) \leq 1$. Thus we can effectively treat λ and c as equal, and ignore $\beta(\lambda, k)$. The properties of $\beta(\lambda, k)$ are given in the following lemma.

Lemma A 1 *Let* $\lambda = \lambda(c)$ *be defined by*

$$\lambda \frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = c, \tag{8}$$

where

$$\begin{aligned}\beta(\lambda,0) &= 1,\\ \beta(\lambda,k) &= 1-e^{-\lambda}(1+\lambda+\cdots+\frac{\lambda^{k-1}}{k-1!}), \qquad k\geq 1.\end{aligned}$$

(o) *For $c>k$ the function $\lambda(c)$ is well defined,*

(i) $c-k<\lambda<c$,

(ii) (a) $\frac{\beta(\lambda,k-1)}{\beta(\lambda,k)}$ *is a monotone decreasing function of λ, tending to 1 as $\lambda\to\infty$,*

(b) $\frac{\beta(\lambda,k-1)}{\beta(\lambda,k)}$ *is a monotone increasing function of k,*

(iii) $\left(\frac{c}{\lambda e}\right)e^{\lambda/c}>1$,

(iv) (a) *$\beta(\lambda,k)$ is a monotone increasing function of λ tending to 1 as $\lambda\to\infty$.*

(b) $\frac{1}{\beta(\lambda,k)}\leq e^{\frac{k^2}{c-k}}$.

(c) *$\beta(\lambda,k)\geq e^{-D}$ where $D=\frac{kc^{k-1}}{(k-1)!}e^{-(c-k)}e^{k^2/(c-k)}$.*

Proof (o) We give a proof restricted to $\lambda\geq 2k$. This is adequate here, as $\lambda>c-k$ by (i), and $c\geq C_k=2(k+1)^3$.

Let $f(x)=x\frac{\beta(x,k-1)}{\beta(x,k)}$, let $\gamma=e^x\beta(x,k)=\sum_{j\geq k}\frac{x^j}{j!}$ and let $\phi(j)=\frac{x^{k-j}}{(k-j)!}$. Then (eventually)

$$\frac{df(x)}{dx}=\frac{1}{\gamma^2}\left((\gamma+\phi(1))(\gamma-x\phi(1))+x\gamma\phi(2)\right).$$

However, denoting $k(k+1)...(k+i)$ by $k^{(i)}$,

$$\begin{aligned}\gamma &\geq \frac{x^k}{(k-1)!}\left(\frac{1}{k}+\frac{x}{k(k+1)}+\cdots+\frac{x^i}{k^{(i)}}+\cdots+\frac{x^{k-1}}{k^{(k-1)}}\right)\\ &\geq x\phi(1).\end{aligned}$$

Because $x\geq 2k$, each of the k terms in the sum (above) is at least $1/k$. Thus $f(x)$ is monotone increasing on $[2k,\infty)$.

(i),(ii) We note that

$$\frac{\beta(\lambda,k-1)}{\beta(\lambda,k)}=1+\frac{k}{\lambda}\frac{1}{\left(1+\frac{\lambda}{k+1}+\frac{\lambda^2}{(k+1)(k+2)}+...+\frac{\lambda^i}{(k+1)^{(i)}}+...\right)}. \qquad (9)$$

(iii) For simplicity denote $\beta(\lambda, i)$ by β_i, and $\frac{\lambda^{k-1}}{k-1!}e^{-\lambda}$ by ℓ then $\beta_{k-1} = \beta_k + \ell$ and

$$\begin{aligned} \left(\frac{c}{\lambda e}\right) e^{\lambda/c} &= \left(e^{\left(\frac{\beta_k}{\beta_{k-1}}-1\right)} \frac{\beta_{k-1}}{\beta_k}\right) \\ &= \left(e^{-\frac{\ell}{\beta_k+\ell}}\right)\left(1+\frac{\ell}{\beta_k}\right) \\ &> \left(1-\frac{\ell}{\beta_k+\ell}\right)\left(1+\frac{\ell}{\beta_k}\right) \\ &= 1. \end{aligned}$$

(iv)(b) This comes from (9) by iterating

$$\beta(\lambda, j-1) \le (1+\frac{j}{\lambda})\beta(\lambda, j).$$

(iv)(c) This follows by applying (iv)(b) to the right hand side of

$$\beta(\lambda, k) \ge \exp\left\{-\frac{(1+\lambda+\cdots+\frac{\lambda^{k-1}}{(k-1)!})e^{-\lambda}}{\beta(\lambda, k)}\right\}.$$

□

Proof of Lemma 3

Let $X = X_i$ be a truncated Poisson random variable with parameters λ, k giving the occupancy of cell i, then $\mathbf{E}X = c$ where $cn = 2m$. Conditioning on $\sum_{i=1}^n X_i = 2m$ in $\mathcal{P}(n, \lambda, k)$, we obtain $\mathcal{O}(n, 2m, k)$. Specifically, if $\mathbf{x} \in \mathcal{P}(n, \lambda, k)$, $\mathbf{x} = (X_1, ..., X_n)$ and $\sum_{i=1}^n X_i = 2m$ so that $\mathbf{x} \in \mathcal{O}(n, 2m, k)$, then

$$\mathbf{Pr}(\mathbf{x}; \mathcal{O}(n, 2m, k)\ ) = \mathbf{Pr}([\mathbf{x} \mid \sum X_i = 2m]; \mathcal{P}(n, \lambda, k)\ ).$$

This is a generalization of the result that a multinomial random variable may be obtained from independent Poisson random variables by conditioning on their sum; and the details are described in, for example, [BFU]. Thus

$$\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)\ ) \le \frac{\mathbf{Pr}(\mathcal{E}; \mathcal{P}(n, \lambda, k)\ )}{\mathbf{Pr}(\sum_{i=1}^n X_i = 2m; \mathcal{P}(n, \lambda, k)\ )}.$$

By the Local Limit Theorem, (see [BFU] and [Du]) $\mathbf{Pr}(\sum_{i=1}^n X_i = 2m)$ is asymptotic to $1/\sqrt{2\pi\sigma^2 n}$, where $\sigma^2 = \mathbf{Var}(X)$ is given by

$$\sigma^2 = \lambda^2 \frac{\beta(\lambda, k-2)}{\beta(\lambda, k)} - c^2 + c, \tag{10}$$

and by (8) and Lemma A1(ii)(b) we see that $\sigma^2 < c$.

An element $\mathbf{d} = (d_i : i = 1, ...n)$ of $\mathcal{O}(n, 2m, k)$, induces a set of configuration multigraphs $M(\mathbf{d}) \subset \mathcal{M}(n, 2m, k)$. The expected number of loops and multiple edges in a configuration multigraph arising from an element of $\mathcal{O}(n, 2m, k)$, is a function of the degree sequence $\mathbf{d}$. The maximum occupancy of any box in $\mathcal{O}(n, 2m, k)$ is $o(\log n)$ with probability of the complementary event $n^{-O(\log\log n)}$. Conditioning on maximum degree $o(\log n)$ and using the methods given in [Bo], the probability there are no loops or multiple edges in such a configuration multigraph is asymptotic to $\exp\{-\zeta/2 - \zeta^2/4\}$, where

$$\zeta = \frac{1}{2m}\sum_{i=1}^{n} d_i(d_i - 1).$$

In $\mathcal{P}(n, \lambda, k)$, the random variable ζ is the sum of independent random variables and is sharply concentrated with expected value

$$\mathbf{E}\zeta = \lambda\frac{\beta(\lambda, k-2)}{\beta(\lambda, k-1)}.$$

Fix $\epsilon > 0$, small. Then with probability $1 - e^{-\Omega(\epsilon^2 n)}$, we have $(1-\epsilon)\mathbf{E}\zeta \le \zeta \le (1+\epsilon)\mathbf{E}\zeta$ in $\mathcal{P}(n, \lambda, k)$ and $\mathcal{O}(n, 2m, k)$. If $\mathcal{S}$ is the subset of $\mathcal{M}(n, 2m, k)$ corresponding to $G_{n,m,k}$, then

$$\begin{aligned}\mathbf{Pr}(\mathcal{S}; \mathcal{M}(n, 2m, k)\ ) &= o(1) + \sum_{\mathbf{d}}(1 + o(1))\exp\{-(\zeta(\mathbf{d})/2 + \zeta(\mathbf{d})^2/4)\mathbf{Pr}(M(\mathbf{d}); \mathcal{M}(n, 2m, k)\ )\} \\ &= (1 + O(\epsilon))\exp\{-(\mathbf{E}\zeta/2 + (\mathbf{E}\zeta)^2/4)\}.\end{aligned}$$

Now

$$\mathbf{Pr}(\mathcal{E}; G_{n,m,k}\ ) = \frac{\mathbf{Pr}(\mathcal{E} \cap \mathcal{S}; \mathcal{M}(n, 2m, k)\ )}{\mathbf{Pr}(\mathcal{S}; \mathcal{M}(n, 2m, k)\ )} \le \frac{\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)\ )}{\mathbf{Pr}(\mathcal{S}; \mathcal{M}(n, 2m, k)\ )}.$$

□

The following lemma, and its proof are due to B. Pittel [Pi], who uses this approach in, for example, [PW] and [Pi1]. The proof technique uses the Local Limit Theorem (see [Gn], [Du]) to avoid a direct application of the saddle point method. The origins of this technique can be traced back to A. I. Kinchin [Ki]. The use of the Local Limit Theorem in conjunction with generating functions for problems of this type was championed by V. F. Kolchin (see [Ko] for a wide ranging discussion).

Lemma A 2 (Pi)

$$\frac{|\mathcal{M}(n, 2m, k)\ |}{|\mathcal{M}(n, 2m)|} = (1 + o(1))\sqrt{\frac{c}{\sigma^2}}\ \left[\left(\frac{c}{\lambda e}\right)^c e^{\lambda}\beta(\lambda, k)\right]^n.$$

Proof Let $f(z) = \sum_{j\geq k} \frac{z^j}{j!}$ so that $f(z) = e^z\beta(z,k)$, and let $[z^t]g(z)$ denote the coefficient of z^t in the power series of $g(z)$.

$$\begin{aligned}\frac{|\mathcal{M}(n,2m,k)\,|}{|\mathcal{M}(n,2m)|} &= \sum_{\substack{b_1,\dots,b_n\geq k\\ b_1+\cdots+b_n=2m}} \binom{2m}{b_1\cdots b_n}\left(\frac{1}{n}\right)^{2m}\\ &= \frac{(2m)!}{n^{2m}}[z^{2m}](f(z))^n.\end{aligned}$$

Let $x > 0$, so that

$$[z^{2m}](f(z))^n = \frac{(f(x))^n}{x^{2m}}[z^{2m}]\left(\frac{f(zx)}{f(x)}\right)^n.$$

Let $Y(x)$ be a random variable chosen so that

$$\mathbf{E}z^Y = \frac{f(zx)}{f(x)},$$

where, such a Y exists as $\frac{f(zx)}{f(x)}\Big|_{z=1} = 1$. Let $Y_1, ..., Y_n$ be independent copies of Y. Then

$$\begin{aligned}[z^{2m}]\left(\frac{f(zx)}{f(x)}\right)^n &= [z^{2m}]\mathbf{E}(z^{Y_1+\cdots+Y_n})\\ &= \mathbf{Pr}(Y_1+\cdots+Y_n = 2m).\end{aligned}$$

Now

$$\mathbf{E}Y = \frac{d}{dz}\mathbf{E}z^Y\Big|_{z=1} = \frac{xf'(x)}{f(x)} = x\frac{\beta(x,k-1)}{\beta(x,k)},$$

so that, if we choose $x = \lambda$, then $\mathbf{E}Y = c$. Similarly, $\mathbf{Var}(Y) = \sigma^2$ is given by (10). By the Local Limit Theorem,

$$\mathbf{Pr}(Y_1+\cdots+Y_n = 2m) \sim \frac{1}{\sqrt{2\pi\sigma^2 n}}.$$

Hence

$$\frac{|\mathcal{M}(n,2m,k)\,|}{|\mathcal{M}(n,2m)|} = \frac{1+o(1)}{\sqrt{2\pi\sigma^2 n}}\frac{(2m)!}{n^{2m}}\frac{(f(\lambda))^n}{\lambda^{2m}},$$

and the result follows. □

Proof of Lemma 4

(i) We have for $j = 0$ and k that

$$|G_{n,2m,j}|2^m m! = (1+o(1))e^{-\theta_j/2-\theta_j^2/4}|\mathcal{M}(n,2m,j)|,$$

where $\theta_0 = c$, and $\theta_k = \lambda\beta(\lambda,k-2)/\beta(\lambda,k-1)$ was shown in the proof of Lemma 3.

(ii) Apply Lemma A1 (i),(iii) and (iv). □

Lemma A 3 *Let t, b be integer, where $t \geq b \geq 0$, and let $a > 0$, then*

$$\sum_{T \geq t} T^b \frac{a^T}{T!} \leq 2e^a (ae^{b/t})^b. \tag{11}$$

Proof Suppose $T \geq t \geq b$, then

$$\frac{T^b}{(T)_b} = \frac{T^b (T-b)!}{T!} \leq 2 \left(1 - \frac{b}{T}\right)^{T-b} e^b \leq 2 \exp\left\{\frac{b^2}{T}\right\}.$$

Thus

$$\begin{aligned} \sum_{T \geq t} T^b \frac{a^T}{T!} &\leq \sum_{T \geq t} 2e^{b^2/T} \left((T)_b \frac{a^T}{T!}\right) \\ &\leq 2e^{b^2/t}\, a^b e^a. \end{aligned}$$

□