

Random walks on edge colored random graphs

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Abstract

We consider random walks on edge coloured random graphs, where the colour of an edge reflects the cost of using it. In the simplest instance, the edges are coloured red or blue. Blue edges are free to use, whereas red edges incur a unit cost every time they are traversed.

1 Introduction

The cover time of a connected graph is the maximum over the start vertex of the expected time for a simple random walk to visit every vertex of the graph. There is a large literature on this subject see for example [1], [13], including [4]–[8] which give precise estimates of cover time for various models of random graphs.

We consider the following scenario: There is a connected graph G in which the edges are colored red and blue. We study the cover time of such a graph when there is a bound on the number red edges that can be used. We can think of blue edges as free to use, whereas red edges represent toll roads, the bound being the maximum budget. The decision to use, or not to use, a toll road can be made in various ways, which we characterize as congestion charging,

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flip walks, oblivious walks and smooth walks. In the simplest case, the **oblivious model**, the random walk uses edges of G regardless of colour until the budget is used up, after which it will be restricted to the blue sub-graph.

Models of red-blue regular graphs. Let G be a connected n -vertex graph, and let $\mathcal{W} = \mathcal{W}(R, B)$ be a simple random walk on G subject to the given constraints on the use of red and blue edges. We use $C_G = C_G(R, B)$ for the constrained cover time of G by the walk $\mathcal{W}(R, B)$, and use \widehat{C}_G for the corresponding unconstrained cover time of G .

Let

$$\sigma_{RB} = \frac{r+b-1}{r+b-2}, \quad \sigma_B = \frac{b-1}{b-2}. \quad (1)$$

One simple model is to take G as the union of a random n -vertex r -regular graph with red edges and a random n -vertex b -regular graph with blue edges. For such graphs, w.h.p. the unconstrained cover time of G (see [4]) is asymptotic to $\widehat{C}_G = \sigma_{RB} n \log n = (r+b-1)/(r+b-2)n \log n$, whereas for $b \geq 3$, the cover time of a walk restricted to the blue edges is asymptotic to $C_G(B) = \sigma_B n \log n = (b-1)/(b-2)n \log n$.

Another simple model is to take G as the union of a blue Hamilton cycle and a random r -regular graph with red edges. This model, based on short cutting a Hamilton cycle, is one of the original Watts-Strogatz type small world models [17]. For $r = 1$ (the union of a Hamilton cycle and a random 1-factor) the diameter was studied by Bollobás and Chung [3]. Subsequently, the fact that 3-regular graphs are Hamiltonian w.h.p. was established by Robinson and Wormald [16]. Thus almost all 3-regular graphs have a decomposition into a blue Hamilton cycle and a random red 1-factor.

Smooth walks. The walk alternates in phases between using edges of either colour (a red/blue phase) and edges of colour blue only (a blue phase). In a red/blue phase the walk is on G , and in a blue phase the walk is on G_B , the subgraph induced by the blue edges. The length of the phases can vary, but to analyse the model, we require that the length of each phase is at least $\omega T_G \log n$ where T_G is a mixing time given by (2), and $\omega \rightarrow \infty$ with n . We call such a walk *smooth*. We say that such a random walk is α -constrained if in total, it is only allowed to use a red edge at most $\gamma(\alpha)n \log n$ times, where $\gamma(\alpha)$ is given in Theorem 1(a) below.

The **congestion pricing model** fits naturally as a special case of smooth walks. Peak and off-peak periods alternate as in the smoothed model. The length C, F (charged, free) of the peak and off peak periods are fixed and in constant proportion. The walk uses the red (charged) edges only in off-peak periods when they are free to use. In peak periods the walk uses only blue (free) edges. We assume there are at most n^θ distinct phases of equal length $C + F$, for some small $\theta > 0$ constant.

Theorem 1. (SMOOTH WALKS) *Let G be the union of a random r -regular graph with red edges and a b -regular graph with blue edges where $r \geq 1$ and $b \geq 2$.*

- (a) *If $b \geq 3$ and $\alpha < 1$ constant, then w.h.p. the cover time of an α -constrained smooth random walk on G with budget $\gamma(\alpha) = (\alpha\sigma_{RB}r)(r + b)n \log n$ is asymptotically equal to $C_G(R, B) = (\alpha\sigma_{RB} + (1 - \alpha)\sigma_B)n \log n$.*
- (b) *If $b = 2$, $r \geq 1$, and the blue edges span a Hamilton cycle then in the congestion pricing model with periods C, F , the cover time of a random walk on G is asymptotically equal to $C_G = (1 + C/F)((r + 1)/r)n \log n$.*

Flip walks. In a flip walk there is no hard bound on the number of times we use a red edge, only a stochastic one. At any step, let ρ_R be the probability of transition down a given red edge adjacent to the current vertex and ρ_B the probability of transition down a given blue edge. Thus the probability the next transition is red is $r\rho_R$ etc. Examples include the following. Let $\rho_R = \alpha/r$, and $\rho_B = (1 - \alpha)/b$ in which case the probability of transition down a red edge at any step is α , and a blue edge is $1 - \alpha$.

Theorem 2. (FLIP WALKS)

- (a) *Simplest case. If $r = 1$, $b = 2$, and $q = 2\rho_B = 1 - \rho_R$, where $0 < \rho_R \leq \rho_B$, the cover time of a flip walk on G is asymptotically equal to $C_G \sim \theta n \log n$ where*

$$\theta = \frac{2}{q(5 - q + \sqrt{9 - 10q + q^2})} + \frac{2}{q(1 - q + \sqrt{9 - 10q + q^2})}.$$

This cover time is minimized at $2n \log n$ when $\rho_R = \rho_B = 1/3$, and all edges have the same transition probability.

- (b) *If $r = 1$, $b \geq 2$ and $0 < \rho_R \leq \rho_B$ then w.h.p. the cover time of a flip walk on G is asymptotically equal to $C_G \sim (1/(1 - f))n \log n$. Here f is the smallest solution in $[0, 1]$ to $F(z) = 0$ where $F(z)$ is given by (24).*

Theorem 2(a) confirms that for the simplest case, the cover time cannot be improved by biasing the walk probabilities. The general solution $r \geq 2, b \geq 2$ seems difficult to obtain.

Oblivious walks. The simplest case. The random walk uses edges of G regardless of colour until the budget is used up, after which it is restricted to the blue subgraph. In the case where $b = 2$, $r \geq 1$ and the budget is less than the unconstrained cover time $C_G \sim (r + 1)/r n \log n$, things go seriously wrong for the walk; see Theorem 3(b)–(c) below.

Theorem 3. (OBLIVIOUS WALKS) *Let $\varepsilon > 0$ constant.*

- (a) Suppose $b \geq 3$. and the budget is $(1 - \varepsilon)\sigma_{RB}n \log n$. The cover time of an oblivious walk on G is asymptotically equal to $(\alpha\sigma_{RB} + (1 - \alpha)\sigma_B)n \log n$ w.h.p., as given in (1) where $\alpha = \frac{1-\varepsilon}{r(r+b)}$.
- (b) Suppose that $b = 2$, $r \geq 1$, the blue edges span a Hamilton cycle, and the budget be at most $(1 - \varepsilon)(r + 1)/rn \log n$. The cover time of an oblivious walk on G is asymptotic to $C_G(R, B) \sim n^2/2$, w.h.p.
- (c) If $b = 2$ and the blue edges span a random 2-factor then w.h.p. an oblivious walk with budget of at most $(1 - \varepsilon)(r + 1)/rn \log n$ steps will fail to cover G .

Related models. The underlying graph of the random walk we consider varies structurally over time (temporally) in a predictable fashion; but within the constraint of a fixed budget. Many authors have considered aspects of this, or related problems. We mention a few below.

For a survey on temporal networks, see Holme [10]. Random walks whose transition probability alters depending on structural condition have been studied extensively. An early model by Kemperman [12], the oscillating random walk, remains one of the simplest and most attractive. Citing from [12]: Let $\{Y_n, n = 0, 1, \dots\}$ denote a stationary Markov chain taking values in \mathbb{R}^d . As long as the process stays on the same side of a fixed hyperplane E , it behaves as a ordinary random walk with jump measure μ or ν , respectively.

As a general instance of an application of random walks with edge costs, Coppersmith, Doyle, Raghavan and Snir, [9], use random walks on graphs with positive real costs on all edges to study the design and analysis of randomized on-line algorithms. Two recent papers by Majumdar et al., [14], [15], which study variable charges for walk transitions, have some bearing on the model we consider. In [14], [15], a one-dimensional walker's position X_t at discrete time t is a positive random variable evolving according to $X_t = X_{t-1} + \eta_t$. Transitions are priced according to their length η , there being one price for short distances and another for longer ones. This reflects how taxi fares are calculated in congested cities¹. The authors compute the average and variance of the distance covered in n steps when the total budget C is fixed.

Proof outline. Our main tool will be the *First Time Visit Lemma*. This technique, introduced in [4], allows us to make very precise estimates of cover time in certain classes of graphs. The technique was subsequently refined in [7], and in [8] where various technical conditions were removed. In outline, the approach is to bound the cover time above and below using

¹[15] The municipality prescribes a threshold speed η_c derived from statistical analyses of local traffic patterns. If the taxi surpasses η_c , the meter tallies the fare based on the distance covered, while a slower pace results in time-based fare computation. This approach ensures that drivers are compensated even when they face prolonged periods of slow progress.

(4) of Lemma 4. Much of the work is in constructing the lower bound on the cover time. On regular graphs the walks we consider have uniform stationary distribution, and are rapidly mixing. The unvisit probability (4) has a parameter $p_v \sim \pi_v/R_v$ (see (3)) where R_v is the expected number of returns to vertex v during the mixing time.

2 Background material

Notation. We use $A_n \sim B_n$ to denote $A_n = (1 + o(1))B_n$ and thus $\lim_{n \rightarrow \infty} A_n/B_n = 1$. We use ω to denote a quantity which tends to infinity with n more slowly than any other variables in the given expression. The expression $f(n) \ll g(n)$ indicates $f(n) = o(g(n))$. The inequality $A \lesssim B$ which stands for $A \leq (1 + o(1))B$ is used to unclutter notation in some places.

2.1 First Time Visit Lemma

Let G denote a fixed connected graph, and let u be some arbitrary vertex from which a walk \mathcal{W}_u is started. Let $\mathcal{W}_u(t)$ be the vertex reached at step t , let P be the matrix of transition probabilities of the walk, and let $P_u^{(t)}(v) = \mathbb{P}(\mathcal{W}_u(t) = v)$. Let π be the steady state distribution of the random walk \mathcal{W}_u . For an unbiased ergodic random walk on a graph G with $m = m(G)$ edges, $\pi_v = \frac{d(v)}{2m}$, where $d(v)$ denotes the degree of v in G .

We denote the steady state probability of a simple random walk on $G = (V, E)$ being at $v \in V$ is $\pi_G(v)$. We let $\mathbb{P}_{u,G}^{(t)}(v)$ denote the probability that a simple random walk on G started at u is at v after t steps. We then define a mixing time T_G by

$$T_G = \min \left\{ \tau : \max_{u,v} \left\{ \frac{|\mathbb{P}_{u,G}^{(\tau)}(v) - \pi(v)|}{\pi(v)} \right\} = o(1) \text{ for } t \geq \tau \right\}. \quad (2)$$

For $t \geq 0$, let $\mathcal{A}_t(v)$ be the event that \mathcal{W}_u does not visit v in steps $T_G, T_G + 1, \dots, t$. The vertex u will have to be implicit in this definition. We will use the following version:

Lemma 4. *Let $R_v \geq 1$ is the expected number of visits by \mathcal{W}_v to v in the time interval $[0, T_G]$. Suppose that $T_G \pi_v = o(1)$ and $R_v = O(1)$. Let*

$$p_v = \frac{\pi_v}{R_v(1 + O(T_G \pi_v))}. \quad (3)$$

Then for all $t \geq T_G$,

$$\mathbb{P}(\mathcal{A}_t(v)) = \frac{(1 + O(T_G \pi_v))}{(1 + p_v)^t} + O(T_G^2 \pi_v e^{-\mu t/2}), \quad (4)$$

where $\mu = \Omega(1/T_G)$.

2.2 Properties of random regular graphs

The use of (4) will require us to verify some facts about random regular graphs. These will be taken from [4]. Let

$$\sigma = \lfloor \log^{1/2} n \rfloor. \quad (5)$$

Say a cycle C is *small* if $|C| \leq \sigma$. A vertex of G is locally tree-like if v is at distance at least σ from the nearest small cycle.

Theorem 5. *Let $s \geq 3$ be a constant and let G be chosen uniformly from the set \mathcal{G}_s of s -regular graphs with vertex set $[n]$. Then w.h.p.*

P1. G is connected.

P2. $T_G = O(\log n)$.

P3. If v is locally tree-like then $R_v = \frac{s-1}{s-2} + o(\sigma^{-1})$.

P4. There are at most $s^{3\sigma}$ non locally-tree-like vertices and if v is not locally-tree-like then $R_v \leq 3$.

P5 The second eigenvalue λ of G is positive and $\lambda \leq 1 - \theta$ for some positive constant θ .

3 Proof of Theorem 1

3.1 Proof of Theorem 1(a)

In this section we prove that if $b \geq 3$ then w.h.p. the cover time of an α -constrained smooth random walk on G with budget $\gamma(\alpha) = (\alpha\sigma_{RB}r)/(r+b)n \log n$ is asymptotically equal to $C_G = (\alpha\sigma_{RB} + (1-\alpha)\sigma_B)n \log n$. It was shown in [4] that the unconstrained cover time of G is $\widehat{C}_G \sim \sigma_{RB}n \log n$ w.h.p. Without any constraint on the choice of the next edge, a walk of length \widehat{C}_G would cross a red edge $\sim \frac{r}{r+b}\sigma_{RB}n \log n$ times. It follows that w.h.p. we will not be able to complete the walk within the budget without having to avoid red edges a significant number of times.

If the phases are $o(T_G)$ in length then there are some technical difficulties that we have not yet managed to overcome, see Remark 1 below.

Remark 1. *There does not seem to be a compelling reason for us to only consider smooth walks. If we allow the walk to switch between allowing red edges and not allowing them,*

then we have no problems with mixing. This is because the steady state probabilities are $1/n$ regardless. The problem arises because R_v will depend heavily on how we switch between the two possibilities.

3.1.1 Upper bound

Suppose now that we start a walk at vertex u and that we alternate between red/blue phases and blue phases. Take $t_0 = 0$, and suppose we change phase at times $t_1, t_2, \dots, t_i, \dots$. Assume that whenever i is even we start a red/blue phase at t_i and switch to a blue phase when i is odd. At each step in a red/blue phase there is an $r/(r+b)$ chance of using a red edge. We also know that this walk will finish at least as early as one that only uses blue edges. This follows from Lemma 4 and the fact that $s/(s-1)$ decreases as s increases. We will assume $t_i - t_{i-1} \geq \omega \log^2 n \gg T_G$ for $1 \leq i < m$, so that

$$t_{RB} = \sum_{i \geq 0} (t_{2i+1} - t_{2i}) \leq \left(1 - \Theta\left(\frac{\log n}{n^{1/2}}\right)\right) \alpha \sigma_{RB} n \log n. \quad (6)$$

Multiplying the RHS of (6) by $r/(r+b)$ and applying the Chernoff bounds yields an upper bound of $\gamma(\alpha)n \log n$ on the number of times a red edge is used.

Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk \mathcal{W}_u . Let U_t be the number of vertices of G which have not been visited by \mathcal{W}_u at step t . We note the following:

$$C_u = \mathbb{E}T_G(u) = \sum_{t > 0} \mathbb{P}(T_G(u) \geq t), \quad (7)$$

$$\mathbb{P}(T_G(u) > t) = \mathbb{P}(U_t > 0) \leq \min\{1, \mathbb{E}U_t\}. \quad (8)$$

It follows from (7), (8) that for all t

$$C_u \leq t + \sum_{s \geq t} \mathbb{E}U_s = t + \sum_{v \in V} \sum_{s \geq t} \mathbb{P}(\mathbf{A}_s(v)). \quad (9)$$

In both phases the random walk is on a regular graph and so $\pi_v = 1/n$ throughout.

We are stopping the RB walk at $t^* \sim \sigma_{RB} n \log n$ when the budget runs out. At that time the walk has made $t_{RB} \sim \alpha \sigma_{RB} n \log n$ RB steps, and $t_B \sim (1-\alpha)\sigma_B n \log n$ B steps, consisting of m phases each of B and RB up to t^* . This is followed by a final 'blue forever' phase of length $t'_B = t - t^*$.

So,

$$\mathbb{P}(\mathbf{A}_t(v)) = \left(1 + O\left(\frac{\log n}{n}\right)\right)^m \exp\left\{-\frac{\pi_v}{\sigma_{RB}} \sum_{i \geq 0} (t_{2i+1} - t_{2i}) - \frac{\pi_v}{\sigma_B} \sum_{i \geq 1} (t_{2i} - t_{2i-1})\right\}. \quad (10)$$

Now assuming that $t = O(n \log n)$ and because each phase has length at least $\omega \log^2 n$ we have $m = o(n/\log n)$, and so we have

$$\begin{aligned} \mathbb{P}(\mathbf{A}_t(v)) &\lesssim \exp \left\{ - \left(\alpha + \frac{1}{\sigma_B} \left(\frac{t}{n \log n} - \alpha \sigma_{RB} \right) \right) \log n \right\} \\ &= \exp \left\{ - \frac{t}{\sigma_B n} + \alpha \sigma_{RB} \left(\frac{1}{\sigma_B} - \frac{1}{\sigma_{RB}} \right) \log n \right\}. \end{aligned} \quad (11)$$

For non locally tree-like vertices, using P4, we can write

$$\mathbb{P}(\mathbf{A}_t(v)) \lesssim e^{-t/3n}. \quad (12)$$

Define a time $t_0 = (1 + \varepsilon)\tau_C$, by

$$t_0 = (1 + \varepsilon) (\alpha \sigma_{RB} + (1 - \alpha) \sigma_B n \log n).$$

We note this value of t_0 , (and t_1 below), is unrelated to our previous use of notation t_i for a smoothed period, but is consistent with [4].

Thus if $\varepsilon = \log^{-1/2} n$ going back to (9) and using P4, (11), we see that for some absolute constants $\gamma_1, \gamma_2 > 0$,

$$\begin{aligned} C_u &\leq t_0 + \sum_{v \in [n]} \sum_{\tau \geq 0} \mathbb{P}(\mathbf{A}_{t_0+\tau}) \\ &\leq t_0 + n e^{-\gamma_1 \log^{1/2} n} \sum_{\tau \geq 0} e^{-\gamma_2 \tau \log n} + (r + b)^{3 \log \log n} \sum_{\tau \geq t_0} e^{-\tau/3n} \\ &= t_0 + o(n). \end{aligned} \quad (13)$$

This confirms the upper bound in Theorem 1(a).

3.1.2 Lower bound

For the lower bound we let $t_1 = \frac{(1-\varepsilon)\alpha(r+b)}{(b-2)(r+b-1)} n \log n$ and argue that $C_u \geq t_1$ w.h.p. Proceeding as in [4], for any vertex u , we can find a set of vertices S such that at time t_1 , the probability the set S is covered by the walk \mathcal{W}_u tends to zero. Hence $T_G(u) > t_1$ **whp** which together with (13) completes the proof of Theorem 1(a).

We construct S as follows. Let σ be given by (5). Let $S \subseteq [n]$ be some maximal set of locally tree-like vertices all of which are at least distance $2\sigma + 1$ apart. Thus $|S| \geq (n - r^{3\sigma})/r^{(2\sigma+1)}$.

Let S_1 denote the subset of S which has not been visited by \mathcal{W}_u after step t_1 .

$$\begin{aligned} \mathbb{E}|S_1| &\sim \sum_{v \in S} \exp \left\{ -\frac{t_1}{\sigma_B n} + \left(1 + O \left(\frac{\log n}{n^{1/2}} \right) \right) \frac{\alpha(r+b) \log n}{r} \left(\frac{1}{\sigma_{RB}} - \frac{1}{\sigma_B} \right) \right\} \\ &\geq (n - r^{3\sigma}) r^{-(2\sigma+1)} \exp \left\{ -\frac{\varepsilon \alpha(r+b) \log n}{(b-2)(r+b-1)} \right\} \rightarrow \infty. \end{aligned} \quad (14)$$

Let $Y_{v,t}$ be the indicator for the event that \mathcal{W}_u has not visited vertex v at time t . Let $Z = \{v, w\} \subset S$. We will show below that

$$\mathbb{E}(Y_{v,t_1} Y_{w,t_1}) \sim \exp \left\{ -2 \left(\frac{t_1}{\sigma_B n} + \left(1 + O \left(\frac{\log n}{n^{1/2}} \right) \right) \frac{\alpha(r+b) \log n}{r} \left(\frac{1}{\sigma_{RB}} - \frac{1}{\sigma_B} \right) \right) \right\}. \quad (15)$$

Thus

$$\mathbb{E}(Y_{v,t_1} Y_{w,t_1}) = (1 + o(1)) \mathbb{E}(Y_{v,t_1}) \mathbb{E}(Y_{w,t_1}). \quad (16)$$

It follows from (14) and (16), that

$$\mathbb{P}(S(t_1) \neq \emptyset) \geq \frac{(\mathbb{E}|S(t_1)|)^2}{\mathbb{E}|S(t_1)|^2} = \frac{1}{\frac{\mathbb{E}|S_{t_1}|(|S_{t_1}|-1)}{(\mathbb{E}|S(t_1)|)^2} + (\mathbb{E}|S_{t_1}|)^{-1}} = 1 - o(1).$$

Going back to (10), we see that with these values of π_Z, R_Z , this completes the proof of Theorem 1(a), apart from (15) which we give next.

Proof of (15). Let Γ be obtained from G by merging v, w into a single node Z . This node has degree $2r$ and every other node has degree r .

There is a natural measure preserving mapping from the set of walks in G which start at u and do not visit v or w , to the corresponding set of walks in Γ which do not visit Z . Thus the probability that \mathcal{W}_u does not visit v or w in the first t steps is equal to the probability that a random walk $\widehat{\mathcal{W}}_u$ in Γ which also starts at u does not visit Z in the first t steps.

That $\pi_Z = \frac{2}{n}$ is clear. We also have

$$R_v \leq R_Z \leq R_v + \sum_{t=2\sigma+1}^T (\pi_w + \lambda^t) = R_v + O(T\lambda^{2\sigma}).$$

The sum here being a bound on the probability that \mathcal{W}_v is at w before the end of the mixing time. The value λ is the second eigenvalue in **P5**. \square

3.2 Congestion model: Theorem 1(b)

Let $t_1 = (1 + \varepsilon)(1 + C/F)\sigma_{RB}n \log n$ where $\varepsilon > 0$ is arbitrary. The amount of time spent in a free phase is $(1 + \varepsilon)\sigma_{RB}n \log n$ and so w.h.p. G will be covered.

For the lower bound, let $t_2 = (1 - \varepsilon)(1 + C/F)\sigma_{RB}n \log n$. We defer the proof that G will not be covered until Section 5.4.

4 Proof of Theorem 2

Stationary distribution and mixing time. Let $d = r + b$ be the total degree of a vertex in the red/blue coloured graph G . Assume $d \geq 3$ and G satisfies the conditions of Section 2.2.

In the flip model, ρ_R (resp. ρ_B) is the probability of transition over a given red edge (resp. blue edge). This corresponds to a weighted graph in which $w(e_R) = \rho_R$, $w(e_B) = \rho_B$ for red and blue edges respectively, all vertices v have weight $w(v) = 1$ and $w(G) = n$. Thus $\pi_{RB}(v) = 1/n$, and G has uniform stationary distribution. For a given set $S \subseteq V$, the conductance (bottleneck ratio) $\Phi_{RB}(S)$ is

$$\Phi_{RB}(S) = \frac{Q(S, S^c)}{\pi(S)} = \frac{\sum_{v \in S, w \in S^c} \pi_v P_{RB}(v, w)}{\sum_{v \in S} \pi_v} = \frac{\sum_{v \in S, w \in S^c} P_{RB}(v, w)}{|S|}.$$

Comparing with a simple random walk on G ,

$$\frac{1}{|S|} \sum_{v \in S, w \in S^c} P_{RB}(v, w) \geq \min(\rho_R, \rho_B) \frac{E(S : S^c)}{d|S|} = \min(\rho_R, \rho_B) \Phi_{SRW}(S),$$

where $E(S : S^c)$ is the number of edges between S and S^c and Φ_{SRW} refers to the conductance of a simple random walk.

As G satisfies **P5**, the conductance Φ_{SRW}^* of a simple random walk on G is constant for some $0 < c < 1$. Assuming $\min(\rho_R, \rho_B)$ is also a positive constant, the conductance Φ_{PB}^* of the flip walk is constant, i.e.,

$$\Phi_{PB}^* = \min_{\pi(S) \leq 1/2} \Phi_{RB} \Phi(S) \geq \min(\rho_R, \rho_B) \Phi_{SRW}^*.$$

It follows that the flip walk on G is rapidly mixing and satisfies (2) as required.

Return probability. The proof of Theorem 2 will follow from the proof in [4], once we establish R_v for locally-tree-like vertices. To do this we argue as follows. The initial formulation is for the general case $r \geq 1, b \geq 2$. We then prove existence of solutions for $r = 1, b \geq 2$, and state the exact value of R_v for the special case $r = 1, b = 2$. The journey to the answer is longer than we might expect.

Consider an infinite d -regular tree T_d , where $r + b = d$ and in which each vertex has r red and b blue edges. For discussion we regard T_d as rooted at a designated vertex v , with all edges

nominally oriented outward from v . With this orientation, a transition away from v (resp. u) at a vertex u , is a transition uw down an out-edge of u . Relative to this orientation, a vertex other than v is red if its in-edge is red, and blue if its in-edge is blue. With the exception of v this induces an alternating pattern of red and blue vertices in T_d , in which e.g., a red vertex has $(r - 1)$ red out-edges, and b blue out-edges.

Recall that, in the flip model, ρ_R (resp. ρ_B) is the probability of transition over a given red edge (resp. blue edge). Let f be the probability of a first return to v in T_d , then the expected number of returns to v is $R_v(T_d) = 1/(1 - f)$. Let $\phi_{v,R}$ (resp. $\phi_{v,B}$) be the probability of a return to v given the particle exits v on a red (resp. blue) edge. Thus

$$f = r\rho_R\phi_{v,R} + b\rho_B\phi_{v,B}.$$

Let u be a red vertex, and uw a red edge pointing away from u and thus away from v . Define $\phi_{RR} = \phi_{RR}^{(u)}(uw)$ by

$$\phi_{RR} = \mathbb{P}(\text{the walk returns to } u \mid u \text{ is red and the walk exits away from } u \text{ on a red edge}),$$

and define $\phi_{RB}, \phi_{BB}, \phi_{BR}$ similarly. Note that $\phi_{RR} = \phi_{BR} = \phi_{v,R} = \phi_R$, say. For a red vertex u , let $\psi_R(u)$ be the probability the walk moves away from v at u , and subsequently returns to u . Thus with $P(u, w)$ the colour dependent transition probability over the edge uw , $\phi(uw)$ the colour dependent return probability, and $N_R^+(u)$ the red out-neighbours of u etc.,

$$\psi_R(u) = \sum_{w \in N^+(u)} P(u, w)\phi(uw) = \sum_{w \in N_R^+(u)} \rho_R\phi_{RR} + \sum_{w \in N_B^+(u)} \rho_B\phi_{RB}.$$

If u is red, there are $r - 1$ red out-neighbours, and b blue out-neighbours, so

$$\psi_R = (r - 1)\rho_R\phi_R + b\rho_B\phi_B.$$

Suppose the transition is away from u over a red edge uw to a red out-neighbour w , then the walk moves back to u with probability ρ_R and so

$$\phi_{RR}^{(u)}(uw) = \rho_R + \psi_R\rho_R + \cdots + (\psi_R)^j\rho_R + \cdots = \frac{\rho_R}{1 - \psi_R}. \quad (17)$$

Here the term $(\psi_R)^j\rho_R$ is the probability that the walk returns to u after making j returns to w that do not include a visit to u .

For a walk arriving at u , three things can happen. It moves directly back towards v (probability $\rho_R > 0$), moves away from u and returns to u later (probability ψ_R), or moves away from u and never returns (probability ξ_R). Thus

$$\rho_R + \psi_R + \xi_R = 1, \quad (18)$$

and $\psi_R \leq 1 - \rho_R < 1$, and similarly for $\psi_B < 1$. This justifies division by $1 - \psi_R$ in (17).

We obtain

$$\psi_R = \frac{(r-1)\rho_R^2}{1-\psi_R} + \frac{b\rho_B^2}{1-\psi_B}, \quad (19)$$

$$\psi_B = \frac{r\rho_R^2}{1-\psi_R} + \frac{(b-1)\rho_B^2}{1-\psi_B}, \quad (20)$$

$$f = \frac{r\rho_R^2}{1-\psi_R} + \frac{b\rho_B^2}{1-\psi_B}. \quad (21)$$

Do (19) and (20) always have solutions ψ_R, ψ_B in $[0, 1]$ and if more than one solution, which to choose? This is discussed next. Assume for now that solutions exists. Then for a given solution ψ_B , using (20) and (21) we see that

$$f = \psi_B + \frac{\rho_B^2}{1-\psi_B}. \quad (22)$$

As $1/R_v = 1 - f + o(1)$ in G , we obtain the required solution to the cover time of G as

$$C_G \sim \frac{1}{1-f} n \log n. \quad (23)$$

General solution in the case $r = 1$. Put $\psi_R = x, \psi_B = z$. As $r = 1$ we have $\rho_R + b\rho_B = 1$. Assume $b \geq 2$ and $\rho_R \leq \rho_B$ so that $0 \leq \rho_R \leq 1/(b+1)$. Use (19) to get

$$\frac{1}{1-x} = \frac{1-z}{1-z-b\rho_B^2},$$

and substitute this for $1/(1-x)$ in (20) giving

$$z = \frac{\rho_R^2(1-z)}{1-z-b\rho_B^2} + \frac{(b-1)\rho_B^2}{1-z}.$$

Put $q = b\rho_B$ and $\rho_R = 1 - q$ where $b/(b+1) \leq q \leq 1$, to consider $F(z) = 0$ where

$$F(z) = z - \frac{(1-q)^2(1-z)}{1-q^2/b-z} - \frac{(b-1)q^2}{b^2(1-z)}. \quad (24)$$

If $\rho_R = 1/(b+1)$ then all edges are equivalent in a $b+1$ regular tree and the roots z^* are $1/(b+1), b/(b+1)$. If $q = 1$ (no useable red edges) then the problem reverts to a b -regular tree, in which case the roots z^* of $F(z)$ are $1/b, (b-1)/b$.

If $b = 2, q = 1, z = 1/2$ is the only root to $F(z) = 0$. Ignoring this case which corresponds to a random walk on the line, we will assume that $q < 1$ and $b \geq 2$. Let $z_0 = 1 - q^2/b$. In the interval $I_0 = [0, z_0]$, $F(0) < 0, F(z_0^-) = -\infty$. It can be checked that $F(z)$ is concave on I_0 . In the interval $I_1 = [z_0, 1]$, $F(z_0^+) = \infty, F(1) = -\infty$ and $F(z)$ is monotone decreasing. So there

are between one and three roots in $[0, 1]$ depending on what happens in I_0 . We will establish that if $b \geq 2$, $q < 1$, then $F(1/2) > 0$ which implies there are two roots in I_0 . We have

$$F(1/2) = \frac{1}{2} - \frac{b(1-q)^2}{b-2q^2} - \frac{2(b-1)q^2}{b^2}.$$

It can be checked that $\partial F/\partial b > 0$ for any fixed q , so as $F(1/2)$ is monotone increasing with b , and $b = 2$ has the largest range of q , (i.e., $2/3 \leq q < 1$), we check this case. After some simplification we obtain that if $b = 2$,

$$F(1/2) = (1-q) \left(\frac{1+q}{2} - \frac{1}{1+q} \right).$$

So $F(1/2, b=2) > 0$ boils down to $(1+q)^2 > 2$. As $q \geq 2/3 = b/(b+1)$, and $(5/3)^2 > 2$, it is indeed the case that $F(1/2) > 0$ as required.

The simplest case: Theorem 2(a). We solve $F(z) = 0$, where $F(z)$ is given by (24), in the simplest case; namely $r = 1$, and $b = 2$. Thus $\rho_R + 2\rho_B = 1$. Put $q = 1 - \rho_R = 2\rho_B$, and $w = 1 - z$ to give, after some rearrangement, $F(w) = N(w)/D(w)$. Here the numerator $N(w)$ is

$$N(w) = w^3 - \frac{q}{2}(4-q)w^2 + \frac{3}{4}q^2w - \frac{1}{8}q^4.$$

It can be checked that $w = q/2$ is a factor of $N(w)$ and so

$$N(w) = (w - q/2)(w^2 + w(q/2)(q-3) + q^3/4).$$

The roots of $F(w) = 0$ are $q/2$, $(q/4)[(3-q) \pm \sqrt{9-10q+q^2}]$. We prove that the smallest root $z \in [0, 1]$ is indeed the correct one. We use $\rho_B + \psi_B + \xi_B = 1$ from (18). As $\rho_B = q/2$ and $\psi_B = z = 1 - w$ we obtain $\xi_B = w - q/2$.

The root $w^- = (q/4)[(3-q) - \sqrt{9-10q+q^2}]$ implies $\xi_B < 0$ for $q < 1$. As ξ_B is a probability, this is impossible. The root $w = q/2$ (or $w^-(q=1)$) imply that $\xi_B = 0$. This corresponds to a recurrent system, an answer we do not accept (see below). Thus the only feasible root is $w^+ = (q/4)[(3-q) + \sqrt{9-10q+q^2}]$ giving $\psi_B = z = 1 - w^+$, this being the smallest root of $F(z) = 0$.

We briefly sketch why the system is not recurrent. For a random walk on the half-line $\{0, 1, 2, \dots\}$ starting from vertex 1, let q, p be the probability of moving left and right respectively at any vertex $i > 0$. Then assuming $q \leq p$, the probability of absorption at the origin 0, is $\pi_0 = q/p$. Thus for our system

$$\pi_0 \leq \max \left\{ \frac{\rho_R}{1 - \rho_R}, \frac{\rho_B}{1 - \rho_B} \right\} = \max \left\{ \frac{\rho_R}{2\rho_B}, \frac{\rho_B}{\rho_B + \rho_R} \right\}.$$

As we assume $0 < \rho_R \leq \rho_B$ it follows that $\pi_0 < 1$, $\xi > 0$ and the system is not recurrent.

Refer to (22) and (23). With $\psi_B = 1 - w$ and $\rho_B = q/2$. the value of $\theta = 1/(1 - f)$ is

$$\theta = \frac{1}{2} \left(\frac{1}{w + q/2} + \frac{1}{w - q/2} \right),$$

The largest value of w among the roots, and hence the smallest value of $z = \psi_B$, gives

$$\theta = \frac{2}{q(5 - q + \sqrt{9 - 10q + q^2})} + \frac{2}{q(1 - q + \sqrt{9 - 10q + q^2})}. \quad (25)$$

The minimum of $\theta(q)$ is at $q = 2/3$, which follows by checking the derivative. This corresponds to $\rho_R = \rho_B = 1/3$. The value $q = 2/3$ gives $\theta = 2$, and hence value of $C_G \sim 2n \log n$ in (23). This confirms the w.h.p. cover time of a random 3-regular graph from [4] as the minimum cover time for this model. The value of θ is plotted in Figure 1.

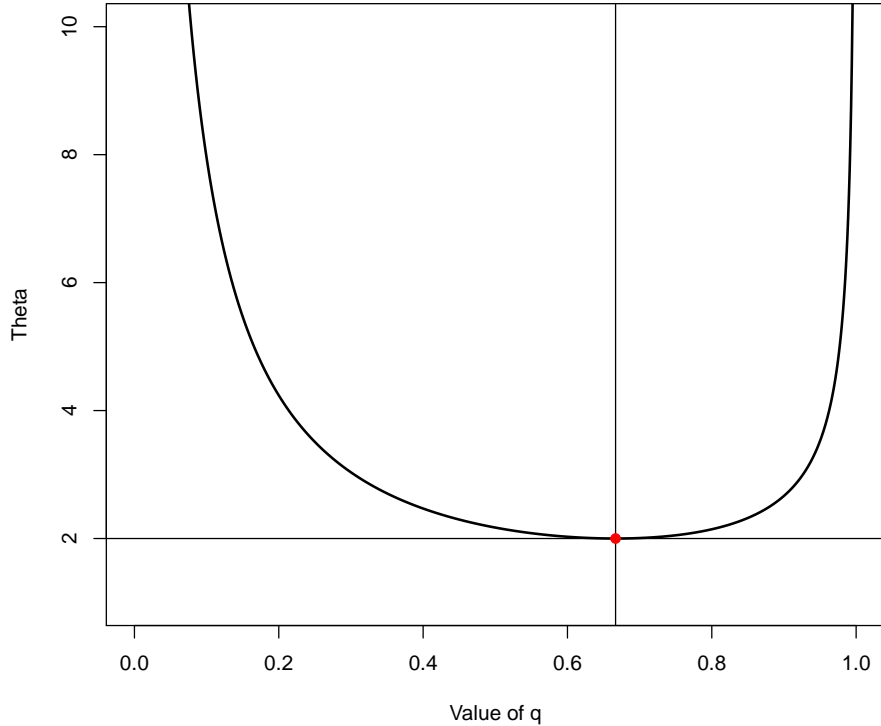


Figure 1: Plot of $\theta = \theta(q)$ from (25) as q varies from 0 to 1. The minimum value of $\theta = 2$ is at $q = 2/3$, giving a cover time of $2n \log n$ for random 3-regular graphs. Values of q for which $\rho_R < \rho_B$ are to the right of the vertical line at $q = 2/3$. The plots diverge at $q = 0, 1$.

5 The oblivious model: Proof of Theorem 3

5.1 Proof of Theorem 3(a).

This follows directly from Theorem 1(a).

5.2 Proof of Theorem 3(b).

We prove that if $b = 2$, and the blue edges span a Hamilton cycle H , then the cover time of an oblivious walk on G with budget $t_1 = (1 - \varepsilon)(r + 1)/rn \log n$ is asymptotic to $n^2/2$.

Divide H into $\ell = n^\nu$ consecutive intervals I_1, I_2, \dots, I_ℓ of length $L = n^{1-\nu}$, where $0 < \nu \leq \varepsilon/3$. Fix one such interval I . Let S_0 be the locally tree-like vertices in I and then choose S by greedily choosing a subset of S_0 where each pair of vertices are at distance $2\sigma + 1$ apart. Note that $|S| \geq n^{1-\nu-o(1)}$.

Let $S_1 = S_1(t_1)$ denote the subset of S which has not been visited by the walk \mathcal{W} at step $t_1 = (1 - \varepsilon)(d - 1)/(d - 2)n \log n$, where $d = r + b$ as usual, and $b = 2$. Let X_v be the indicator that vertex $v \in S$ is unvisited at step t_1 . Thus applying Lemma 4 as in Section 3.1, we obtain

$$\begin{aligned} \mathbb{E}|S_1| &= \sum_{v \in S} X_v = \sum_{v \in S} \exp \left\{ -(1 + \eta) \frac{R_v}{n} (1 - \varepsilon) \frac{d - 1}{d - 2} n \log n \right\} \\ &= \frac{n^{1-\nu}}{d^{3\sigma}} n^{-1+\varepsilon+o(1)} = n^\delta, \end{aligned}$$

where $\delta = \varepsilon - \nu + o(1)$. Here $\eta = o(1)$ combines the errors arising from estimating $R_v = (d - 2)/(d - 1)(1 + o(1))$, the convergence via (2) of \mathcal{W} to $\pi_v = 1/n$, and the approximation of $p_v = \pi_v/R_v(1 + O(T\pi_v))$ in Lemma 4.

We next prove that $|S_1(t_1)| > 0$ by adapting the approach used in Section 3.1.2. When calculating the variance of $|S_1|$ by contraction of pairs $u, v \in S_1$, the main error ν comes from the change in R_v due to the probability that the walk travels between the pair of contracted vertices u, v in the mixing time. From [4] this is

$$\beta = \sum_{t=2\sigma}^{T_G} (\pi_v + \lambda^t) = o(1/\log n) \tag{26}$$

assuming $\lambda = 1 - \theta$ for some $\theta > 0$ constant, and σ is given by (5).

For $X = \sum X_v$,

$$\begin{aligned} \mathbb{E}X(X-1) &= \sum_{u,v \in S} \mathbb{E}X_u X_v \\ &= \sum_{u,v \in S} \exp \left\{ -(1+\eta) \frac{2}{n} R_{(u,v)} (1+\beta) (1-\varepsilon) \frac{d-1}{d-2} n \log n \right\} \\ &= n^{2\delta} n^{O(\beta/n)} = (\mathbb{E}S_1)^2 \left(1 + O\left(\frac{1}{n}\right) \right), \end{aligned}$$

where β is defined in (26). So,

$$\begin{aligned} \mathbf{Var}|S_1| &= \mathbb{E}X(X-1) + \mathbb{E}|S_1| - (\mathbb{E}|S_1|)^2 \\ &= \mathbb{E}|S_1| \left(1 + O\left(\frac{\mathbb{E}|S_1|}{n}\right) \right) \\ &= \mathbb{E}|S_1|(1 + o(1)). \end{aligned}$$

Using the Chebychev Inequality we deduce that

$$\mathbb{P}(S_1 = \emptyset) < \mathbb{P}(|S_1 - \mathbb{E}S_1| \geq \mathbb{E}S_1/2) \leq \frac{5}{\mathbb{E}S_1} = O\left(\frac{1}{n^\delta}\right).$$

Let $\nu = \varepsilon/3$. The expected number of segments with $S_1 = \emptyset$ is at most

$$O\left(\frac{\ell}{n^\delta}\right) = O\left(\frac{n^\nu}{n^{\varepsilon-\nu+o(1)}}\right) = \frac{1}{n^{\nu+o(1)}}.$$

We conclude that w.h.p. each segment I has at least one unvisited vertex in $S_1(I)$ when the budget runs out at t_1 . Thus we have to effectively walk around the cycle to reach them all. This requires $\sim n^2/2$ time w.h.p. Let $C_{OBV}(\varepsilon)$ be the cover time of an oblivious walk with budget at most $t_1 = (1-\varepsilon)C(G)$, then

$$C_{OBV}(\varepsilon) \approx C(C_n) \sim \frac{n^2}{2}.$$

5.3 Proof of Theorem 3(c).

We first observe that the space $\mathcal{F}(2, r)$ of graphs formed by the union of a random 2-factor and a random r -regular graph is contiguous to the space $\mathcal{G}(r+2)$ of random $(r+2)$ -regular graphs, see for example Wormald [18]. So, the Red/Blue walk has mixing time $O(\log n)$ w.h.p. (we apply contiguity to the property of having a small second eigenvalue as in P5).

We next prove a basic lemma on random 2-factors.

Lemma 6. *W.h.p. a random 2-factor contains at least 2 cycles of length at least $cn/(\log^2 n)$ for some absolute constant $c > 0$.*

Proof. We begin by arguing that the number of cycles in a random 2-factor is $O(\log n)$ w.h.p. It follows that w.h.p. there is at least one cycle of length $\Omega(n/\log n)$, among which we choose the longest. We then argue that w.h.p. the maximum length of this cycle is at most $\ell = n - n/\log n$. As at least $n/\log n$ vertices remain after removing this cycle it follows that there is another cycle of length at least $\Omega(n/\log^2 n)$.

Number of cycles. Let S_n denote the set of permutations of $[n]$, and let $S' \subseteq S_n$ be the subset with minimum cycle length 3. We can obtain a random 2-factor F by choosing a random permutation from S' . This is not done uniformly, but we choose permutation π with probability proportional to $\frac{1}{2^\ell}$ where $\ell = \ell(\pi)$ is the number of cycles in π . Having chosen π we obtain our 2-factor by ignoring orientation. A 2-factor F with $\ell = \ell(F)$ cycles arises from 2^ℓ distinct permutations and so a 2-factor F is chosen with probability $2^{\ell(F)} \cdot \frac{1}{2^{\ell(F)}} \cdot \frac{1}{M}$, where $M = \sum_{\pi \in S'} \frac{1}{2^{\ell(\pi)}}$. Now the proof of Proposition 2(a) of Karp [11] implies that

$$\mathbb{P}(\ell(\pi) \geq a \log n) \leq \mathbb{P}(\text{Bin}(a \log n, 1/2) \leq \log_2 n) \leq n^{-a/3} \quad \text{for } a \geq 2. \quad (27)$$

So, $\mathbb{P}(\ell(\pi) \geq 2 \log n) \leq n^{-2/3}$. Because $|S'|/|S_n| = c \sim e^{-3/2}$ this also holds for S' . Summing over S' ,

$$M = \sum_{\ell(\pi) < 2 \log_2 n} \frac{1}{2^{\ell(\pi)}} + \sum_{\ell(\pi) \geq 2 \log_2 n} \frac{1}{2^{\ell(\pi)}} \geq \frac{cn!(1 - O(n^{-2/3}))}{n^2}.$$

And

$$\mathbb{P}(\ell(F) \geq 3 \log_2 n) = \sum_{m \geq 3 \log_2 n} \frac{|\{\pi \in S' : \ell(\pi) = m\}|}{2^m M} = O(n^{-1}).$$

Longest cycle. We use the configuration model of Bollobás [2]. In this model the expected number of cycles of length at least $\ell \geq \ell_0 = n - n/\log n$ is

$$\begin{aligned} \sum_{k=\ell_0}^n \binom{n}{k} \frac{(k-1)!}{2} \cdot 2^k \cdot \frac{(2n-2k)!}{(n-k)!2^{2n-k}} \cdot \frac{n!2^n}{(2n)!} &= \sum_{k=\ell_0}^n \frac{2^{2k}}{2k} \frac{n!n!}{(2n)!} \binom{2n-2k}{n-k} \\ &= O\left(\frac{1}{\sqrt{n}}\right) + O(\sqrt{n}) \sum_{k=\ell_0}^{n-1} \frac{1}{\sqrt{n-k}} \frac{1}{k} \\ &= O\left(\frac{1}{\sqrt{n}}\right) + O(\sqrt{\log n}) \log\left(\frac{n}{n-n/\log n}\right) \\ &= O\left(\frac{1}{\sqrt{\log n}}\right). \end{aligned}$$

□

Given Lemma 6 we proceed as follows: let C_1, C_2 be two of the cycles of length at least $\Omega(n/\log^2 n)$, as promised by Lemma 6. As in Section 5.2, let $L = n^{1-\nu}$ where $\nu > 0$. Partition each of the cycles C_1, C_2 into intervals of length L . We know that w.h.p. at time $t_1 = (1 - \varepsilon)(d - 1)/(d - 2)n \log n$, (where $d = r + b$) there will be unvisited vertices in C_1, C_2 and the result follows.

5.4 Finishing the proof of Theorem 1(b). The lower bound

For the lower bound, let $t_2 = (1 - \varepsilon)(1 + C/F)\sigma_{RB}n \log n$. The length C, F of the charged and free periods are fixed and in constant proportion. The t_2 steps are divided into $\tau = n^\theta$ phases of equal length $C + F = n^{1-\theta+o(1)}$. The total amount of time spent in free phases is $(1 - \varepsilon)\sigma_{RB}n \log n$.

In phase $1 \leq i \leq \tau$, during a charged period C the walk will w.h.p. cover an interval J_i of the Hamilton cycle of length $n^{(1-\theta)/2+o(1)}$. Let I_1, I_2, \dots, I_ℓ be intervals of length $L = n^{1-\nu}$, where $\nu = \varepsilon/3$ as in Section 5.2. In total, the intervals J_1, J_2, \dots, J_τ cover at most $n^\theta n^{(1+\theta)/2+o(1)}$ vertices.

We require $n^{\theta+(1+\theta)/2+o(1)} \ll L = n^{1-\varepsilon/3}$, which is satisfied by $\theta = (1 - \varepsilon)/4$. With this value of θ , w.h.p. each interval I_j $j = 1, \dots, \ell$ will contain a vertex unvisited during the free phases. This completes the proof of Theorem 1(b).

References

- [1] David Aldous and James Allen Fill. Reversible Markov Chains and Random Walks on Graphs. Unfinished monograph, 2002 (recompiled version, 2014).
- [2] B. Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled graphs. *European Journal on Combinatorics* 1, 311-316, (1980).
- [3] B. Bollobás and F. R. K. Chung. The diameter of a cycle plus a random matching. *SIAM Journal on Discrete Mathematics* 1, 328–333 (1988).
- [4] C. Cooper and A. M. Frieze, The cover time of random regular graphs, *SIAM Journal on Discrete Mathematics*, 18 (2005) 728-740.
- [5] C. Cooper and A. M. Frieze, The cover time of random digraphs, Stationary distribution and cover time of random walks on random digraphs, *Journal of Combinatorial Theory B* 102 (2012) 329-362.

- [6] C. Cooper and A. M. Frieze, The cover time of random geometric graphs, *Random Structures and Algorithms* 38 (2011) 324-349.
- [7] C. Cooper and A. M. Frieze, The cover time of the giant component of $G_{n,p}$, *Random Structures and Algorithms*, 32 (2008) 401–439.
- [8] C. Cooper, A. M. Frieze and T. Radzik, The cover times of random walks on hypergraphs. *Theoretical Computer Science*, 509 (2013) 51–69.
- [9] D. Coppersmith, P. Doyle, P. Raghavan, M. Snir, Random Walks on Weighted Graphs, and Applications to On-line Algorithms (Preliminary Version). STOC 1990: 369-378
- [10] P. Holme, Modern temporal network theory: A colloquium, *Eur. Phys. J. B* 88 (2015) 234.
- [11] R. Karp, A patching algorithm for the non-symmetric traveling salesman problem, *SIAM Journal on Computing* 8 (1979) 561–573.
- [12] J. H. B. Kemperman. The oscillating random walk. *Stochastic Processes and their Applications* 2, 1-29, (1974).
- [13] David A Levin and Yuval Peres. *Markov Chains and Mixing Times*. American Mathematical Society., 2017.
- [14] S. N. Majumdar, F. Mori, and P. Vivo. Cost of Diffusion: Nonlinearity and Giant Fluctuations, *Physical Review Letters* 130, 237102 (2023).
- [15] S. N. Majumdar, F. Mori, and P. Vivo, Nonlinear-cost random walk: Exact statistics of the distance covered for fixed budget, *Physical Review E* 108, 064122 (2023).
- [16] R. Robinson and N. Wormald, Almost all cubic graphs are Hamiltonian, *Random Structures and Algorithms* 3 (1992) 117-125.
- [17] Watts, D. J.; Strogatz, S. H. (1998). Collective dynamics of 'small-world' networks, *Nature* 393 (6684) 440–442.
- [18] N. Wormald, Models of random regular graphs, *Surveys in Combinatorics* (1999) 239-298.