Shortest paths with a cost constraint: a probabilistic analysis

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Abstract

We consider a constrained version of the shortest path problem on the complete graphs whose edges have independent random lengths and costs. We establish the asymptotic value of the minimum length as a function of the cost-budget within a wide range.

1 Introduction

Let the edges of the complete graph K_n be given independent random edge lengths $w(e)$ and random costs $c(e)$ for $e \in E_n = E(K_n)$. Suppose further that we are given a budget c_0 and we need to find a path P from vertex 1 to vertex 2 of minimum length $w(P) = \sum_{e \in P} w(e)$ whose cost $c(P) = \sum_{e \in P} c(e)$ satisfies $c(P) \leq c_0$. More precisely, let $\mathcal{P}$ denote the set of paths from 1 to 2 in K_n. We wish to solve

> **CSP:** minimize $w(P)$ subject to $P \in \mathcal{P}, c(P) \leq c_0$.

This is a well studied problem, at least in the worst-case, see for example Chen and Nie [3], Climaco and Martins [4], Machuca, Mandow, Pérez de la Cruz and Ruiz-Sepulveda [8], Nielsen, Pretolani and Anderesen [10], Pascoal, Captivo and Clímaco [11].

In this paper we consider the case where $w(e), c(e), e \in E_n$ are independent random variables and we let $L_n = L_n(c_0)$ denote the random minimum length of a within budget shortest path. Also, let H_n denote

the *hop-count* (number of edges) in the shortest such path. In particular we will assume that $w(e), c(e)$ are independent copies of the uniform $[0,1]$ random variable U. In a recent paper Frieze, Pegden, Sorkin and Tkocz [5] considered a slightly more general setting, but were only able to bound L_n w.h.p. between two values. In the simpler setting of this paper we are able to get an asymptotically correct estimate of L_n.

Notation: we say that $A_n \lesssim B_n$ if $A_n \leq (1 + o(1))B_n$ as $n \to \infty$; $A_n \gtrsim B_n$ if $A_n \geq (1 + o(1))B_n$ as $n \to \infty$; $A_n \approx B_n$ if $A_n \lesssim B_n$ and $A_n \gtrsim B_n$.

We prove the following:

Theorem 1. *Suppose that $w(e), c(e), e \in E_n$ are independent copies of a uniform random variable on $[0,1]$. Suppose that*

$$\frac{1}{\sqrt{2}} \frac{\log^2 n}{n} \leq c_0 \leq \frac{1}{2\sqrt{2}}. \tag{1}$$

Then w.h.p.

$$L_n \approx \frac{\log^2 n}{4 c_0 n} \qquad and \qquad H_n \approx \frac{\log n}{2}.$$

The main new ideas of the paper are in the proof of Theorem 1. With a little effort this theorem can be generalised to prove the following:

Theorem 2. *Suppose now that $w(e), c(e), e \in E_n$ are independent copies of U^γ where $\gamma > 0$. Suppose that*

$$\frac{a_1 \log^2 n}{n} \leq c_0 \leq a_2 \qquad for\ constants\ a_1, a_2\ dependent\ on\ \gamma.$$

Then w.h.p.

$$L_n \approx \frac{\gamma \log^2 n}{4\Gamma\left(\frac{1}{\gamma} + 1\right)^2 c_0 n^\gamma} \qquad and \qquad H_n \approx \frac{\gamma \log n}{2}.$$

2 Outline of paper

We will obtain an estimate of L_n in two distinct ways and combine them to give us what we need. In Section 3 we use the first moment method to get a lower bound and in Section 4 we use Lagrangean Duality to obtain another bound. We combine the two bounds and finish the proof of Theorem 1 in Section 5. We then give a sketch proof of Theorem 2 in Section 6.

3 First Moment

The goal of this section is to prove the following lemma about a high probability bound on the product $w(P)c(P)$ for *every* path $P \in \mathcal{P}$.

Lemma 3. *Suppose that $w(e), c(e), e \in E_n$ are independent copies of U. W.h.p. for every path $P \in \mathcal{P}$, we have*

$$w(P)c(P) \gtrsim \frac{\log^2 n}{4n}.$$

Corollary 4. *For every c_o, $L_n \gtrsim \frac{\log^2 n}{4nc_0}$ w.h.p.*

Proof. We have, $L_n = \min\{w(P) : P \in \mathcal{P}, c(P) \le c_0\} \ge \min\left\{w(P)\frac{c(P)}{c_0} : P \in \mathcal{P}, c(P) \le c_0\right\} \gtrsim \frac{\log^2 n}{4nc_0}$. $\qquad\square$

For the proof of Lemma 3, we need a bound on events that for a fixed path P, we have $w(P)c(P) \le t$.

Lemma 5. *Let S and T be independent copies of $U_1 + \cdots + U_k$, where $U_1, \ldots, U_k$ are i.i.d. copies of a uniform random variable on $[0, 1]$. Then for $0 < t < k^2$, we have*

$$\mathbb{P}\left(ST \le t\right) \le \frac{t^k}{k!^2}\left(k\log\left(\frac{k^2}{t}\right) + 2\frac{k!}{k^k}\right).$$

Proof. We have $\mathbb{P}\left(S \le x\right) \le \frac{x^k}{k!}$ for every $x \ge 0$ (e.g. by looking at the volume of the orthogonal simplex of side-length x). Moreover, $S \le k$, so $\mathbb{P}\left(S \le x\right) = 1$ for every $x \ge k$. Therefore, using independence,

$$\mathbb{P}\left(ST \le t\right) = \mathbb{E}_T\mathbb{P}_S\left(S \le \frac{t}{T}\right) = \mathbb{E}_T\left(\mathbb{P}_S\left(S \le \frac{t}{T}\right)1\left(\frac{t}{T} < k\right)\right) + \mathbb{E}_T 1\left(\frac{t}{T} \ge k\right)$$

$$\le \mathbb{E}_T\left(\frac{1}{k!}\left(\frac{t}{T}\right)^k 1\left(T > \frac{t}{k}\right)\right) + \mathbb{P}_T\left(T \le \frac{t}{k}\right). \tag{2}$$

For the second term, we use again $\mathbb{P}_T\left(T \le \frac{t}{k}\right) \le \frac{1}{k!}\left(\frac{t}{k}\right)^k$, whereas for the first one, writing

$$T^{-k} = \int_0^\infty k u^{-k-1} 1\left(u > T\right) \mathrm{d}u$$

yields

$$\mathbb{E}_T\left(\frac{1}{k!}\left(\frac{t}{T}\right)^k 1\left(T > \frac{t}{k}\right)\right) = \frac{t^k}{k!}\mathbb{E}_T\left(\int_0^\infty k u^{-k-1} 1\left(T > \frac{t}{k}, T < u\right)\mathrm{d}u\right)$$

$$\le \frac{t^k}{k!}\int_{t/k}^\infty k u^{-k-1}\mathbb{P}\left(T < u\right)\mathrm{d}u$$

$$\le \frac{t^k}{k!}\left(\int_{t/k}^k k u^{-k-1}\frac{u^k}{k!}\mathrm{d}u + \int_k^\infty k u^{-k-1}\mathrm{d}u\right)$$

$$= \frac{t^k}{k!^2}\left(k\log\left(\frac{k^2}{t}\right) + \frac{k!}{k^k}\right).$$

Putting these together finishes the proof. $\qquad\square$

Proof of Lemma 3. For constant $\beta > 0$ to be chosen soon, we let

$$\mathcal{P}_\beta = \left\{P \in \mathcal{P} : w(P)c(P) \le \frac{\beta\log^2 n}{n}\right\}.$$

Then,

$$\mathbb{E}(|\mathcal{P}_\beta|) \le \sum_{1 \le \ell \le n} n^{\ell-1}\frac{1}{\ell!^2}\left(\frac{\beta\log^2 n}{n}\right)^\ell\left(\ell\log\left(\frac{\ell^2 n}{\beta\log^2 n}\right) + 2\frac{\ell!}{\ell^\ell}\right). \tag{3}$$

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Explanation: we choose the $\ell-1$ internal vertices and order them in $\binom{n}{\ell-1}(\ell-1)! \leq n^{\ell-1}$ ways to create a path P of edge-length ℓ. We then use Lemma 5 to bound the probability that $w(P)c(P) \leq \frac{\beta \log^2 n}{n}$, i.e. $P \in \mathcal{P}_\beta$.

Let u_ℓ denote the summand in (3). Note that for large enough n,

$$\ell \log\left(\frac{\ell^2 n}{\beta \log^2 n}\right) + 2\frac{\ell!}{\ell^\ell} \leq \ell \log\left(\frac{n^3}{\beta \log^2 n}\right) + 2 \leq 3\ell \log n.$$

Using $\ell! \geq (\ell/e)^\ell$, putting $\ell = \alpha \log n$ and $\Delta = \alpha(\log(\beta/\alpha^2) + 2) - 1$, we have

$$n^{\ell-1}\frac{1}{\ell!^2}\left(\frac{\beta \log^2 n}{n}\right)^\ell \leq n^{\ell-1}\left(\frac{e}{\ell}\right)^{2\ell}\left(\frac{\beta \log^2 n}{n}\right)^\ell = e^{\Delta \log n}.$$

Looking at $\frac{\partial \Delta}{\partial \alpha} = \log\frac{\beta}{\alpha^2}$, we see that for a fixed β, Δ is maximized when $\alpha = \sqrt{\beta}$, giving $\Delta \leq 2\sqrt{\beta} - 1$. Therefore, we choose β such that, say $2\sqrt{\beta} - 1 = -(\log n)^{-1/2}$, that is

$$\beta = \frac{1}{4}\left(1 - \frac{1}{\sqrt{\log n}}\right)^2.$$

Then

$$\sum_{1 \leq \ell \leq \log^2 n} u_\ell \leq \sum_{1 \leq \ell \leq \log^2 n} e^{-\sqrt{\log n}} \cdot 3\ell \log n \leq 3\log^5 n\, e^{-\sqrt{\log n}} = o(1).$$

It remains to note that for $\ell > \log^2 n$,

$$n^{\ell-1}\frac{1}{\ell!^2}\left(\frac{\beta \log^2 n}{n}\right)^\ell \leq n^{-1}\left(\frac{e^2\beta \log^2 n}{\ell^2}\right)^\ell \leq n^{-1}\left(\frac{e^2\beta \log^2 n}{\log^4 n}\right)^\ell \leq n^{-1}e^{-\ell} \leq n^{-1}e^{-\log^2 n},$$

thus

$$\sum_{\log^2 n < \ell \leq n} u_\ell \leq \sum_{\log^2 n < \ell \leq n} n^{-1}e^{-\log^2 n} \cdot 3\ell \log n \leq 3n(\log n)e^{-\log^2 n} = o(1).$$

We conclude that $\mathbb{P}\left(|\mathcal{P}_\beta| > 0\right) \leq \mathbb{E}|\mathcal{P}_\beta| = o(1)$ with $\beta = \frac{1}{4}\left(1 - \frac{1}{\sqrt{\log n}}\right)^2$. $\qquad\square$

Remark 1. *Corollary 4 can also be obtained by a much simpler first moment argument counting the number of paths P such that $w(P) \leq w$ and $c(P) \leq c$ with $wc \gtrsim \frac{\log^2 n}{4n}$ (see [5]). Lemma 3 will be crucial however in Section 5.*

4   The dual

In this section, motivated by the Lagrangean dual to CSP, we consider the following random variable

$$\psi(\lambda) = \min\left\{w(P) + \lambda c(P) : P \in \mathcal{P}\right\},$$

where λ is a positive parameter chosen later. In words, $\psi(\lambda)$ is the the minimum length of a path from 1 to 2 in K_n when edge lengths are independent copies of $W = W(\lambda) = U_1 + \lambda U_2$. Here U_1, U_2 are independent copies of U. The goal of this section is to establish the following upper bound on $\psi(\lambda)$ for a specific *optimal* choice of λ (see Remark 3 below).

Lemma 6. *Let c_0 satisfy (1). Let $\lambda^* = \frac{\log^2 n}{4c_0^2 n}$. Then*

$$\psi(\lambda^*) \lesssim \frac{\log^2 n}{2c_0 n} \qquad w.h.p. \tag{4}$$

We shall need the following result of Bahmidi and van der Hofstadt from [1].

Theorem 7 ([1]). *Let $s > 0$ be a constant. Let $L_{s,n}$ be the length of a shortest path from 1 to 2 in the complete graph K_n when edge lengths are independent copies of ξ^s, where ξ is an exponential mean 1 random variable. We have the following convergence in distribution*

$$n^s L_{s,n} - \frac{1}{\Gamma(1+1/s)^s} \log n \xrightarrow{d} Z, \tag{5}$$

for some random variable Z. Moreover, for the hop-count $H_{s,n}$,

$$H_{s,n} \approx s \log n \qquad w.h.p.$$

In fact, we shall only need the following simple consequence of (5).

Claim. $L_{s,n} \approx \frac{1}{\Gamma(1+1/s)^s} \frac{\log n}{n^s}$ w.h.p.

Proof. For every sequence of numbers $a_n \to 0$, from (5), we get

$$a_n \left(n^s L_{s,n} - \frac{1}{\Gamma(1+1/s)^s} \log n \right) \to 0 \qquad \text{in probability.}$$

Choosing, say $a_n = (\log n)^{-1/2}$, we get

$$\mathbb{P}\left((\log n)^{-1/2} \left| n^s L_{s,n} - \frac{1}{\Gamma(1+1/s)^s} \log n \right| > 1 \right) \to 0,$$

or, in other words,

$$L_{s,n} = \frac{\log n}{\Gamma(1+1/s)^s n^s} \left[1 + \theta_n (\log n)^{-1/2} \Gamma(1+1/s)^s \right]$$

with $\mathbb{P}(|\theta_n| < 1) \to 1$. $\qquad \square$

Heuristically, the idea is that the density of $W(\lambda)$ near the origin behaves like the density of $(2\lambda\xi)^{1/2}$, hence $\psi(\lambda)$ is asymptotic to $(2\lambda)^{1/2} L_{1/2,n}$ whose asymptotic behaviour is in turn governed by (5). To make this rigorous, we need the following lemma.

Lemma 8. *Let $L_{1/2,n}$, $\tilde{L}_{1/2,n}$ be the length of a shortest path from 1 to 2 in the complete graph K_n when edge lengths are independent copies of $\xi^{1/2}$, $\tilde{\xi}^{1/2}$ respectively, where ξ is an exponential mean 1 random variable and*

$$\tilde{\xi} = \begin{cases} \xi, & \xi \leq \frac{\log^2 n}{n}, \\ \infty, & \xi > \frac{\log^2 n}{n}. \end{cases}$$

Then $L_{1/2,n} = \tilde{L}_{1/2,n}$ w.h.p.

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Proof. Now the claim applied with $s = \frac{1}{2}$ implies that w.h.p. the shortest path from 1 to 2 has length $\lesssim \frac{1}{\sqrt{2}} \frac{\log n}{\sqrt{n}}$ if edge lengths are given by $\xi^{1/2}$. This clearly implies that w.h.p. *the* shortest path P from 1 to 2 contains no edge with $\xi^{1/2} > \frac{\log n}{\sqrt{n}}$. Replacing ξ by $\tilde{\xi}$ can only increase path lengths and by the previous sentence, P will w.h.p. still have the same length. This implies the lemma. $\qquad\square$

We proceed with the proof of Lemma 6.

Proof of Lemma 6. We split the argument into two cases depending on the value of λ^*.

Case 1. $\lambda^* \geq 1$. For $t \leq 1$, we have

$$\mathbb{P}(W \leq t) = \mathbb{P}(U_1 + \lambda^* U_2 \leq t) = \frac{t^2}{2\lambda^*} \geq 1 - e^{-\frac{t^2}{2\lambda^*}} = \mathbb{P}\left((2\lambda^*\xi)^{1/2} \leq t\right). \tag{6}$$

Observe that for $\tilde{\xi}$ from Lemma 8, we have

$$\mathbb{P}\left(\tilde{\xi} \leq t\right) = \begin{cases} \mathbb{P}\left(\xi \leq t\right), & t \leq \frac{\log^2 n}{n}, \\ \mathbb{P}\left(\xi \leq \frac{\log^2 n}{n}\right), & t > \frac{\log^2 n}{n}, \end{cases}$$

for every $t \geq 0$. Therefore, the following comparison holds

$$\mathbb{P}(W \leq t) \geq \mathbb{P}\left((2\lambda^*\tilde{\xi})^{1/2} \leq t\right), \tag{7}$$

for every $t \geq 0$ as long as $\frac{1}{2\lambda^*} \geq \frac{\log^2 n}{n}$, equivalently $c_0 \geq \frac{\log^2 n}{2^{1/2}n}$, which is assumed in (1). This means that W is stochastically dominated by $(2\lambda^*\tilde{\xi})^{1/2}$. As a result, w.h.p.

$$\psi(\lambda^*) \leq \sqrt{2\lambda^*}\tilde{L}_{1/2,n} = \sqrt{\frac{\log^2 n}{2c_0^2 n}}L_{1/2,n},$$

where the equality follows from Lemma 8. The claim made after Theorem 7 gives $L_{1/2,n} \approx \frac{\log n}{\sqrt{2n}}$, which finishes the argument.

Case 2. $\lambda^* \leq 1$. We repeat the whole argument of Case 1. The only change is that now (6) holds for all $t \leq \lambda^*$ instead of all $t \leq 1$, thus to establish (7) for all $t \geq 0$, we need $\frac{\lambda^*}{2} \geq \frac{\log^2 n}{n}$, equivalently, $c_0 \leq \frac{1}{2\sqrt{2}}$. $\qquad\square$

Remark 2. *An inspection of the proof shows that the implicit $o(1)$ term in (4) does not depend on c_0.*

Remark 3. *The value λ^* was chosen so as to minimize $\psi(\lambda) - \lambda c_0$ (the dual lower bound on CSP), where we put $\psi(\lambda) = \sqrt{\lambda}\frac{\log n}{\sqrt{n}}$ (heuristically $\psi(\lambda) \approx \sqrt{2\lambda}L_{1/2,n} \approx \sqrt{\lambda}\frac{\log n}{\sqrt{n}}$).*

5   Proof of Theorem 1

In view of Corollary 4, we need to upper bound L_n, or in other words, show that w.h.p. there is a path P with $w(P) \lesssim \frac{\log^2 n}{4nc_0}$ and $c(P) \leq c_0$.

Let $\lambda^* = \frac{\log^2 n}{4c_0^2 n}$. By the definition of $\psi(\lambda^*)$, we get a path P of length $w = w(P)$ and cost $c = c(P)$ that w.h.p. satisfies

$$wc \gtrsim \frac{\log^2 n}{4n} \qquad \text{from Lemma 3,} \tag{8}$$

$$w + \frac{\log^2 n}{4c_0^2 n} c \lesssim \frac{\log^2 n}{2c_0 n} \qquad \text{from Lemma 6.} \tag{9}$$

The implicit $o(1)$ terms here do not depend on c_0 (which is clear for (8) and is justified by Remark 2 for (9)). Combining (8) and (9) yields

$$(1 - o(1))\frac{\log^2 n}{4n} \le wc \le \left[\frac{\log^2 n}{2c_0 n}(1 + o(1)) - \frac{\log^2 n}{4c_0^2 n}c\right]c = \frac{\log^2 n}{4nc_0}\left(2(1 + o(1)) - \frac{c}{c_0}\right)c,$$

thus, in terms of $r = \frac{c}{c_0}$,

$$1 - o(1) \le 2(1 + o(1))r - r^2 \quad \text{or} \quad 1 - \varepsilon \le 2(1 + \varepsilon)r - r^2$$

for some $\varepsilon = \varepsilon(n) \to 0$. Re-arranging gives

$$(r - 1 - \varepsilon)^2 \le 3\varepsilon + \varepsilon^2 \text{ and so } c \le c_0(1 + 2\varepsilon^{1/2}).$$

Note now that (9) implies that

$$w \le \frac{\log^2 n}{4c_0 n}\left(2 + o(1) - \frac{c}{c_0}\right) \approx \frac{\log^2 n}{4c_0 n}. \tag{10}$$

Now let $\widehat{c}_0 = c_0(1 - 2\varepsilon^{1/2})$ and repeat the above analysis with $\widehat{c}_0$ replacing c_0. Then we see that w.h.p. there is a path of length at most $\frac{\log^2 n}{4\widehat{c}_0 n} \approx \frac{\log^2 n}{4c_0 n}$ and cost at most $\widehat{c}_0(1 + 2\varepsilon^{1/2}) \le c_0$. This completes the proof of Theorem 1.

6 More general distributions

The goal is to sketch a proof of Theorem 2. We first have to generalise Lemma 5. For this we need the following lemma.

Lemma 9. *Let $\gamma > 0$. Let $U_1, U_2, \ldots, U_k$ be independent copies of a uniform random variable on $[0, 1]$. Then, for $u \ge 0$, we have*

$$\mathbb{P}(U_1^\gamma + U_2^\gamma + \cdots + U_k^\gamma \le u) \le \frac{u^{k/\gamma}\Gamma\left(\frac{1}{\gamma} + 1\right)^k}{\Gamma\left(\frac{k}{\gamma} + 1\right)}.$$

Proof. We have

$$\mathbb{P}(U_1^\gamma + U_2^\gamma + \cdots + U_k^\gamma \le u) = \text{Vol}\left\{x \in [0, 1]^k : \sum_{i=1}^k x_i^\gamma \le u\right\}$$

$$\le \text{Vol}\left\{x \in [0, \infty)^k : \sum_{i=1}^k x_i^\gamma \le u\right\} = u^{k/\gamma}v_{k,\gamma}$$

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where $v_{k,\gamma} = \text{Vol}\left\{x \in [0,\infty)^k : \sum_{i=1}^k x_i^\gamma \le 1\right\}$. A standard computation leads to a closed expression,

$$\left(\int_0^\infty e^{-t^\gamma}dt\right)^k = \int_0^\infty \cdots \int_0^\infty e^{-x_1^\gamma - \cdots - x_k^\gamma}dx_1 \ldots dx_k = \int_0^\infty \cdots \int_0^\infty \int_{s > x_1^\gamma + \cdots + x_k^\gamma} e^{-s}dsdx_1 \ldots dx_k$$

$$= \int_{s>0} e^{-s}\text{Vol}\left\{x \in [0,\infty)^k : \sum_{i=1}^k x_i^\gamma < s\right\}ds = \int_{s>0} e^{-s}s^{k/\gamma}v_{k,\gamma}ds,$$

and thus

$$v_{k,\gamma} = \frac{\left(\int_0^\infty e^{-t^\gamma}dt\right)^k}{\int_{s>0} e^{-s}s^{k/\gamma}ds} = \frac{\Gamma\left(\frac{1}{\gamma}+1\right)^k}{\Gamma(\frac{k}{\gamma}+1)}.$$

$\square$

Given this we have

Lemma 10. *Let $\gamma > 0$. Let S and T be independent copies of $U_1^\gamma + \cdots + U_k^\gamma$, where $U_1, \ldots, U_k$ are i.i.d. copies of a uniform random variable on $[0,1]$. Then for $0 < t < k^2$, we have*

$$\mathbb{P}\left(ST \le t\right) \le \frac{t^{k/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^{2k}}{\Gamma\left(\frac{k}{\gamma}+1\right)^2}\left(\frac{k}{\gamma}\log\left(\frac{k^2}{t}\right) + 2\frac{\Gamma\left(\frac{k}{\gamma}+1\right)}{k^k\Gamma\left(\frac{1}{\gamma}+1\right)^k}\right).$$

Proof. We repeat the proof of Lemma 5. The bound in (2) becomes

$$\mathbb{E}_T\left(\frac{\left(\frac{t}{T}\right)^{k/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^k}{\Gamma\left(\frac{k}{\gamma}+1\right)}\mathbb{1}\left(T > \frac{t}{k}\right)\right) + \frac{\left(\frac{t}{k}\right)^{k/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^k}{\Gamma\left(\frac{k}{\gamma}+1\right)}$$

$$\le \frac{t^{k/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^k}{\Gamma\left(\frac{k}{\gamma}+1\right)}\left(\frac{k}{\gamma}\frac{\Gamma\left(\frac{1}{\gamma}+1\right)^k}{\Gamma\left(\frac{k}{\gamma}+1\right)}\log\left(\frac{k^2}{t}\right) + \frac{1}{k^{k/\gamma}}\right) + \frac{\left(\frac{t}{k}\right)^{k/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^k}{\Gamma\left(\frac{k}{\gamma}+1\right)}$$

$$= \frac{t^{k/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^{2k}}{\Gamma\left(\frac{k}{\gamma}+1\right)^2}\left(\frac{k}{\gamma}\log\left(\frac{k^2}{t}\right) + 2\frac{\Gamma\left(\frac{k}{\gamma}+1\right)}{k^{k\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^k}\right).$$

$\square$

Proof of Theorem 2 (Sketch). We define

$$\mathcal{P}_\beta = \left\{P \in \mathcal{P} : w(P)c(P) \le \frac{\beta\log^2 n}{n^\gamma}\right\}.$$

Using Lemma 10, (3) becomes,

$$\mathbb{E}(|P_\beta|)) \le \sum_{\ell \ge 1} n^{\ell-1}\frac{(\beta\log^2 n)^{\ell/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^{2\ell}}{n^\ell\Gamma\left(\frac{\ell}{\gamma}+1\right)^2}\left(\frac{\ell}{\gamma}\log\left(\frac{\ell^2}{t}\right) + 2\frac{\Gamma\left(\frac{\ell}{\gamma}+1\right)}{\ell^\ell\Gamma\left(\frac{1}{\gamma}+1\right)^\ell}\right).$$

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We deduce from this that w.h.p.

$$w(P)c(P) \gtrsim \frac{\log^2 n}{4\Gamma\left(\frac{1}{\gamma}+1\right)^{2\gamma} n^\gamma}. \tag{11}$$

To consider the dual problem we use that if U_1, U_2 are independent copies of U, then

$$\mathbb{P}(U_1^\gamma + \lambda U_2^\gamma \le t) = \frac{t^{2/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)^2}{\lambda^{1/\gamma}\Gamma\left(\frac{2}{\gamma}+1\right)}$$

valid for $0 < t < 1 \le \lambda$, see equation (36) of [6].

As in Section 4, thanks to (5) (with $s = \gamma/2$) and stochastic dominance (an analogue of Lemma 8), we obtain

$$\psi(\lambda) \lesssim \frac{\sqrt{\lambda^*}\log n}{\Gamma(1+\frac{1}{\gamma})^\gamma n^{\gamma/2}} = \frac{\log^2 n}{2c_0\Gamma(1+\frac{1}{\gamma})^{2\gamma}n^\gamma} \tag{12}$$

with $\lambda^* = \frac{1}{4c_0^2\Gamma(1+\frac{1}{\gamma})^{2\gamma}}\frac{\log^2 n}{n^\gamma}$ (chosen to minimise $\frac{\sqrt{\lambda}\log n}{\Gamma(1+\frac{1}{\gamma})^\gamma n^{\gamma/2}} - \lambda c_0$). Applying the analogous argument in Section 5 to (11), (12) we see finally that w.h.p.

$$L_n \approx \frac{\gamma \log^2 n}{4\Gamma\left(\frac{1}{\gamma}+1\right)^{2\gamma} c_0 n^\gamma} \quad \text{and} \quad H_n \approx \frac{\gamma \log n}{2}.$$

This finishes the proof. □

7 Final Remarks

We first observe that our proof shows that w.h.p. the duality gap between the maximum dual value and the optimal value of the solution to the constrained shortest path problem is within $o(1)$ of the optimal value to the latter problem.

The imposed range (1) on c_0 is a by-product of our proof. It is likely off by a factor $\log n$ on both sides: a lower bound on c_0 of $\approx \frac{\log n}{n}$ comes from the unconstrained minimum cost of a path, whereas if $c_0 \ge \log n$, then w.h.p. the unconstrained minimum length path will be within cost budget.

A coupling argument of Janson [7] can be used to derive a generalisation of Theorem 2 to the case where $w(e), c(e)$ have the distribution function $F_w(t) = \mathbb{P}(X \le t)$, of a random variable X, that satisfies $F(t) \approx at^{1/\gamma}, \gamma > 0$ as $t \to 0$. This argument is spelled out in detail in Section 4.1 of [6].

We would next like to mention the fact that the approach of Beier and Voeking [2] can be applied to solve the computational problem in polynomial expected time. This paper was the first (and only?) paper to give a polynomial expected time algorithm for solving random 0-1 knapsack problems. In Theorem 2 of this paper, they give a significant generalisation which opens the door for solving the constrained shortest path problem.

Theorem 2 of that paper is

Theorem 11. *Let $S_1, S_2, \ldots, S_m$ be a fixed but arbitrary sequence of subsets of $[N]$. Suppose that profits are chosen according to the uniform distribution over $[0,1]$. Let q denote the number of dominating sets over $S_1, S_2, \ldots, S_m$. Then $\mathbb{E}(q) = O(N^3)$.*

To unpack this, we first observe that q determines the running time of an algorithm of Nemhauser and Ullman [9] that can be used to solve the knapsack problem. Here q is the number of sets among $S_1, S_2, \ldots, S_m$ that are not dominated by any other set. Here S_i dominates S_j if it has smaller cost and larger profit. To apply the theorem we let $N = \binom{n}{2}$ and $[N]$ be associated with the edge set of K_n. Then we let the profit $p(e)$ of edge e be equal to $1 - w(e)$. We then apply the theorem separately for each $\ell = 1, 2, \ldots, n-1$ and let the S_i correspond to the set of edges in the ℓ-edge paths from 1 to 2. In this way we can solve the constrained shortest path problem in $O(n^7)$ expected time.

The paper [5] allowed multiple constraints and it is a challenge to tighten the result there to get an asymptotic result, as we did here.

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