Probabilistic Analysis of a Parallel Algorithm for Finding Maximal Independent Sets

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ABSTRACT

We consider a natural parallel version of the classical greedy algorithm for finding a maximal independent set in a graph. This version was studied in Coppersmith, Raghavan, and Tompa* and they conjecture there that its expected running time on random graphs *of arbitrary edge density of O* (log *n*). We prove that conjecture.

Key Words: average case complexity, independent sets, parallel algorithm.

1. INTRODUCTION

In this note we consider the problem of finding the lexicographically first maximal independent set (LFMIS) in a random graph. Coppersmith, Raghavan, and Tompa [1] describe a parallel version of the standard greedy algorithm for this problem: Suppose we are given a graph $G = (V, E)$, $V = [n] = \{1, 2, \ldots, n\}$. For $Z \subseteq V$ we let

$$\Gamma^+(Z) = \{x \notin Z : xz \in E \text{ for some } z < x, z \in Z\},$$

and

$$\Gamma^-(Z) = \{x \notin Z : xz \in E \text{ for some } z > x, z \in Z\}.$$

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* Proceedings of the 28th Annual IEEE Symposium on Foundations of Computer Science, 1987, pp. 260–269.

Note that we have implicitly oriented the edges from low to high.

algorithm PARALLEL GREEDY (G);
  begin
    GIS ← ∅;
    until G has no vertices **do**
      begin
        let $A = \{a : \Gamma^-(a) = \emptyset\}$;
        GIS ← GIS ∪ A;
        remove $A \cup \Gamma(A)$ from G
      end
    output GIS
  end

It is easy to see (Ref. 1, Lemma 2.1) that GIS is the LFMIS. Cook [2] showed that the problem of computing the LFMIS of a graph is complete for P and so is not in NC unless NC = P. PARALLEL-GREEDY can be implemented on a CRCW PRAM in $O(1)$ time per iteration if one processor is allocated to each edge of G.

Coppersmith, Raghavan, and Tompa showed that if $T(n, p)$ denotes the *expected* number of iterations $\tau = \tau(G)$ when $G = G_{n,p}$ then $T(n, p) = O(\frac{(\log n)^2}{\log \log n})$. ($G_{n,p}$ is the random graph with vertex set $[n]$ where each edge occurs independently with probability $p = p(n)$.)

They conjecture that $T(n, p) = O(\log n)$ and it is the aim of this article to prove it. We also prove a lower bound $T(n, p) = \Omega(\frac{\log n}{\log \log n})$ for a range of values of p. More precisely,

Theorem 1.   (a) $\frac{\alpha \log n}{4 \log \log n} \leq T(n, p)$ *for* $\frac{1}{n} \leq p \leq \frac{1}{n^\alpha}$ *where* $0 < \alpha \leq 1$ *is constant*
             (b) $T(n, p) = O(\log n)$.
             The hidden constant in (b) *is independent of* p.

Note that our inequalities are only claimed for n large.

2. LOWER BOUND PROOF

Let $m = \lfloor p^{-1} \rfloor$ and consider the subgraph H of G induced by $[m]$. If H contains a component which is a *directed path of length* $l - 1$, then clearly $\tau \geq \frac{1}{2}l$. (The direction of an edge ij, $i < j$ is from i to j.) Now let

$$l = \left\lfloor \frac{2 \log m}{3 \log \log m} \right\rfloor$$

and Z_l = the number of components of H which are directed paths of length l. We show

$$\Pr(Z_l \neq 0) \geq \frac{1}{1 + m^{-1/4}} \tag{1}$$

and the lower bound follows.

Now

$$E(Z_l) = \binom{m}{l} p^{l-1}(1-p)^{l(m-l)+\binom{l}{2}-l+1}$$

But

$$\binom{m}{l} = \frac{m^l e^{l-l^2/2m+O(l^3/m^2)}}{\sqrt{2\pi l}\, l^l}$$

and

$$(1-p)^{l(m-l)+\binom{l}{2}-l+1} = e^{-l+l^2/2m+O(l/m)}$$

and so

$$E(Z_l) \geq m^{1/3} \tag{2}$$

Furthermore

$$E(Z_l(Z_l - 1)) = \binom{m}{l}\binom{m-l}{l} p^{2l-2}(1-p)^{2l(m-2l)+\binom{2l}{2}-2l+2}$$

$$= \frac{\binom{m-l}{l}}{\binom{m}{l}} (1-p)^{-l^2} E(Z_l)^2$$

$$\leq \left(1 - \frac{l}{m}\right)^l (1-p)^{-l^2} E(Z_l)^2$$

$$\leq E(Z_l)^2 . \tag{3}$$

We can now use the useful inequality (Schwarz)

$$\Pr(Z_l \neq 0) \geq \frac{E(Z_l)^2}{E(Z_l^2)}$$

$$\geq \frac{1}{1 + E(Z_l)^{-1}} \quad \text{by (3)}$$

$$\geq \frac{1}{1 + m^{-1/3}} \quad \text{by (2)}$$

and (1) follows. ∎

3. UPPER BOUND PROOF

We shall, as in Ref. 1 consider a modified algorithm which *always* takes at least as long as PARALLEL-GREEDY. It is similar to the one described there. In what follows we may take $r = 5$.

algorithm MODIFIED PARALLEL-GREEDY (G);
  begin
     $X := V$; $Y := \text{GIS} := \emptyset$;
     while $|X| \geq m$ **do**
L1: **begin**
        $Z := m$ lowest numbered elements of X; $X := X \backslash Z$
        run PARALLEL-GREEDY for r iterations on $Y \cup Z$
        and let A be the independent set constructed;
        $X := X - (A \cup \Gamma^+(A))$; $Y := (Y \cup Z) - (A \cup \Gamma^+(A))$;
        $\text{GIS} := \text{GIS} \cup A$
L2: **end**
L3: run PARALLEL-GREEDY to completion on $X \cup Y$
  end

We claim that MODIFIED PARALLEL-GREEDY (MPG) constructs the LFMIS of G and always requires at least as many iterations as PARALLEL-GREEDY. This is because the difference between the modified algorithm and the original algorithm is that it does not necessarily add vertices to GIS the first time they become sinks. Instead we may have to wait until they become members of Z first. This does not change what goes into GIS, but instead delays the time when subsequent members of GIS become sinks.

We will show first under the assumption

$$p \leq (\log n)^{-7}$$

that with sufficiently high probability

there are at most $\lceil \log_{10/9} n \rceil$ executions of the main loop L1–L2, and

$$(4)$$

at most $\frac{3}{2} \lceil \log_{10/9} n \rceil$ *iterations of* PARALLEL-GREEDY *are needed to execute statement* L3.

$$(5)$$

Now let X_t, Y_t, $Z_t \subseteq X_t$ be the values of X, Y, Z at the start of the tth execution of the main loop A-B of MPG. Let A_t denote the set A constructed during the tth iteration.

The key to an analysis of MPG is an understanding of the distribution of the edges contained in $\cup_{i=1}^{k} Z_i$ for $k = 1, 2, \ldots$.

Suppose we condition on the values of $Z_1, Z_2, \ldots, Z_k$. We claim that

the edges contained in each Z_i are unconditioned, i.e., are chosen independently with probability p.

$$(6a)$$

a possible edge uv, $u \in Z_i$, $v \in Z_j$, $i < j$ will occur independently with probability 0 $(u \in A_i)$ or p $(u \notin A_i)$.

$$(6b)$$

Knowing this we first prove

Lemma 1. *If m is large and $t \leq m^{1/4}$ then*

$$\Pr\left(|Y_t| \geq 2\left(\frac{e^2}{2r}\right)^r m\right) \leq \frac{200 r^4 t^3}{m} .$$

Proof. Let $v \in Y_t$ and suppose $v \in Z_a$, $a \leq t - 1$. We claim there exists $b \leq a$ and a directed path with head v, of length greater than $2r(t - b)$, contained in $\cup_{i=b}^{a} Z_i$. Now either Z_a contains a directed path of length greater than $2(t - a)r$ with head v, in which case we take $b = a$, or not. In the latter case there is a vertex $v' \in Z_c \cap Y_a$, where $c < a$, and a directed path of length greater than $2r(t - a)$ from v' to v which is contained in $\cup_{i=c}^{a} Z_i$. For otherwise, at the start of iteration a the longest directed path with head v contained in $Y_a \cup Z_a$ is of length at most $2r(t - a)$. But then v will be eliminated by the end of iteration $t - 1$, contradiction. Inductively, for some $d \leq c$, v' is the head of a directed path of length greater than $2r(a - d)$ which is contained in $\cup_{i=d}^{c} Z_i$. The result now follows with $b = d$. (The base case, $t = 2$, for this induction is of course identical to the case where $b = a$ above.)

Let ξ_b denote the number of paths of length *exactly* $2r(t - b)$ contained in $\cup_{i=b}^{t-1} Z_i$. It follows that

$$|Y_t| \leq \sum_{b=1}^{t-1} \xi_b .$$

Now it follows from Eq. (6) that ξ_b is dominated stochastically by the number of *directed* paths η_b of length $2r(t - b)$ in the random graph $G_{m(t-b),p}$.

We prove in the Appendix that

$$\Pr\left(\eta_b \geq \left(\frac{e^2}{2r}\right)^{r(t-b)} m\right) \leq \frac{200 r^4 t^2}{m}, \quad 1 \leq b \leq t - 1. \tag{7}$$

Hence

$$\Pr\left(|Y_t| \geq m \sum_{k=1}^{t-1} \left(\frac{e^2}{2r}\right)^{kr}\right) \leq \frac{200 r^4 t^3}{m}$$

and the lemma follows. ∎

If $|Y_t|$ is small compared with m, then we can show that $|X_{t+1}| \leq \frac{9}{10}|X_t|$ with high probability.

Lemma 2. *If* $|Y_t| < 2(\frac{e^2}{2r})^r m$ *and* $|X_t| \geq m$ *then*

$$\Pr\left(|X_{t+1}| > \frac{9}{10}|X_t|\right) < e^{-\alpha m}$$

for some constant $\alpha > 0$.

Proof. We can proceed somewhat similarly to Lemma 2.2 of Ref. 1. Let $\beta = 2(\frac{e^2}{2r})^r$ and assume $|Y_t| \leq \beta m$. Let

$$Z_t' = \{z \in Z_t: \not\exists y < z, \, y \in Y_t \cup Z_t \text{ and } yz \in E\}$$

and

$$X_t' = \{x \in X_t - Z_t: \exists z \in Z_t' \text{ such that } zx \in E\} \, .$$

Then clearly $X_{t+1} \subseteq X_t - (Z_t' \cup X_t')$. Now if $z \in Z_t$ we have

$$\Pr(z \in Z_t') \ge (1 - p)^{(1+\beta)m} \ge 0.236 \ldots$$

and since the events $\{z \in Z_t'\}$ for $z \in Z_t$ are independent, we have

$$\Pr(|Z_t'| < \tfrac{1}{5}m) \le e^{-\alpha'm} \quad \text{for some } \alpha' 0.$$

(Here we use the fact that the tails of the binomial distribution are exponentially small.)

On the other hand, for $x \in X_t - Z_t$ we have

$$\Pr(x \not\in X_t' \mid |Z_t'| \ge \tfrac{1}{5}m) \le (1 - p)^{\frac{1}{5}m} \le 0.82$$

and since, given Z_t', the events $\{x \in X_t'\}$ for $x \in X_t - Z_t$ are independent

$$\Pr(|X_t'| \le \tfrac{1}{10}|X_t| \mid |Z_t'| \ge \tfrac{1}{5}m) \le e^{-\alpha''m} \quad \text{for some } \alpha'' > 0 \, .$$

The result follows.   ∎

Now let $l = \lceil \log_{10/9} n \rceil \le m^{1/4}$ (since $p \le (\log n)^{-7}$). For there to be more than l executions of the main loop of MPG we must have

$$|Y_t| \ge 2\left(\frac{e^2}{2r}\right)^r m \text{ for some } t \le l$$

or

$$|Y_t| < 2\left(\frac{e^2}{2r}\right)^r m \text{ and } |X_{t+1}| > \tfrac{9}{10}|X_t| \text{ for some } t \le l \, .$$

But, by Lemmas 3.1 and 3.2 the probability of one of these events is at most

$$\frac{200r^4 l^4}{m} + l e^{-\alpha m} = O((\log n)^{-3}) \, . \tag{8}$$

Suppose now that $\tau \le l$ iterations of the main loop of MPG suffice. Let $\lambda = \lambda(\tau) =$ the length of the longest directed path in $\cup_{i=1}^{\tau} Z_i$. Clearly the execution of Statement L3 of MPG requires at most $\lambda/2$ iterations of PARALLEL-GREEDY. But

$$\Pr\left(\exists \tau \le l \colon \lambda(\tau) \ge 3l\right) \le \sum_{t=1}^{l} \binom{mt}{3l} p^{3l-1}$$

$$\le m \sum_{t=1}^{l} \left(\frac{et}{3l}\right)^{3l}$$

$$\le m l m^{-2.5}$$

$$\le m^{-1}.$$

Together with Eq. (8) we have

Pr(MPG requires more than $(r + \frac{3}{2}) \log_{10/9} n$ iterations
of PARALLEL-GREEDY) $= O((\log n)^{-3})$.

Hence

Pr(PARALLEL-GREEDY requires more than
$(r + \frac{3}{2}) \log_{10/9} n$ iterations) $= O((\log n)^{-3}$. \hfill (9)

Now it follows easily from calculations done in Ref. 1

$$\Pr\left(\text{PARALLEL-GREEDY requires more than}\right.$$

$\dfrac{c(\log n)^2}{\log \log n}$ iterations$\Big) = O(1/n)$, for some absolute constant $c > 0$. \hfill (10)

It follows immediately from Eqs. (9) and (10) that we can write

$\tau(n, p) \le c \log n$ for some absolute constant $c > 0$,
provided $p \le (\log n)^{-7}$. \hfill (11)

We now consider $p \ge (\log n)^{-7}$. For this we introduce another modification of PARALLEL-GREEDY which we call MPG2. Let $m' = \lceil p^{-2} \rceil$.

Algorithm MPG2
   begin $X := [n]$; GIS $:= \emptyset$;
        while $X \ne \emptyset$ **do**
L1:   **begin**
           $Z := \min \{m', |X|\}$ lowest numbered elements of X;
           run PARALLEL-GREEDY to completion on Z and
           let A be the independent set constructed;
           $X := X - (A \cup \Gamma^+(A))$;
           GIS $:= $ GIS $\cup A$
L2:   **end**
   end

One can see as for MPG that MPG2 constructs the LFMIS of G and requires at least as many iterations as PARALLEL-GREEDY.

We will assume from now on that $p \le (2 \log 1/p)^{-7}$. This is true for p sufficiently small, say $p \le p_0$. For $p > p_0$ we know that with probability $1 - o(1/n)$ the size of GIS is less than say $2 \log_b n$ ($b = \frac{1}{1-p}$) and since PARALLEL-GREEDY finds at least one new member of GIS at each stage, it requires $O(\log n)$ iterations when $p > p_0$.

Let σ denote the number of iterations of the main loop L1–L2 of MPG2. Define X_t, Y_t, Z_t in analogy to MPG. Next let ζ_t, $t = 1, 2, \ldots, \sigma$ denote the number of iterations of PARALLEL-GREEDY required for the tth execution of L1–L2, and let $\zeta_t = 0$ for $t \ge \sigma + 1$.

Then

$$T(n, p) \le E\left(\sum_{t=1}^{n} \zeta_t\right). \tag{12}$$

Note by Eq. (11)

$$E(\zeta_t) \le c \log 1/p, \quad t = 1, 2, \ldots, n. \tag{13}$$

[The constant here is (roughly) twice that in (11). Also $E(\zeta_\sigma) = E_{m''}(T(m'', p))$ where $m'' \le m'$ is a random variable. Clearly, $T(m'', p) \le T(m', p)$ always.]

Now let $\sigma_0 = \frac{3 \log n}{\log 1/p} + 1$; we will show that

$$\Pr(\sigma > \sigma_0) \le \frac{1}{n}. \tag{14}$$

Hence

$$E\left(\sum_{t=1}^{n} \zeta_t\right) = E\left(\sum_{t=1}^{\sigma_0} \zeta_t\right) + E\left(\sum_{t=\sigma_0+1}^{n} \zeta_t\right)$$

$$\le \sum_{t=1}^{\sigma_0} E(\zeta_t) + n \Pr(\sigma > \sigma_0)$$

$$\le \sigma_0 c \log 1/p + 1$$

$$= O(\log n).$$

We will therefore have finished the proof of our theorem once we have completed the proof of Eq. (14).

Proof of Eq. (14). We will show in the Appendix that if $t \le \sigma$, $b = \frac{1}{1-p}$, $\xi \ge 2$, $\alpha = \log_b (m'p + 1) - 2 \log_b \log_e (m'p + 1)$, then

$$\Pr(|A_t| = \alpha - \log_b \xi) \le \begin{cases} e^{-\xi/p} & \xi \le \dfrac{m'p + 1}{4(\log_e (m'p + 1))^2} \\ e^{-1/4p^2} & \text{otherwise}. \end{cases} \tag{15}$$

Thus if $x \in X_t - Z_t$ and $\xi_0 = \frac{m'p + 1}{4(\log_e (m'p + 1))^2}$ then

$$\Pr(x \in X_{t+1}) = E((1-p)^{|A_t|})$$

$$\leq (1-p)^{\alpha - \log_b 2} + \sum_{k=\lceil \alpha - \log_b \xi_0 \rceil}^{\lfloor \alpha - \log_b 2 \rfloor} e^{-\frac{b^{\alpha - k}}{p}} + \alpha e^{-\frac{1}{4p^2}}$$

$$\leq \frac{1}{\xi_0} + \int_2^{\infty} e^{-\frac{\xi}{p}} \frac{d\xi}{\xi \log_e b} + \frac{1}{p^2} e^{-\frac{1}{4p^2}}$$

$$\leq \frac{1}{\xi_0} + \int_2^{\infty} e^{-\frac{\xi}{2p}} d\xi + \frac{1}{p^2} e^{-\frac{1}{4p^2}}$$

$$= \frac{1}{\xi_0} + 2pe^{-\frac{1}{p}} + \frac{1}{p^2} e^{-\frac{1}{4p^2}}$$

$$\leq p^{2/3} .$$

Hence if $|X_t| > m'$ then

$$E(X_{t+1} | |X_t|) \leq p^{2/3}|X_t| . \tag{16}$$

So if we define $x_t = |X_t|$ for $t \leq \sigma$ and $x_t = 0$ otherwise, then

$$E(x_t) \leq np^{\frac{2(t-1)}{3}} . \tag{17}$$

This is true for $t = 1$ and assuming that it is true for t we have

$$E(x_{t+1}) = \sum_{k=0}^{n} E(x_{t+1} | x_t = k) \Pr(x_t = k) \leq \sum_{k=0}^{n} p^{2/3}k \Pr(x_t = k) \quad \text{by (16)}$$
$$= p^{2/3}x_t$$

and Eq. (17) follows. Note that the argument here works regardless of $t \leq \sigma$ or $t > \sigma$.

It follows from Eq. (17) that

$$E(x_{\sigma_0}) \leq \frac{1}{n}$$

and so

$$\Pr(x_{\sigma_0} \geq 1) \leq \frac{1}{n}$$

and Eq. (14) follows. ∎

We have thus verified the conjecture of Ref. 1. It would be interesting to see if the same result can be proved for random regular graphs.

Acknowledgments

APPENDIX

Proof of Eq. (7). We first note that

$$E(\eta_b) = \binom{m(t-b)}{2r(t-b)+1} p^{2r(t-b)}$$

and so, since $t^2 = o(m)$ in our range of interest,

$$\frac{m}{4\sqrt{rt}} \left(\frac{1}{r}\right)^{2r(t-b)} \le E(\eta_b) \le m \left(\frac{e}{2r}\right)^{2r(t-b)} . \tag{A1}$$

Furthermore, where $s_0 = 2r(t-b)$ and $m_0 = m(t-b)$, we have

$$E(\eta_b(\eta_b - 1)) \le E(\eta_b)\left(E(\eta_b) + \sum_{s=1}^{s_0-1} \sum_{l=1}^{s} \binom{s_0}{2l-1}\binom{m_0}{s_0-s-l} p^{s_0-s}\right). \tag{A2}$$

In the summation s denotes the number of edges that a general path of length s_0 has in common with a given fixed such path, l denotes the number of subpaths made up by these common edges.

Now let $u_l = \binom{s_0}{2l-1}\binom{m_0}{s_0-s-l}$. Then

$$\frac{u_{l+1}}{u_l} = \frac{(s_0 - 2l)(s_0 - 2l + 1)(s_0 - s - 1)}{2l(2l+1)(m_0 - s_0 + s + l + 1)} \le \frac{s_0^3}{m_0} .$$

But since $t^3 = o(m)$ we have

$$E(\eta_b(\eta_b - 1)) \le E(\eta_b)\left(E(\eta_b) + 2\sum_{s=1}^{s_0-1} s_0 \binom{m_0}{s_0-s} p^{s_0-s}\right)$$

$$= E(\eta_b)^2 \left(1 + 2s_0 \sum_{s=1}^{s_0-1} v_s\right)$$

where

$$v_s = \frac{\binom{m_0}{s_0-s}}{\binom{m_0}{s_0+1}} p^{-s} .$$

Observe now that

$$\frac{v_{s+1}}{v_s} = \frac{s_0 - s}{(m_0 - s_0 + s + 1)p} \le \frac{s_0}{m_0 p} \le 2r .$$

Also $v_1 \le \frac{5r^2}{m}$. We deduce then that

$$E(\eta_b(\eta_b - 1)) \le E(\eta_b)^2 \left(1 + \frac{10r^2 s_0^2}{m}(2r)^{s_0}\right)$$

and hence

$$\frac{\text{Var}(\eta_b)}{E(\eta_b)^2} \le \frac{10r^2 s_0^2}{m}(2r)^{s_0} + \frac{1}{E(\eta_b)}$$

$$\le \frac{10r^2 s_0^2}{m}(2r)^{s_0} + \frac{4\sqrt{rt}}{m} r^{s_0}$$

$$\le \frac{11r^2 s_0^2}{m}(2r)^{s_0}.$$

Now the Chebycheff inequality yields

$$\Pr(\eta_b \ge \alpha m) \le \frac{\text{Var}(\eta_b)}{(\alpha m - E(\eta_b))^2} \quad \text{if } \alpha m \ge E(\eta_b)$$

$$\le \frac{44r^2 s_0^2}{\alpha^2 m}\left(\frac{e^2}{2r}\right)^{s_0} \quad \text{if } \alpha \ge 2\left(\frac{e}{2r}\right)^{s_0}.$$

Putting $\alpha = \left(\frac{e^2}{2r}\right)^{s_0/2}$ yields Eq. (7). ∎

Proof of (15). Now A_t is the set of vertices picked by the ordinary sequential greedy algorithm for finding a maximal independent set. Assume without loss of generality that $t = 1$ and $S = \{s_1 = 1, s_2, \ldots, s_k\} \subseteq [m']$. Then, where $s_{k+1} = m' + 1$,

$$\Pr(A_t = S) = (1-p)^{\binom{k}{2}} \prod_{i=1}^{k} (1 - (1-p)^i)^{s_{i+1} - s_i - 1}$$

$$\le (1-p)^{\binom{k}{2}} \prod_{i=1}^{k} (1 - (1-p)^k)^{s_{i+1} - s_i - 1}$$

$$= (1-p)^{\binom{k}{2}}(1 - (1-p)^k)^{m'-k}.$$

Hence

$$\Pr(|A_t| = k) \le \binom{m'-1}{k-1}(1-p)^{\binom{k}{2}}(1 - (1-p)^k)^{m'-k}.$$

Let now $\lambda = m'p + 1$ and $k = \log_b \lambda - 2\log_b \log_e \lambda - \log_b \xi$, $\xi \ge 2$. Assume first that $\xi \le \frac{\lambda}{4(\log_e \lambda)^2}$. We then have

$$\Pr\left(|A_t|=k\right) \leq \left(\frac{m'e}{\log_b \lambda}\right)^{\log_b \lambda} \left(\frac{(\log_e \lambda)^2 \xi}{\lambda}\right)^{\frac{k-1}{2}} \left(1 - \frac{(\log_e \lambda)^2 \xi}{\lambda}\right)^{m'-k}$$

$$\leq \left(\frac{m'e}{\log_b \lambda}\right)^{\log_b \lambda} \left(1 - \frac{(\log_e \lambda)^2 \xi}{\lambda}\right)^{m'}$$

$$\leq \left(\frac{3\lambda}{\log_e \lambda}\right)^{\log_b \lambda} \exp\left\{-\xi m' \frac{(\log_e \lambda)^2}{\lambda}\right\}$$

$$\leq \exp\left\{\frac{\log_e \lambda}{p} \left(\log_e \lambda + \log_e 3 - \log_e \log_e \lambda\right)\right.$$

$$\left. - \xi m' \frac{(\log_e \lambda)^2}{\lambda}\right\}$$

$$\leq \exp\left\{\frac{(\log_e \lambda)^2}{p}\left(1 - \frac{\xi}{2p}\right)\right\}$$

$$\leq e^{-\frac{\xi}{p}} \text{ as } p \text{ is small.}$$

Finally, suppose $\xi \geq \frac{\lambda}{4(\log_e \lambda)^2}$, so that $k \leq \log_b 4$. Then

$$\Pr\left(|A_t|=k\right) \leq \left(\frac{m'e}{\log_b \lambda}\right)^{\log_b \lambda} \left(\frac{3}{4}\right)^{(m'-k)}$$

$$\leq \left(\frac{6}{p\log_e \lambda}\right)^{\log_b \lambda} \left(\frac{3}{4}\right)^{m'}$$

$$\leq \exp\left\{\frac{1}{p}\log_e \lambda\left(\log_e 6 + \log_e \frac{1}{p}\right) - \frac{1}{p^2}\log_e \frac{4}{3}\right\}$$

$$\leq e^{-\frac{1}{4p^2}}. \qquad \square$$

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