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ON PATCHING ALGORITHMS FOR RANDOM ASYMMETRIC TRAVELLING SALESMAN PROBLEMS

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Let the arc-lengths L_{ij} of a complete digraph on n vertices be independent uniform [0, 1] random variables. We consider the patching algorithm of Karp and Steele for the travelling salesman problem on such a digraph and give modifications which tighten the expected error. We extend these ideas to the k-person travelling salesman problem and also consider the case where cities can be visited more than once.

Key words: Travelling salesman problem, probabilistic analysis, heuristics.

1. Introduction

Let $L=\|L_{ij}\|$ be an $n\times n$ real matrix. The *travelling salesman problem* (TSP) is that of computing the minimum value of $L_{i_1i_2}+L_{i_2i_3}+\cdots+L_{i_ni_1}$ over all permutations $i_1, i_2, \ldots, i_n$ of $V_n=\{1,2,\ldots,n\}$. It is a very well studied NP-hard problem. (See [9].)

This paper considers random models of the problem and describes heuristics which are very nearly optimal with very high probability.

The *assignment problem* (AP) is a closely related problem and can be described as follows: let S_n be the set of permutations of V_n. For $\sigma\in S_n$ let $L_\sigma=\sum_{i\in V_n} L_{i\sigma(i)}$. Then AP is defined to be

$$\begin{aligned}&\text{minimize} \quad L_\sigma\\ &\text{subject to} \quad \sigma\in S_n.\end{aligned} \tag{1.1}$$

Let $\sigma^*=\sigma^*(L)$ denote the optimal solution to (1.1) and $A(L)=L_{\sigma^*}$ denote its value. (In our random models the probability of alternate optima is zero.)

Now AP can be solved in $O(n^3)$ time (see, for example, [8, 10]) and is used extensively in branch and bound algorithms for solving the TSP (see, for example, [2]). To see the relationship we define the digraph D_σ with vertex set V_n and arcs $E_\sigma=\{(i,\sigma(i)): i\in V_n\}$. (We will use standard graph-theoretic terminology throughout the paper without comment.) Now it is easy to show that D_σ is a set of k_σ vertex disjoint cycles covering V_n. If $k_\sigma=1$ then D_σ is a cycle through V_n and

σ is called a *cyclic* permutation. Let T_n denote the set of cyclic permutations of V_n. Then the TSP can be rephrased as

$$\begin{aligned} &\text{minimize} \quad L_\tau \\ &\text{subject to} \quad \tau \in T_n. \end{aligned} \tag{1.2}$$

Now let $\tau^* = \tau^*(L)$ denote the optimum solution to (1.2) and let $B(L)$ denote its value. Clearly $A(L) \leqslant B(L)$ and furthermore $A(L)$ is usually 'very close' to $B(L)$.

Karp [6] gave some theoretical justification for the latter remark. Let L_{ij}, $i, j \in V_n$, be independent uniform [0, 1] random variables and consider the random variables $A_n = A(L)$ and $B_n = B(L)$. Karp described an $O(n^3)$ algorithm which, starting with an optimal solution to AP, 'patches' together the cycles of σ^* to make a *tour* $t \in T_n$ of length $B_{n,1}$ and proved

$$1 \leqslant E(A_n) \leqslant E(B_n) \leqslant E(B_{n,1}) = E(A_n) + O((\log n)^{3/2} n^{-0.24}).$$

(Here, and throughout, $\log n$ denotes $\log_e n$.)

Karp and Steele [7] considerably simplified and strengthened this result by constructing a simpler $O(n^3)$ algorithm which produces a tour τ_2 of length $B_{n,2}$ where

$$E(B_{n,2}) = E(A_n) + O(n^{-1/2}).$$

This raises the questions:

(i) What is the magnitude of $E(B_n - A_n)$?

(ii) How good can be expected length of a tour found by a polynomial time algorithm be?

By making extensive use of the ideas of Karp [6] and Karp and Steele [7] we improve their results and show

Theorem 1.1. *There is an* $O(n^3)$ *time algorithm which constructs a tour of length* $\bar{B}_n$ *satisfying*

$$E(\bar{B}_n) = E(A_n) + O((\log n)^4/(n \log\log n)), \tag{1.3a}$$

$$\Pr(\bar{B}_n - A_n \geqslant c_1(\log n)^4/(n \log\log n)) = o(1/n^2). \tag{1.3b}$$

(In this paper $c_1, c_2, \ldots$, will, without further remark, denote unspecified absolute constants. If required, appropriate numerical values for them can easily be deduced from the proofs.) We then consider a variant of the TSP in which we ask for a minimum length closed walk through V_n, i.e., each $v \in V_n$ may be visited more than once if this improves matters. This is equivalent to replacing L by $\hat{L}$ in (1.2), where $\hat{L}_{ij}$ is the minimum length of a path from i to j using L_{ij} as arc length.

We positively answer an open question of [7] by proving the following.

Theorem 1.2. *There is an* $O(n^3)$ *time algorithm which constructs a tour of length* $\hat{B}_n$ *satisfying*

$$1 \leqslant E(A(\hat{L})) \leqslant E(B(\hat{L})) \leqslant E(\hat{B}_n) = E(A(\hat{L})) + O((\log n)^2/n), \tag{1.4a}$$

$$\Pr(\hat{B}_n - A(\hat{L}) \geqslant c_2(\log n)^2/n) = o(1/n^2). \tag{1.4b}$$

The techniques we have developed can be applied to the *k-person TSP* (k-TSP) (see [3]). The problem is to find k cycles $C_1, C_2, \ldots, C_k$ with vertices in V_n. Each cycle must include vertex n and each vertex other than n must lie on exactly one cycle. More formally we require that

$$V(C_i) \cap V(C_j) = \{n\} \quad (1 \leq i < j \leq k) \tag{1.5a}$$

($V(C_i)$ is the vertex set of C_i for $i = 1, 2, \ldots, k$),

$$\bigcup_{i=1}^{k} V(C_i) = V_n, \tag{1.5b}$$

$$|V(C_i)| \geq 2 \quad (1 \leq i \leq k). \tag{1.5c}$$

(1.5c) will hold with probability tending to 1, without being 'forced', and its assumption clarifies part of the analysis. The objective is to

$$\begin{aligned} &\text{minimize} \quad \max\{L(C_i): i = 1, 2, \ldots, k\} \\ &\text{subject to} \quad (1.5) \end{aligned} \tag{1.6}$$

where $L(C_i)$ is the length of cycle C_i using L_{ij} as arc lengths. Let $B_k(L)$ denote the minimum value in (1.6). The problem we use as the basis for our heuristic in this case will be denoted by AP_k. Here we seek cycles $C_1, C_2, \ldots, C_l$ ($l \geq k$) satisfying

$$V(C_i) \cap V(C_j) = \{n\} \quad (1 \leq i < j \leq k), \tag{1.7a}$$

$$\bigcup_{i=1}^{l} V(C_i) = V_n, \tag{1.7b}$$

$$V(C_i) \cap V(C_j) = \emptyset \quad (1 \leq i \leq l,\ k < j \leq l,\ i < j), \tag{1.7c}$$

$$|V(C_i)| \geq 2 \quad (1 \leq i \leq k). \tag{1.7d}$$

Thus we have k cycles through vertex n and $l - k$ cycles covering the vertices left uncovered by $C_1, C_2, \ldots, C_k$. The objective now is to

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{l} L(C_i) \\ &\text{subject to} \quad (1.7). \end{aligned} \tag{1.8}$$

We let $A_k(L)$ denote the minimum value in (1.8). Assuming once again that the arc lengths L_{ij} are independent uniform [0, 1] random variables we prove the following.

Theorem 1.3. *Let $k \geq 1$ be constant. There is an $\mathrm{O}(n^3)$ time algorithm which constructs a solution to TSP_k of value $\bar{B}_{n,k}$ such that*

$$\begin{aligned} 1/k \leq E(A_k(L))/k \leq E(B_k(L)) \leq E(\bar{B}_{n,k}) \\ = E(A_k(L))/k + \mathrm{O}((\log n)^6/n), \end{aligned} \tag{1.9a}$$

$$\Pr(\bar{B}_{n,k} \geq A_k(L)/k + c_3(\log n)^6/n) = \mathrm{o}(1/n^2). \tag{1.9b}$$

We prove Theorems 1.1-1.3 in the next three sections. The reader should observe in what follows that, since we are mainly interested in asymptotic results, we often use, without comment, inequalities which are true only for sufficiently large values of n.

2. Proof of Theorem 1.1

Following the approach of [7] our algorithm 'patches' the cycles of the optimum solution to AP together. The 'large' ones are dealt with as in [7]. Our contribution is a procedure for patching in the 'small' cycles at smaller expected cost. It has similarities to the algorithm of Angluin and Valiant [1].

Algorithm 2.1.
Step 1.
Solve AP;
Let $C_1, C_2, \ldots, C_s$ be the cycles of D_{σ^*}, where $|C_1| \geq |C_2| \geq \cdots \geq |C_r| \geq n/\log n > |C_{r+1}| \geq \cdots \geq |C_s|$.
Step 2.
$\hat{C}_1 := C_1$;
for $i=2$ **to** r **do**
suppose that the edges of C_i are, in sequence, $e_1, e_2, \ldots$;
A: **for** $j=1$ **to** $|C_i|$ **do**
try to 'patch' C_i to $\hat{C}_{i-1}$, creating $\hat{C}_i$, by deleting $e_j=(y,z)$ and an edge (u,v) of $\hat{C}_{i-1}$, and adding the two edges (y,v), (u,z) if they are both of length $\leq 16(\log n)^2/n$. (See Figure 1.)

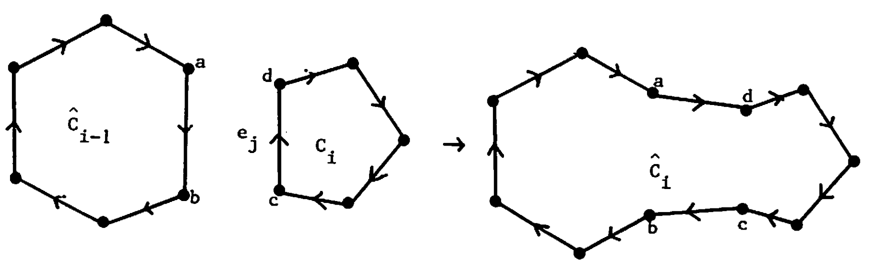

Fig. 1.

At the end of Step 2 we (should) have a large cycle $\hat{C}_r$ plus small cycles $C_{r+1}, C_{r+2}, \ldots, C_s$. We continue to 'absorb' the C_i into one cycle.
Step 3.
begin
for $i=r+1$ **to** s **do**
(arbitrarily) choose $e=(y,z) \in C_i$;

construct a digraph $\tilde{H}$ with vertices $\hat{C}_{i-1} \cup \{x\}$, where if $v \in \hat{C}_{i-1}$ then there is an edge (x, v) of length $L_{y,v}$ and an edge (v, x) of length $L_{v,z}$. (Distances within $\hat{C}_{i-1}$ are as before.)

Call DHAM (below) to try to find a hamilton cycle in $\tilde{H}$ using 'few' (i.e. $O(\log n/(\log\log n)))$ edges of length at most $35(\log n)^2/n$. (See Figure 2).

If successful we can replace vertex x by the path $(C_i - e)$ to create a cycle $\hat{C}_i$ through the vertices of $\hat{C}_{i-1}$ and C_i;

end

Procedure DHAM(C, x);

begin

let $E = \{(u, v)\colon \{u, v\} \subseteq C \cup \{x\}$ and $L_{u,v} \leq 35(\log n)^2/n\}$;

let $\phi : C \to C$ be such that the cycle $C = (v, \phi(v), \phi^2(v), \dots)$ for any $v \in C$ (we have used the same names for cycles and their vertex sets here, hopefully without confusion);

for each $(x, v) \in E$ create a path P of length $|C|$ with endpoints x and end$(P) = \phi^{-1}(v)$. (See Figure 3.)

$\boldsymbol{P}_0 := \{$paths created$\}$; $E_0 := \{$end$(P)\colon P \in \boldsymbol{P}_0\}$—in general we construct paths P from vertex x to vertex end(P);

for $t = 1$ **to** $T = \lceil \log n/(4 \log\log n) \rceil$ **do**

begin

for each path $P \in \boldsymbol{P}_{t-1}$ check for the existence of edge (end$(P), x)$

if one exists **then terminate** {a cycle has been found}

else for each path $P \in \boldsymbol{P}_{t-1}$ create all paths obtainable as follows.

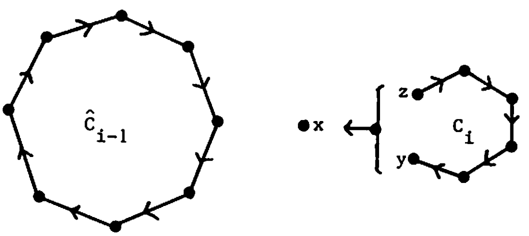

Fig. 2.

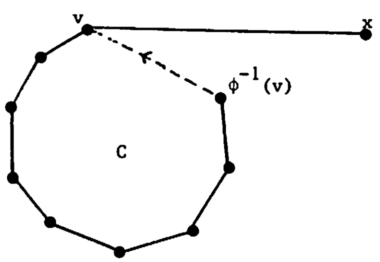

Fig. 3.

Let $P=(x=x_0, x_1, \ldots, x_m)$;
if $(x_m, x_i), (x_{i-1}, x_j) \in E$ where $1 \leq i < j$
then let ROTATE$(P, i, j) = (x_0, x_1, \ldots, x_{i-1}, x_j, x_{j+1}, \ldots, x_m, x_i, x_{i+1}, \ldots, x_{j-1})$. (See Figure 4.)
$\boldsymbol{P}_t :=$ {paths created in this stage}; $E_t := \{\text{end}(P)\colon P \in \boldsymbol{P}_t\}$

end

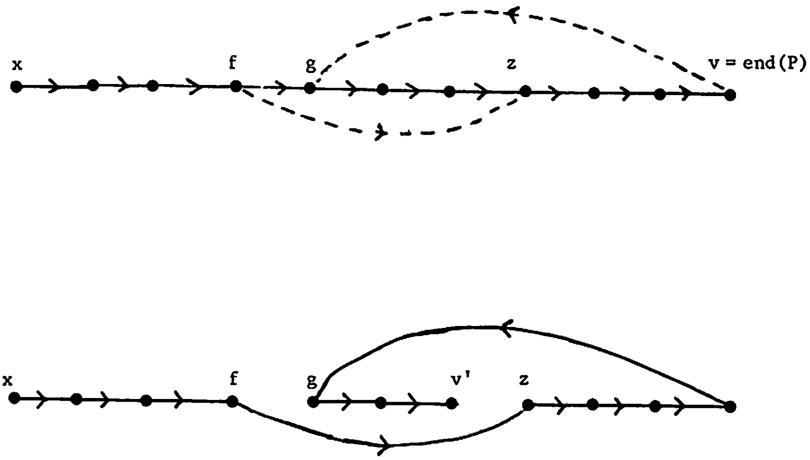

Fig. 4.

Analysis of Algorithm 2.1

The first lemma describes the likely cycle structure of the digraph $D_{\sigma^*} = (V_n, E_{\sigma^*})$. Most of it is stated in [7], with or without proof, but we give a full proof here for completeness.

Lemma 2.1. (a) *Each* $\sigma \in S_n$ *is equally likely to be optimal in* AP.
(b) $\Pr(s \geq 6 \log_2 n) = o(1/n^2)$.
(c) $\Pr(|C_{r+1}| + \cdots + |C_s| \geq 4n/(\log\log n)) = o(1/n^2)$.

Proof. (a) Let $P = \|P_{ij}\|$ be a 0–1 permutation matrix with $P_{ij} = 1$ iff $j = \sigma(i)$, where $\sigma \in S_n$. Then PL has the same distribution as L and σ_1 is optimal for L iff $\sigma_2 = \sigma\sigma_1$ is optimal for PL.

(b) If C is the cycle of D_{σ^*} containing vertex 1, then

$$\Pr(|V(C)| = k) = 1/n \quad \text{for } k = 1, 2, \ldots, n.$$

This follows from the observation that there are $\binom{n-1}{k-1}(k-1)!(n-k)!$ permutations in which $|V(C)| = k$. Furthermore, removal of the cycle C containing vertex 1 leaves the cycle structure of a random permutation of a set of size $n-|V(C)|$. It follows therefore that the following algorithm constructs a sequence $c_1, \ldots, c_s$ with the same distribution as that of the cycle sizes in D_{σ^*}.

Algorithm CYCLESIZES.
begin
  $t := 1$; $m := n$;
  repeat
    c_t is chosen randomly from $[m]$;
    $m := m - c_t$; $t := t+1$
  until $m = 0$
end

Let $s_0 = \lfloor 6 \log_2 n \rfloor$. Consider a random 0-1 sequence $x_1, x_2, \ldots, x_{s_0}$ generated as follows: run algorithm CYCLESIZES and suppose it produces a sequence of length s. Let

$$x_t = \begin{cases} 0 & \text{if } t \leq s \text{ and } c_t \leq \lceil \frac{1}{2} m \rceil, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\xi = \sum_{t=1}^{s_0} x_t$ dominates the binomial random variable $B(s_0, \frac{1}{2})$. Furthermore if $s > s_0$ then $\xi \leq \log_2 n$. Thus

$$\Pr(s > s_0) \leq \Pr(\xi \leq \log_2 n) \leq \sum_{t=0}^{\lceil \log_2 n \rceil} \binom{s_0}{t} 2^{-s_0} = o(n^{-2}).$$

(c) For $K \subseteq V_n$, $|K| = k \geq a = \lfloor 4n/\log\log n \rfloor$, the probability that K is the union of cycles of σ^* is, by (a), $k!(n-k)!/n! = 1/\binom{n}{k}$. Conditional on this event, the probability that all of these cycles have fewer than $b = \lfloor n/\log n \rfloor$ vertices is no more than $\lfloor k/b \rfloor !^{-1}$. To see this suppose $m \geq b$ and let σ_m be a random permutation of V_m. Let u_m be the probability that all cycles in σ_m are of size at most b. We need to show that $u_m \leq \lfloor m/b \rfloor !^{-1}$. This is true for $m = b$ and so inductively assume it is true for some $m \geq b$. Then $u_{m+1} \leq \lfloor (m+1)/b \rfloor !^{-1}$ follows immediately from

$$u_{m+1} \leq (b/(m+1)) \max\{u_m, u_{m-1}, \ldots, u_{m+1-b}\}.$$

But this follows directly from the remarks at the start of (b) above. Thus

$$\Pr((\text{c}) \text{ fails}) \leq \sum_{k=a}^{n} \lfloor k/b \rfloor !^{-1} = o(1/n^2). \qquad \square$$

The next lemma claims a pessimistic assumption about the distribution of the lengths of arcs not in E_{σ^*}. This was proved in [7], and is a key insight that makes much of the analysis possible. We require a similar result for Theorem 1.3, and this is proved later as Lemma 4.2.

Lemma 2.2. *Conditional on an event of probability* $1 - o(1/n^2)$ *we may assume that the lengths of arcs not in* E_{σ^*} *are independent uniform random variables in the range* $[\alpha_n, 1]$ *where* $\alpha_n = 15(\log n)^2/n$. $\square$

We turn next to the analysis of Step 2 of Algorithm 2.1. We first prove that not too many arcs have their lengths examined in loop A.

Lemma 2.3. $\Pr(\exists i \leqslant r : j$ *reaches* $3n/(\log n)^2$ *in loop* A$) = o(1/n^2)$.

Proof. If $(y, z) \in C_i$ and $(u, v) \in \hat{C}_{i-1}$ in Step 2 then Lemma 2.2 implies that

$$\Pr(L_{y,v}, L_{u,z} \leqslant 16(\log n)^2/n) \geqslant (\log n)^4/n^2.$$

The lengths of arcs (y, v) and (u, z) are only examined once during Step 2, for each possible y, z, u and v. Hence for any given i, the probability that j reaches $3n/(\log n)^2$ is no more than

$$(1-(\log n)^4/n^2)^{(3n/(\log n)^2)(n/\log n)} \leqslant 1/n^3.$$

Multiplying by at most $6\log_2 n$ (Lemma 2.1(b)) yields the result. □

Let X_i be the set of edges of C_i that are examined for deletion in Step 2. Let $X = \bigcup_{i=1}^{r} X_i$ and V_X be the set of vertices incident with an arc of X. If both $u, v \notin V_X$ and $(u, v) \notin A_{\sigma^*}$ then we have no information about $L_{u,v}$ other than that given by Lemma 2.2. Hence with probability $1-o(1/n^2)$ we complete Step 2 with a cycle $\hat{C}_r$, $|\hat{C}_r| \geqslant n - 4n/\log\log n$, and a set of vertices V_X, $|V_X| = O(n/\log n)$, such that the lengths of the arcs which are not incident with V_X or contained in E_{σ^*} can still be treated as independent uniform $[\alpha_n, 1]$ random variables.

We now analyse Step 3 of Algorithm 2.1, and, in particular, procedure DHAM. Let us consider the digraph H with vertex set $V(H) = V(\hat{C}_r) - V_X$ and arcs $E(H)$ where $(x, y) \in E(H)$ if $L_{x,y} \leqslant 35(\log n)^2/n$. Let $p = 20(\log n)^2/(n - 15(\log n)^2)$. Our assumption enables us to treat H as the random digraph $D_{h,p}$ where $h = |V(H)| = n - o(n)$ and each possible arc exists independently with probability p. To analyse Step 3 of Algorithm 2.1 we bound, from below, the expected growth rate of the endpoint set E_t in DHAM. We do this by considering the H-neighbourhoods of 'small' vertex sets.

For $T \subseteq V(H)$, let $N^+(T) = \{w : w \notin T \text{ and } (v, w) \in E(H)\}$.

Lemma 2.4. *Let* Π_1 *be the event*

$$\exists\, T \subseteq V(H):\quad 1 \leqslant |T| \leqslant n/(\log n)^3 \text{ and } |(|N^+(T)| - 20(\log n)^2|T|)| \geqslant (\log n)^2|T|.$$

Then $\pi_1 = \Pr(\Pi_1) = o(1/n^2)$.

Proof. If $|T| = t$ and $w \notin T$, then let $p_1 = \Pr(w \in N^+(T)) = 1-(1-p)^t$. Note $p_1 \sim pt$. Now $|N^+(T)|$ is distributed as the binomial random variable $B(h-t, p_1)$. We make use of the following 'large deviation' inequality for sums of bounded random variables.

Let $Z_1, Z_2, \dots, Z_m$ be independent random variables with $0 \leq Z_j \leq 1$ for $j = 1, 2, \dots, m$, and $\mu = E(Z_1 + Z_2 + \cdots + Z_m)/m)$. Then as a simple corollary of Theorem 1 of Hoeffding [5],

$$\Pr(|Z_1 + Z_2 + \cdots + Z_m - m\mu| \geq \varepsilon m\mu) \leq 2\,\mathrm{e}^{-\varepsilon^2 m\mu/3} \quad (0 < \varepsilon < 1). \tag{2.1}$$

Hence, since $B(n, p) = Z_1 + Z_2 + \cdots + Z_m$, where $\Pr(Z_j = 1) = p$ for $j = 1, 2, \dots, m$,

$$\pi_1 \leq \sum_{t=1}^{\lfloor n/(\log n)^3 \rfloor} 2 \binom{h}{t} \mathrm{e}^{-c_4(\log n)^2 t},$$

since $E(|N^+(T)|) = (h - t)p_1 \sim 20(\log n)^2 t$. Thus

$$\pi_1 \leq \sum_{t=1}^{\infty} 2n^t\, \mathrm{e}^{-c_4(\log n)^2 t} = \mathrm{o}(n^{-2}). \qquad \square$$

Continuing the analysis of Step 3, consider a certain partition of the cycle $\hat{C}_r$. Let $\lambda = \lceil (\log n)^2/\log\log\log n \rceil$ and partition $\hat{C}_r$ into paths $Q_1, Q_2, \dots, Q_\lambda$ each containing $\lfloor h/\lambda \rfloor$ or $\lceil h/\lambda \rceil$ vertices of H. Let S_i denote the vertices of Q_i that are in H for $i = 1, 2, \dots, \lambda$. The following lemma shows that, with high probability, no vertex will have 'too many' of its out-neighbours (in H) concentrated in 'too few' of the S_i's. This is used later to show that there are always sufficiently many arcs which are 'useful' in DHAM.

Lemma 2.5. *Let* $[\lambda] = \{1, 2, \dots, \lambda\}$, *and let* Π_2 *be the event*

$$\exists\, v \in V(H),\ I \subseteq [\lambda]: \quad |I| \leq \tfrac{11}{20}\lambda \ \text{ and } \ \Big|N^+(v) \cap \bigcup_{i \in I} S_i\Big| \geq 12(\log n)^2.$$

Then $\pi_2 = \Pr(\Pi_2) = \mathrm{o}(1/n^2)$.

Proof. We can clearly restrict attention to $|I| = \lfloor \frac{11}{20}\lambda \rfloor$. For a fixed $v \in V(H)$ and $I \subseteq [\lambda]$, it follows that $|N^+(v) \cap \bigcup_{i \in I} S_i|$ is dominated by the random variable $B(\lceil \frac{11}{20}n \rceil + \lambda, p)$. Applying (2.1) we see that

$$\Pr\left(\Big| N^+(v) \cap \bigcup_{i \in I} S_i \Big| \geq 12(\log n)^2 \right) \leq 2\,\mathrm{e}^{-(\delta^2/3)(\lceil 11n/20 \rceil + \lambda)p}$$

where $(1 + \delta)(\lceil \frac{11}{20}n \rceil + \lambda)p = 12(\log n)^2$. Now $\delta = \frac{1}{11} - \mathrm{o}(1)$, and so after multiplying by the numbers of possible v and I we obtain

$$\pi_2 \leq n 2^{\lambda+1}\, \mathrm{e}^{-0.03(\log n)^2} = \mathrm{o}(n^{-2}). \qquad \square$$

We can now put these results together to prove that DHAM almost always works.

Lemma 2.6. $\Pr(\exists\, i$: $r + 1 \leq i \leq s$ *and* DHAM *fails to patch* C_i *into* $\hat{C}_{i-1}) = \mathrm{o}(1/n^2)$.

Proof. We can assume, by Lemma 2.4, that

$$\emptyset \neq T \subseteq V(H),\ |T| \leq n/(\log n)^3 \quad \text{implies} \quad |N^+(T)| \geq 19(\log n)^2 |T|, \tag{2.2}$$

and, by Lemma 2.5,

$$v\in V(H),\ I\subseteq[\lambda],\ |I|\leqslant\tfrac{11}{20}\lambda \quad\text{implies}\quad \left|N^+(v)\cap\bigcup_{i\in I}S_i\right|\leqslant 12(\log n)^2. \tag{2.3}$$

We shall use (2.2) and (2.3) to show that, in an execution of DHAM, $|E_t|$ grows rather rapidly. To this end, fix i ($r+1\leqslant i\leqslant s$). Let $\hat{E}_t=E_t\cap V(H)$. We show first that

$$\Pr(|\hat{E}_0|\leqslant 18(\log n)^2)=o(1/n^3). \tag{2.4}$$

Note that prior to the construction of P_0, the arc lengths $L_{x,v}$ are assumed to be independent and uniform in $[\alpha_n, 1]$, by Lemma 2.2.

Let

$$W_1=\{v\in V(H): L_{x,v}\leqslant 35(\log n)^2/n\}\quad\text{and}\quad W_2=\phi^{-1}(W_1).$$

(Recall that the arcs of the cycle C in DHAM are of the form $(v,\phi(v))$.) Now $|W_1|$ ($=|W_2|$) is distributed as $B(h,p)$ and so, using (2.1), we have

$$\Pr(|W_1|\leqslant 19(\log n)^2)=o(1/n^3).$$

Now $\hat{E}_0\supseteq W_2\cap V(H)$ and thus if we prove that

$$|W_2-V(H)|=o((\log n)^2) \tag{2.5}$$

then (2.4) follows. Now, because $T=O(\log n/\log\log n)$ and DHAM is only called $O(\log n)$ times,

> $\hat{C}_{i-1}$ is obtained from $\hat{C}_r$ by deleting $O((\log n)^2/\log\log n)$ arcs, permuting the pieces, and then adding arcs or paths to make a cycle. (2.6)

Now $\hat{C}_r$ contains at most r arcs (u,v) with $u\notin V(H)$ and $v\in V(H)$ and (2.6) implies that $\hat{C}_{i-1}$ contains $o((\log n)^2)$ more. Thus, since $r=O(\log n)$, (2.5) follows.

We show next that for, $1\leqslant t\leqslant T$, either we terminate DHAM (having found a cycle), or

$$|\hat{E}_t|\leqslant n/6(\log n)^5\quad\text{implies}\quad|\hat{E}_{t+1}|\geqslant 36(\log n)^4|\hat{E}_t|. \tag{2.7}$$

The aim here is to show that the number of end-points in $\hat{E}_t$ grows so rapidly that we can (almost always) find an arc from one of these back to x to close the cycle. We prove (2.7) in two parts, corresponding to the two new arcs used in the rotation step. (See Figure 4.) We show that most paths have 'many' such candidate arcs. The proof involves splitting paths in the middle so that f (of Figure 4) is in the first half-path and z in the second. There are, however, some technicalities in applying this idea. Suppose then that $|\hat{E}_t|\leqslant n/6(\log n)^5$. For each $v\in\hat{E}_t$ choose any one path $P_v\in\boldsymbol{P}_t$ with $v=\text{end}(P_v)$. Consider $\{P_v: v\in\hat{E}_t\}$. If $P_v=(x=x_0,x_1,\ldots,x_m=v)$, let $P'_v=(x_0,x_1,\ldots,x_{\lfloor m/2\rfloor})$ and $P''_v=(x_{\lfloor m/2\rfloor+1},\ldots,x_m)$. We may assume $m\geqslant n-4n/\log\log n$, by Lemma 2.1(c). Let $\Gamma_t=N^+(\hat{E}_t)$ and (see Figure 4),

$$\Phi_t=\{f\in V(H): \exists g\in\Gamma_t \text{ and } v\in\hat{E}_t \text{ such that } (v,g)\in E(H)$$
$$\text{and } (f,g) \text{ is an arc of } P_v\}.$$

Let

$$\Gamma_t' = \{g \in \Gamma_t \colon g \in P_v' \text{ and } (v, g) \in E(H) \text{ for some } v \in \hat{E}_t\}$$

and $\Gamma_t'' = \Gamma_t - \Gamma_t'$. We define Φ_t' and Φ_t'' by replacing Γ_t in the definition of Φ_t by Γ_t' and Γ_t'' respectively. We now show that

$$|\Phi_t'| \geq 6(\log n)^2 |\hat{E}_t|. \tag{2.8}$$

To prove this, fix $v \in \hat{E}_t$ and note that (2.6) implies P_v'' contains vertices from at most $\frac{1}{2}\lambda + O((\log n)^2/\log\log n)$ distinct S_i. Hence (2.3) implies that $|\Gamma_t''| \leq 12(\log n)^2 |\hat{E}_t|$. But (2.2) implies that $|\Gamma_t| \geq 19(\log n)^2 |\hat{E}_t|$ and hence $|\Gamma_t'| \geq 7(\log n)^2 |\hat{E}_t|$. Furthermore, letting

$$\Delta = \{(v, g) \colon v \in \hat{E}_t, g \in P_v' \cap V(H) \text{ and } f \notin V(H) \text{ or } (f, g) \notin \hat{C}_r,$$
$$\text{where } f \text{ precedes } g \text{ on } P_v\},$$

$$|\Phi_t'| \geq |\Gamma_t'| - |\Delta| \geq 7(\log n)^2 |\hat{E}_t| - O((\log n)^2/\log\log n) |\hat{E}_t|,$$

using (2.6) and the observation immediately following it, which implies (2.8).

An analogous argument now yields

$$|\hat{E}_{t+1}| \geq 6(\log n)^2 |\Phi_t'|. \tag{2.9}$$

Indeed, for each $f \in \Phi_t'$, choose a unique $v(f)$ such that f lies on $P_{v(f)}$ and such that if z is the successor of f on $P_{v(f)}$ then $(v(f), z) \in E(H)$. Then f has at most $12(\log n)^2$ out-neighbours in $P_{v(f)}'$. Since Φ_t' has at least $19(\log n)^2 |\Phi_t'|$ out-neighbours altogether, we have

$$|W| \geq 7(\log n)^2 |\Phi_t'|,$$

where

$$W = \{w \in V(H) \colon \exists f \in \Phi_t', (f, w) \in E(H) \text{ and } w \in P_{v(f)}''\}.$$

But

$$|\hat{E}_{t+1}| \geq |W| - |\Delta'|$$

where

$$\Delta' = \{(f, w) \colon f \in \Phi_t', w \in V(H) \cap P_{v(f)}'' \text{ and } u \notin V(H) \text{ or } (u, w) \notin \hat{C}_r,$$
$$\text{where } u \text{ precedes } w \text{ on } P_{v(f)}\}.$$

Since

$$|\Delta'| = O((\log n)^2/\log\log n) |\Phi_t'|)$$

we have (2.9).

If $|\Phi_t'| > \lfloor n/(\log n)^3 \rfloor$ we choose a subset of Φ_t' of this size in its place, since the statement of Lemma 2.4 requires this minor technicality. Now, obviously, (2.8) and (2.9) yield (2.7).

Thus failure of DHAM under these circumstances implies that there exists $\tau \leq T$ such that $|\hat{E}_\tau| \geq 6n/\log n$. But then

$$\Pr(L_{v,x} > 35(\log n)^2/n \text{ for all } v \in \hat{E}_\tau) \leq (1-p)^{6n/\log n} \leq n^{-120},$$

and the lemma follows. □

We are now in a position to complete the proof of Theorem 1.1. Lemmas 2.1 to 2.6 clearly show that the Algorithm 2.1 succeeds with probability $1 - o(1/n^2)$. We have only to establish the remaining claims of Theorem 1.1.

Size of error in tour length. We compute an upper bound to the cost of arcs added to the AP solution. With the required probability, Step 2 adds arcs of total length at most $32s(\log n)^2/n = O((\log n)^3/n)$ and Step 3 adds arcs of total length at most $70sT(\log n)^2/n = O((\log n)^4/(n \log\log n))$. This proves (1.3b) and (1.3a) follows immediately.

Running time. Steps 1 and 2 can easily be executed in $O(n^3)$ time. We will show that (with high probability) each execution of DHAM requires $O(n^{2+o(1)})$ time and this clearly suffices. A calculation using (2.1), as in Lemma 2.4, shows that with probability $1 - o(1/n^2)$ each $v \in V_n$ is incident with at most $36(\log n)^2$ arcs (v, w) for which $L_{v,w} \leq 35(\log n)^2/n$. It follows in this case that in any execution of DHAM, $|P_0| \leq 36(\log n)^2$, and $|P_{t+1}| \leq 1296(\log n)^4 |P_t|$ for $0 \leq t \leq T$. Hence a simple calculation shows that at most $n^{1+o(1)}$ paths are produced. Since we can create a copy of ROTATE(P, i, j) in $O(n)$ time for any P, i and j, this completes the proof of Theorem 1.1. (If we *destroy* P it only takes $O(1)$ time to execute ROTATE, assuming that P is kept as a doubly-linked list. The derivation of the sets $P_0, P_1, \ldots,$ defines, in a natural way, a 'tree of paths' and if we explore this tree depth-first, rather than breadth-first as intimated, we could reduce the running time of Step 3 to $O(n^{1+o(1)})$ time.) □

3. Proof of Theorem 1.2

While, at first sight, this may look more difficult to prove than Theorem 1.1 (owing to the non-independence of the arc lengths $\hat{L}$), it is in fact a simply corollary of a result of Frieze and Grimmett [4].

Lemma 3.1 [4]. $\Pr(\exists i, j \in V_n: \hat{L}_{ij} \geq (13 \log n)/n) = o(n^{-2})$. □

Algorithm 3.1.
Step 1.
Solve AP {with $\hat{L}$ in place of L};
let $C_1, C_2, \ldots, C_s$ be the cycles of D_{σ^*}.

Step 2.
$\tilde{C} := C_1$;
for $i=2$ to s **do**
begin
patch C_i into $\tilde{C}$ by deleting any arc (x, y) of $\tilde{C}$, any arc (u, v) of C_i and adding arcs (x, v), (u, y).
end
end of Algorithm 3.1

To prove Theorem 1.2 we note that Lemma 2.1 still holds, since for any permutation matrix P, $\hat{L}$ and $P\hat{L}$ have the same distribution. Thus, with probability $1-o(n^{-2})$, there are only. $O(\log n)$ cycles. Now Lemma 3.1 shows that *all* edges, in particular the patching edges, will be 'short'. Finally note that $E(A(\hat{L}))\geqslant 1$ since $A(\hat{L})\geqslant \sum_{i=1}^{n} \min\{L_{ij} : j\neq i\}$. □

4. Proof of Theorem 1.3

Our approach takes the following lines. We solve AP_k and transform the solution to a tour of V_n as in Section 2. Next, we delete n from the tour and break the path obtained into k paths of roughly equal length. We then make 'small' changes to these paths, using 'few' rotations, so that we can join n to the endpoints of each path by short arcs.

Algorithm 4.1.
Step 1.
Solve AP_k;
let $C_1, C_2, \ldots, C_l$ be the cycles in the solution, where $|C_1| = \max\{|C_i|: i=1,2,\ldots,l\}$;
Step 2.
$\hat{C}_1 := C_1$;
for $i=2$ **to** l **do**
patch C_i into $\hat{C}_{i-1}$ as in Step 3 of Algorithm 2.1, using arcs of length at most $\beta_n = \alpha_n + 200(\log n)^3/n$. (If C_i and $\hat{C}_{i-1}$ intersect (in n) then delete the 2 arcs (y, n) and (n, z) of C_i incident with n to make the initial path.)
{We now have a cycle $\hat{C}_i = (n, i_1, i_2, \ldots, i_{n-1}, n)$ through V_n.}
Step 3.
Let P be the path $(i_1, i_2, \ldots, i_{n-1})$;
partition P into k consecutive sub-paths $P_1, P_2, \ldots, P_k$, each with length in the range $L(P)/k \pm \beta_n$. {This is (almost) always possible because arcs in P have length at most β_n.}
Step 4.
{ We give only an outline of the remainder of the procedure.}
for $i=1$ **to** k **do**

begin

let path P_i have endpoints x_1 and x_2; keeping x_1 fixed carry out rotations as in DHAM, using arcs of length at most $\mu = 5000k(\log n)^5/n$, until a path $\hat{P}_i$ is constructed, with endpoints x_1 and y, and for which $L_{y,n} \leqslant \mu$. {Report failure if t exceeds T in the main loop.}

Now keeping n as a fixed endpoint, carry out rotations as in DHAM, starting with the path $(\hat{P}_i, n)$, until a path $\tilde{P}_i$ created with endpoints z and n for which $L_{n,z} \leqslant \mu$. {In these rotations we must use the in-neighbours of the path endpoints in place of out-neighbours, etc.}

The ith cycle output is $(n, \tilde{P}_i, n)$.

end

end of Algorithm 4.1

Analysis

We first consider the cycle structure of the solution to AP_k.

Lemma 4.1. (a) *Each feasible solution to* AP_k *is equally likely to be optimal.*

(b) $\Pr(l \geqslant 6\log_2 n) = o(1/n^2)$.

(c) $\Pr(A_k(L) \leqslant 1-\varepsilon) = o(1/n^2)$ *for any fixed* $\varepsilon > 0$.

Proof. Define an $(n+k-1)\times(n+k-1)$ matrix $\tilde{L}$ as follows: Let $\tilde{L}_{ij} = L_{ij}$, $\tilde{L}_{i,n+t} = L_{in}$, $\tilde{L}_{n+t,j} = L_{nj}$ and $\tilde{L}_{n+t,n+t} = \infty$, for $1 \leqslant i,j \leqslant n$ and $0 \leqslant t < k$. Each permutation σ of V_{n+k-1} in which

$$\sigma(n+t) < n \quad \text{for } 0 \leqslant t < k \tag{4.1}$$

gives rise to a feasible solution $f(\sigma)$ of AP_k on replacing the arcs of D_σ of the form $(i, n+t)$ or $(n+t, j)$ by (i, n) and (n, j) respectively. Each solution of AP_k comes from $(k!)^2$ such permutations and the mapping preserves solution cost if $\tilde{L}$ is used for σ and L is used for $f(\sigma)$.

(a) Each $\sigma \in V_{n+k-1}$ gives rise to a permutation matrix P_σ in the usual way. If σ_1, σ_2 satisfy (4.1) then $P_{\sigma_1} = PP_{\sigma_2}Q$ for permutation matrices P and Q where P permutes the rows $1, 2, \ldots, n$ among themselves and the rows $n, n+1, \ldots, n+k-1$ among themselves. Q satisfies an analogous restriction with respect to columns. Since $P\tilde{L}Q$ has the same distribution as $\tilde{L}$, we deduce (a) as in Lemma 2.1(a).

(b) The number of permutations satisfying (4.1) is $(n-1)(n-2)\cdots(n-k)\times(n-1)!$ and so it follows that

$$\Pr(\sigma \in S_{n+k-1} \text{ satisfies } (4.1)) = 1 - o(1).$$

We then proceed as in Lemma 2.1(b) and observe that $f(\sigma)$ has at most $(k-1)$ more cycles than σ.

(c) The solution to AP_k contains an arc leaving each $i \in V_n$. The cost of this arc is at least $\zeta_i = \min\{L_{ij} : j \in V_n\}$. Define random variables

$$z_i = \min\{(n/(2\log n)\zeta_i, 1\} \quad \text{for } i = 1, 2, \ldots, n.$$

Clearly $z_i \in [0, 1]$ and a straightforward computation gives $E(z_i) = 1/(2 \log n) + o(1/n)$. Now (2.1) gives a superpolynomially small probability for the event $\{(2 \log n)/n) \sum_{i=1}^{n} z_i < 1 - \varepsilon\}$ and (c) follows, as ζ_i dominates $((2 \log n)/n) z_i$. □

We now give a high probability upper bound on the length of arcs in the optimum solution to AP_k.

Lemma 4.2. *With probability* $1 - o(1/n^2)$ *all arcs used in the optimum solution to* AP_k *are of length at most* $15(\log n)^2/n$.

Proof. Following [7], we show that, if an optimal solution were to contain any arc which is 'too long', then there are almost always sufficiently many useful short arcs that the solution can be improved by constructing a suitable alternating path. Thus, let $\tilde{D}$ be the bipartite graph $(R, C, \tilde{E})$ where R and C are disjoint copies of V_{n+k-1} and

$$\tilde{E} = \{(i, j) \in R \times C: \tilde{L}_{ij} \leq (5 \log n)/n\},$$

where $\tilde{L}$ is defined as in the proof of Lemma 4.1. For $S \subseteq R$ and $t = 1, 2$, let

$$N_t(S) = \{j \in C: |\{i \in S: (i, j) \in \tilde{E}\}| \geq t\}$$

and define $N_t(S)$ similarly for $S \subseteq C$. For $\sigma \in S_{n+k-1}$ and $S \subseteq R$ let

$$N_\sigma(S) = \{j \in C: \exists i \in S \text{ such that } (i, j) \in \tilde{E} \text{ and } j \neq \sigma(i)\}$$

and define $N_\sigma(S)$ similarly for $S \subseteq C$.

We wish to show that

$$\Pr(\exists\, \sigma \in S_{n+k-1} \text{ and } S \subseteq R \text{ or } S \subseteq C: 1 \leq |S| \leq \tfrac{1}{4} n \text{ and } |N_\sigma(S)| \leq 2|S|) = o(1/n^2), \tag{4.2a}$$

$$\Pr(\exists\, \sigma \in S_{n+k-1} \text{ and } S \subseteq R \text{ or } S \subseteq C: \tfrac{1}{4} n \leq |S| \text{ and } |N_\sigma(S)| \leq \tfrac{3}{4} n) = o(1/n^2). \tag{4.2b}$$

By noting that, for all σ,

$$|N_\sigma(S)| \geq \max\{|N_1(S)| - |S|, |N_2(S)|\},$$

we can demonstrate (4.2) by showing

$$\Pr(\exists\, S \subseteq R \text{ or } S \subseteq C: 1 \leq |S| \leq \tfrac{1}{4} n \text{ and } |N_1(S)| \leq 3|S|) = o(1/n^2), \tag{4.3a}$$

$$\Pr(\exists\, S \subseteq R \text{ or } S \subseteq C: \tfrac{1}{4} n \leq |S| \text{ and } |N_2(S)| \leq \tfrac{3}{4} n) = o(1/n^2). \tag{4.3b}$$

Proof of (4.3a). For $v \in R$ let $\tilde{d}(v)$ be its degree in $\tilde{D}$. Then, letting $p = (5 \log n)/n$, when n is large

$$\begin{aligned}
&\Pr(\exists\, v \in R\colon \tilde{d}(v) \leqslant \tfrac{1}{2}\log n) \\
&\quad \leqslant (n+k-1) \sum_{t=0}^{\lfloor \log n/2 \rfloor} \binom{n+k-1}{t} p^t (1-p)^{n+k-1-t} \\
&\quad \leqslant 2n \binom{n+k-1}{\lfloor \frac{1}{2}\log n \rfloor} p^{\lfloor \log n/2 \rfloor} (1-p)^{n+k-1-\lfloor \log n/2 \rfloor} \\
&\quad \leqslant 3n(enp/\lfloor \tfrac{1}{2}\log n \rfloor)^{\lfloor \log n/2 \rfloor} n^{-5} = o(n^{-2}).
\end{aligned}$$

This clearly disposes of the case $|S| \leqslant \frac{1}{6}\log n$ in (4.3a).

For $S \subseteq R$, $|S| = s \geqslant \frac{1}{6}\log n$ we have

$$\Pr(|N_1(S)| \leqslant 3s) \leqslant \binom{n}{3s} (1-p)^{(s-k)(n-3s)}.$$

Hence

$$\begin{aligned}
&\Pr(\exists\, S \text{ violating the condition in (4.3a)}) \\
&\quad \leqslant \sum_{s=\lceil (\log n)/6 \rceil}^{\lfloor n/4 \rfloor} \binom{n}{s}\binom{n}{3s} (1-p)^{(s-k)(n-3s)} + o(1/n^2) \\
&\quad \leqslant \sum_{s=\lceil (\log n)/6 \rceil}^{\lfloor n/4 \rfloor} \left(\frac{ne}{s}\right)^s \left(\frac{ne}{3s}\right)^{3s} e^{-5(s-k)\log n(1-3s/n)} + o(1/n^2) = o(1/n^2).
\end{aligned}$$

(The sum is readily proved negligible by splitting its range at, say, $s = n/\log n$.)

Proof of (4.3b). Let R_1 and C_1 denote the disjoint copies of V_{n-1} contained in R and C respectively. If $S \subseteq R$ and $|S| \geqslant \frac{1}{4}n$ then $|S \cap R_1| \geqslant \frac{1}{5}n$ for large n. Thus, if $v \in C_1$,

$$\Pr(v \notin N_2(S)) \leqslant (1-p)^{\lceil n/5 \rceil} + \lceil \tfrac{1}{5}n \rceil p (1-p)^{\lceil n/5 \rceil - 1} \leqslant (2\log n)/n.$$

Hence

$$\Pr(|N_2(S)| \leqslant \tfrac{3}{4}n) \leqslant \binom{n}{\lceil \frac{1}{4}n \rceil} ((2\log n)/n)^{\lceil n/4 \rceil}$$

and so

$$\begin{aligned}
&\Pr(\exists\, S \text{ violating the condition in (4.3b)}) \\
&\quad \leqslant 2^{n+k-1} \binom{n}{\lceil \frac{1}{4}n \rceil} ((2\log n)/n)^{\lceil n/4 \rceil} = O(e^{-(n\log n)/5}).
\end{aligned}$$

Thus we have proved that the two events of (4.2) have small enough probability that we may assume they do not occur. Suppose now that the optimum solution to AP_k contains an arc of length exceeding $15(\log n)^2/n$. Equivalently the optimum solution σ^* to AP using $\tilde{L}$ contains an edge (r, c), $r \in R$, $c \in C$, with $\tilde{L}_{r,c} > 15(\log n)^2/n$. We define a sequence $X_0, X_1, \ldots,$, of subsets of R by $X_0 = \{r\}$ and $X_{i+1} = \sigma^{*-1}(N_{\sigma^*}(X_i))$ for $i \geq 0$ and a sequence of subsets $Y_0, Y_1, \ldots,$ of C by $Y_0 = \{c\}$ and $Y_{i+1} = \sigma^*(N_{\sigma^*}(Y_i))$. The existence of i, j such that $N_{\sigma^*}(X_i) \cap Y_j \neq \emptyset$ implies that $\tilde{D}$ has an alternating cycle with at most $2(i+j+1)$ edges and containing (r, c). Now (4.2a) implies that $|X_{i+1}| \geq 2|X_i|$ so long as $|X_i| \leq \frac{n}{4}$. Together with (4.2b) this implies that if $i_0 = \lceil \log_2 n \rceil - 1$ then $|X_{i_0}| \geq \frac{3}{4}n$. Similarly $|Y_{i_0}| \geq \frac{3}{4}n$ and so $N_{\sigma^*}(X_{i_0}) \cap Y_{i_0} \neq \emptyset$. But now, if we add and delete edges on the implied alternating path, the solution value *drops* by at least $15(\log n)^2/n - (10 \log_2 n \log n)/n > 0$, a contradiction. □

Corollary 4.3. *With probability of error* $o(n^{-2})$, *we may assume that the lengths of arcs not in the optimum solution to* AP_k *are dominated by independent uniform random variables in the range* $[\alpha_n, 1]$ *where* $\alpha_n = 15(\log n)^2/n$.

Proof. With probability $1 - o(n^{-2})$, we can construct the optimal solution by ignoring all arcs of length exceeding α_n. Thus, in this case, the lengths of arcs not in the optimal solution are either at most α_n or are only known to be uniform (and independent) in $[\alpha_n, 1]$. In either case, the lengths are clearly dominated by independent uniform $[\alpha_n, 1]$ random variables. □

The rest of the analysis is very similar to that of Theorem 1.1. The conclusions to Lemmas 2.4 and 2.5 continue to hold. We can assume that $m = |\hat{C}_i| \geq n/(6 \log_2 n)$ throughout, in which case $200(\log n)^3/n \geq 20(\log m)^2/m$ and the analysis of Lemma 2.6 is valid for this case. So with probability $1 - o(1/n^2)$ we arrive at the end of Step 2 with a cycle $\tilde{C}$ through V_n with length at most $\gamma_n = 6T \log_2 n\beta_n$ more than the optimal solution to AP_k.

At this stage all arcs of $\tilde{C}$ are of length at most β_n and so Step 4 succeeds in the construction of the paths $P_1, P_2, \ldots, P_k$. We deduce also, from Lemma 4.1(c), that in this case

$$P_i \text{ has at least } n_0 = 0.9/(\beta_n k) \geq c_6 n/(\log n)^3 \text{ arcs for } i = 1, 2, \ldots, k. \quad (4.4)$$

The analysis of Step 4 is in a similar vein to the proof of Lemma 2.6. We give only a bare outline. We choose μ as the upper bound for arc lengths, since $\mu \geq 20(\log n_0)^2/n_0$ and each sequence of rotations starting with a P_i or $\hat{P}_i$ is in a digraph which contains a random digraph with the same arc density as that used in the proof of Lemmas 2.4–2.6. Therefore, conditional on events with probability $1 - o(1/n^2)$, we can be sure that the cardinality of the set of 'free' path-endpoints grows, at least, by a factor proportional to $(\log n)^2$. Thus eventually, with sufficiently high probability, a short connection to or from n can be made.

An examination of the algorithm reveals that the dominant factor in the total increase in cost over that for AP_k will be, from Step 4,

$$O(kT\mu) = O((\log n)^6/(n \log \log n)),$$

and Theorem 1.3 follows. □

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