

The bright side of simple heuristics for the TSP

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Abstract

The greedy and nearest-neighbor TSP heuristics can both have $\log n$ approximation factors from optimal in worst case, even just for n points in Euclidean space. In this note, we show that this approximation factor is only realized when the optimal tour is unusually short. In particular, for points from any fixed d -Ahlfor's regular metric space, our results imply that the greedy and nearest-neighbor heuristics have *additive* errors from optimal on the order of the *optimal* tour length through *random* points in the same space, for $d > 1$.

1 Introduction

Papadimitriou [7] showed that finding an optimum Traveling Salesperson Tour is NP-hard even for points in Euclidean space, while Arora [1] and Mithcell [6] give polynomial-time approximation schemes for the Euclidean TSP. In practice these have resisted efficient implementations, and Euclidean TSP approximation still leans heavily on heuristics which are not known to be asymptotically optimal. For metric TSP, Christofides algorithm achieves an approximation ratio of 1.5, which saw slight improvement with the recent breakthrough of Karlin, Klein, Gharan, and Shayan [4].

Perhaps the simplest heuristics to find a tour through n points are the Nearest Neighbor heuristic and the Greedy heuristic. At each step the Greedy heuristic chooses the shortest available edge which would not create any vertices of degree 3 or close a cycle except on the n th step. It continues

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until it has created a tour. The Nearest Neighbor heuristic starts by choosing some vertex y_1 and builds a sequence of paths $P_i = (y_1, y_2, \dots, y_i)$ where y_{i+1} minimises the distance $\text{dist}(y_i, y)$ over all $y \in X_n \setminus \{y_1, y_2, \dots, y_i\}$ and finishes with the tour $(y_1, y_2, \dots, y_n, y_1)$.

For n points in an arbitrary metric space, each of these heuristics is known to give a tour within $\log n$ of optimal [2, 3], and examples are known which realize these approximation ratios, even just in Euclidean space. But our main result implies that for n points in the unit square whose optimal tour has length $\Omega(\sqrt{n})$ (as is the typical case), the Greedy and Nearest Neighbor heuristics will both return a tour whose length is within a constant factor of optimal.

We will prove our results not just for full-dimensional Euclidean space but for any sufficiently regular metric space with dimension $d > 1$; the point of this generality is to emphasize that for greedy or nearest-neighbor algorithms to have poor approximation ratios on some input, it is really necessary that the input admits an unexpectedly short tour given the space its points are taken from, rather than, say, just because the input was actually chosen from a lower dimensional subset of the space than expected.

A metric space \mathcal{M} equipped with a measure μ is *d-Ahlfors's regular* if there are constants C, D so that

$$Cr^d \leq \mu(B(p, r)) \leq Dr^d \tag{1}$$

for all $p \in \mathcal{M}$ and $0 < r \leq \text{diam}(\mathcal{M})$. Here $B(p, r)$ is the ball of radius r centred at p . Simple examples of regular metric spaces include subspaces of Euclidean space like unit cubes under Lebesgue measure (having integer dimensions), d -dimensional manifolds, or fractals like the Sierpinski gasket under the Hausdorff measure (having intermediate dimensions)—for example, the metric space induced in Euclidean space by any fractal generated by an iterative function system satisfying the open set condition is Ahlfors's regular for some d , for the Hausdorff measure of appropriate dimension (e.g., see Section 8.3 of [5]).

We will prove the following about optimal TSP tours in Ahlfors's-regular spaces:

Theorem 1. *Suppose x_1, x_2, \dots is a sequence of i.i.d points drawn from a d -Ahlfors's regular probability measure on the metric space \mathcal{M} . Then there exists a constant A_1 so that for $X_n = \{x_1, \dots, x_n\}$, the length of the optimal tour through X_n has length at least $A_1 n^{1-\frac{1}{d}}$ for all sufficiently large n , with probability 1.*

Our main result for the nearest-neighbor and greedy heuristics is then the following:

Theorem 2. *If the bounded metric space \mathcal{M} admits a d -Ahlfors regular probability measure then there is a constant A_2 and an n_0 such that for any n points in \mathcal{M} with $n \geq n_0$, the nearest-neighbor and greedy algorithms produce a tour of length at most $A_2 n^{1-\frac{1}{d}}$.*

2 Proofs

Proof of Theorem 1. Let D be the constant from (1) guaranteed to exist for (\mathcal{M}, μ) . Let $r = (\frac{1}{Dn})^{1/d}$. For any fixed i , let Z_i be the indicator for the event \mathcal{E}_i that x_i is the unique point from X_n in $B(x_i, r)$. Then

$$\Pr(\mathcal{E}_i) \geq (1 - Dr^d)^{n-1} \geq e^{-1}.$$

Let $Z = Z_1 + \dots + Z_n$. Thus $\mathbb{E}(Z) \geq e^{-1}n$. Let \mathcal{B}_n be the event that there exists i such that $B(x_i, 2r)$ contains more than $\gamma = \log^2 n$ points from X_n other than x_i . Then

$$\Pr(\mathcal{B}_n) \leq n \Pr(\text{Bin}(n, D(2r)^d) \geq \gamma) \leq \binom{n}{\gamma} (D(2r)^d)^\gamma \leq \left(\frac{2^d e}{\gamma}\right)^\gamma \leq n^{-\log n}.$$

If \mathcal{B}_n does not occur then changing the value of one x_i only changes the value of Z by at most γ . Indeed, moving a point can only remove it from γ balls and so moving a ball can only increase Z by γ . On the other hand, suppose there are λ points $x_j, j \in \Lambda$ such that (i) $Z_j = 1$ and (ii) there is a point x within r of each point in Λ , so that moving x_i to x decreases Z by λ . Now $j \in \Lambda$ implies that the ball $B(x_j, 2r)$ contains at least λ points and so $\lambda \leq \gamma$.

If \mathcal{B}_n does occur then Z could change by at most n . We will now use Warnke's *typical bounded differences inequality* [8] to show that Z is concentrated around its mean.

Theorem 3 (Warnke). *Let $X = (X_1, \dots, X_N)$ be a family of independent random variables with X_k taking values in a set Λ_k . Let $\Gamma \subseteq \prod_{j \in [N]} \Lambda_j$ be an event and assume that the function $f : \prod_{j \in [N]} \Lambda_j \rightarrow \mathbf{R}$ satisfies the typical Lipschitz condition: there are numbers $c_k, k \in [N]$ and $d_k, k \in [N]$ such that whenever x, y differ only in the k th coordinate, we have*

$$|f(x) - f(y)| \leq \begin{cases} c_k & \text{if } x \in \Gamma. \\ d_k & \text{otherwise.} \end{cases}$$

Then for all numbers $\gamma_k, k \in [N]$ with $\gamma_k \in (0, 1)$,

$$\Pr(|f(X) - \mathbb{E}(f(X))| \geq t) \leq 2 \exp \left\{ -\frac{t^2}{2 \sum_{k \in [N]} (c_k + \gamma_k (d_k - c_k))^2} \right\} + \Pr(X \notin \Gamma) \sum_{k \in [N]} \gamma_k^{-1}.$$

We will apply this theorem with $f = Z, N = n, X = \{x_1, \dots, x_n\}, \Gamma = \mathcal{B}_n^c$ and $c_k = \gamma, d_k = n, \gamma_k = n^{-2}$ for $k \in [n]$. This yields

$$\Pr(Z \leq \mathbb{E}(Z) - n^{2/3}) \leq 2 \exp \left\{ -\frac{n^{4/3}}{2n(\log^2 n + 1)^2} \right\} + n^{3-\log n} = o(1).$$

So, w.h.p. there are at least $n/3$ of the x_i that are at least r from their nearest neighbor. Theorem 1 follows immediately from the Borel-Cantelli Lemma. \square

Proof of Theorem 2. Consider any nearest-neighbor or greedy tour x_1, \dots, x_n through the point-set $X = \{x_1, \dots, x_n\} \in \mathcal{M}$. We define a sequence of open balls B_1, \dots, B_{n-1} , where B_i is centered at x_i and has radius $\text{dist}(x_i, x_{i+1})$. Observe that when the edge from $\{x_i, x_{i+1}\}$ is selected, there can be no other vertices x_j which would be available for selection but are closer to x_i than $\text{dist}(x_i, x_{i+1})$. This implies that the family $\mathcal{B}_n = \{B_i\}$ has the following property:

(\star) For any distinct balls $B_i, B_j \in \mathcal{B}_n$, we have either that B_i does not contain the center of B_j or that B_j doesn't contain the center of B_i (according to whether $i < j$ or $j < i$, respectively).

Now we partition \mathcal{B}_n into sets $\mathcal{B}_1, \mathcal{B}_2, \dots$, where each \mathcal{B}_j consists of every ball in \mathcal{B}_n whose radius r satisfies $\frac{1}{2^j} < r \leq \frac{1}{2^{j-1}}$.

Now each family \mathcal{B}_i consists of balls whose radii differ by at most a factor of 2. In particular, as (\star) implies that the distance between the center of two balls in \mathcal{B}_n is at least the minimum of the radii of the two balls, within each family \mathcal{B}_i , we know that the distance between the centers of two balls is at least half the maximum of the radii of the two balls. In particular, if we define families $\tilde{\mathcal{B}}_i$ by rescaling the balls in each family \mathcal{B}_i by a factor of $\frac{1}{2}$, then each family $\tilde{\mathcal{B}}_i$ is a family of disjoint balls, each of measure at least $C2^{-kd}$. As such, we have from the condition (1) that

$$|\mathcal{B}_k| \leq C^{-1} 2^{kd}. \quad (2)$$

In particular, we can bound the total length L of the nearest neighbor tour by the radii $r(B)$ of the balls $B \in \mathcal{B}_n$ as follows:

$$L \leq \sum_{B \in \mathcal{B}_n} r(B) = \sum_{k \geq 1} \sum_{B \in \mathcal{B}_k} r(B) \leq C^{-1} \sum_{k=1}^{k_0} 2^{kd} \cdot 2^{-(k-1)} \leq C^{-1} \frac{2^{(k_0+1)(d-1)+2}}{2^{d-1} - 1}, \quad (3)$$

where k_0 is smallest integer for which the bound $C^{-1}2^{k_0d}$ on $|\mathcal{B}_{k_0}|$ from (2) exceeds n . We have thus that for any $d > 1$ and a constant C_1 depending on the metric space \mathcal{M} but not the point set X , that

$$L \leq C_1 n^{1-\frac{1}{d}},$$

proving the theorem. □

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