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PARTITIONING RANDOM GRAPHS INTO LARGE CYCLES

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Let $r \geqslant 1$ be a fixed positive integer. We give the limiting distribution for the probability that the vertices of a random graph can be partitioned equitably into r cycles.

1. Introduction

The question of the existence of large cycles in random graphs is one of the basic problems of the subject. The threshold for the existence of hamilton cycles has been established by Komlós and Szemerédi [11]. There are now several alternative proofs and generalisations of this result—see for example Bollobás [1], Bollobás and Frieze [2], Fenner and Frieze [5, 6], Frieze [7, 8] and Luczak [12].

As usual, let $G_{n,m}$ denote a random graph chosen uniformly from the set of graphs with vertex set $V_n = \{1, 2, \ldots, n\}$ and having m edges.

Let

$$m = \tfrac{1}{2}n \log n + \tfrac{1}{2}n \log \log n + c_n n. \tag{1.1}$$

(In what follows when naming a cycle C, we also allow C to represent the set of vertices.) Now let $r \geqslant 1$ be a fixed positive integer. Let a graph (V_n, E) have property A_1 if it contains r cycles $C_1, C_2, \ldots, C_r$ which partition V_n and $|C_i| = \lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Clearly property A_i is the same as Hamiltonicity. A simple necessary condition for property A_r is minimum degree at least 2, and in this paper we show that this is almost always sufficient.

Theorem 1.1.

$$\lim_{n\to\infty} \Pr(G_{n,m} \in A_r)$$

$$= \lim_{n\to\infty} \Pr((G_{n,m}) \textit{ has minimum degree at least } 2) = \begin{cases} 0, & \textit{if } c_n \to -\infty, \\ e^{-e^{-2c}} & \textit{if } c_n \to c, \\ 1, & \textit{if } c_n \to +\infty. \end{cases}$$

Recently Bollobás, Fenner and Frieze [3] have designed a polynomial time algorithm which gives a constructive proof of Komlós and Szemerédi's result. (See also Frieze [9, 10]). It could be applied here to give a constructive proof of Theorem 1.1. The paper is made simpler by giving a non-constructive version.

We note the following Corollary of Theorem 1.1. It follows directly from McDiarmids percolation theorem [13].

Corollary 1.2. *Let* $c_n \to \infty$ *in* (1.1). *Then*

$$\lim_{n\to\infty} \Pr(D_{n,2m} \in A_r) = 1,$$

where $D_{n,2m}$ *is a random digraph with vertices* V_n *and* $2m$ *arcs.*

Since the results for $r = 1$ are already known we shall assume $r \geqslant 2$ from now on.

2. Notation

For convenience, we gather together in this section some notation that is used throughout the paper.

$$G = G_{n,m} = (V_n, E).$$

For $S,\ T \subseteq V_n$ let $N(S, T) = \{w \in T - S\colon \exists v \in S \text{ such that } (v, w) \in E\}$, and let $N(S) = N(S, V_n)$, $N(v, T) = N(\{v\}, T)$, $N(v) = N(\{v\})$ for $v \in V_n$.

$$n_1 = \lfloor n/r \rfloor \quad \text{and} \quad n_2 = \lceil n^{1-1/r+1/r^3} \rceil.$$

Let $\theta = n - rn_1$ and

$$S_i = \begin{cases} \{(i-1)n_1 + 1, \ldots, in_1\}, & i = 1, 2, \ldots, r-\theta, \\ \{(i-1)n_1 + i - r + \theta, \ldots, in_1 + i - r + \theta\}, & i = r - \theta + 1, \ldots, r. \end{cases}$$

For $v \in V_n$ define $a(v)$ by $v \in S_{a(v)}$.

Define $Y_i \subseteq S_i$, $|Y_i| = n_2$, $i = 1, 2, \ldots, r$ by

$$Y_i = \begin{cases} \{(i-1)n_1 + 1, \ldots, (i-1)n_1 + n_2\}, \\ \{(i-1)n_1 + i - r + \theta, \ldots, (i-1)n_1 + i - r + \theta + n_2 - 1\}, \end{cases}$$
$$i = 1, 2, \ldots, 1, r - \theta,$$
$$i = r - \theta + 1, \ldots, r.$$

and let $Y = \bigcup_{i=1}^{r} Y_i$.

$$X_0 = \{v \in V_n\colon |N(v)| \leqslant \alpha_r \log n\} \quad \text{and}$$
$$X_0' = \{v \in V_n\colon |N(v)| \leqslant \alpha_r \log n + 1\},$$

where $\alpha_r \log(e/\alpha_r) = 1/(2r^2)$.

$$X_1 = \{v \in V_n: \exists i \leqslant r \text{ such that } |N(v, S_i)| \leqslant (\alpha_r/r)\log n\}$$

and

$$X_1' = \{v \in V_n: \exists i \leqslant r \text{ such that } |N(v, S_i)| \leqslant (\alpha_r/r)\log n + 1\}$$

and note that $X_0 \subseteq X_1$.

For $v, w \in V$, $d(v, w) =$ the minimum number of edges in a path from v to w.

For $S \subseteq V$, $G[S] = (S, E_S)$ is the subgraph of G induced by S. Here $E_S = \{e \in E: e \subseteq S\}$.

$\delta(G) =$ the minimum vertex degree of G.

We say that an event E_n occurs *almost surely* (a.s.) if $\lim_{n\to\infty} \Pr(E_n) = 1$.

3. Typical structure of $G_{n,m}$

The following lemma describes the a.s. properties of $G_{n,m}$ that are needed in what follows:

Lemma 3.1. *If $c_n \to c$ in (1.1), then $G = G_{n,m}$ a.s. satisfies the following*:

$$|X_0'| \leqslant n^{1/2r} \quad \textit{and} \quad |X_1'| \leqslant n^{1-1/r+2/(3r^3)}, \tag{3.1a}$$

$$v \in X_0, \qquad w \in X_1 \cup Y \quad \textit{implies} \quad d(v, w) > \log\log n, \tag{3.1b}$$

$$X \subseteq X_1 \cup Y, \qquad |X| \geqslant r+1 \textit{ implies there exists } v, w \in X \textit{ such that } d(v, w) > \log\log n, \tag{3.1c}$$

$$\textit{If } C \textit{ is a cycle of } G,\ |C| \leqslant \log\log n \textit{ and } v \in X_1 \cup Y, \textit{ then } d(v, C) > \log\log n, \tag{3.1d}$$

$$|N(v)| \leqslant 4\log n, \quad \textit{for } v \in V_n, \tag{3.1e}$$

$$1 \leqslant i \leqslant r \textit{ and } X \subseteq S_i - X_1, \qquad |X| \leqslant n/\log n, \textit{ implies } |N(X, S_i)| \geqslant (\alpha_r/3r)\,|X| \log n, \tag{3.1f}$$

$$S, T \subseteq V, \qquad S \cap T = \emptyset, \qquad |S|, |T| \geqslant n/\log\log n \textit{ implies } |\{e = (v, w) \in E: v \in S, w \in T\}| \geqslant \sqrt{n}. \tag{3.1g}$$

The proof of this lemma follows standard lines. Similar results with different constants have been proved in related papers—see for example [3, 7 or 8]—an outline proof is given in an appendix.

Let $\Gamma_0 = \{G_{n,m}: (3.1) \text{ is satisfied}\}$.

4. Partitioning the vertex set

We first describe how to partition V_n into sets $T_1, T_2, \ldots, T_r$ such that the graphs $G_i = G[T_i]$ are a.s. Hamiltonian for $i = 1, 2, \ldots, r$.

We start with the partition $S_1, S_2, \ldots, S_r$. Some vertices in $X_1 - X_0$ will have too few ($\leqslant \alpha_r \log n/r$) neighbours within their prescribed subset. We move these (Step (a)) to subsets where they have enough neighbours. We deal with the vertices of low degree (X_0) by moving their neighbours into the same subset (Step (b)). The subsets will now have got slightly out of balance and their sizes are re-adjusted by moving elements of Y (Step (c)). At the end of this process the sets $T_1, T_2, \ldots, T_r$ produced a.s. induce subgraphs in which all sets of vertices having $s \leqslant (\alpha_r/9r)n$ vertices have at least $2s$ neighbours (Lemma 4.1). Posa's theorem implies that if one of these subgraphs is not hamiltonian then it contains a large number of maximal paths which cannot be closed by an edge. The edge colouring argument of [5] can be used to finish the proof.

Suppose $c_n \to c$ and $\delta(G_{n,m}) \geqslant 2$.

begin
Step (a)
  $T_i := S_i$, for $i = 1, 2, \ldots, r$;
  $A := \{v \in X_1 - X_0 \colon |N(v, S_{a(v)})| \leqslant \alpha_r \log n/r\}$;
  for $v \in A$ define $b(v)$ by
  $|N(v, S_{b(v)})| = \max\{|N(v, S_i)| \colon i = 1, 2, \ldots, r\}$;
  for $v \in A$ do
    begin
      $T_{a(v)} := T_{a(v)} - \{v\}$; $T_{b(v)} := T_{b(v)} \cup \{v\}$;
      end

Step (b)
  for $i = 1$ to r do
  for $v \in T_i \cap X_0$ do
    begin
      for $j = 1$ to r do $T_j := T_j - N(v)$;
      $T_i := T_i \cup N(v)$
      end;

Remark. If $G \in \Gamma_0$, then $v, w \in X_0$ implies $N(v) \cap N(w) = \emptyset$, by (3.1b).

Step (c)
  $I := \{i \leqslant r \colon |T_i| > |S_i|\}$
  for $i \in I$ choose a subset $B_i \subseteq Y_i - (X_1 \cup N(X_1))$ of size $k_i = |T_i| - |S_i|$ and let $T_i := T_i - B_i$. To be specific choose the k_i smallest elements.

Let $B = \bigcup_{i \in I} B_i$. Partition B into sets W_j, $j \notin I$, where $|W_j| = |S_j| - |T_j|$. To be specific ensure that $j < j'$ implies $\max W_j < \min w_{j'}$.
for $j \notin I$ do $T_j := T_j \cup W_j$.
end

Remark. If $G_{n,m} \in \Gamma_0$, then at the start of Step (c)

$$k_i < |Y_i - (X_1 \cup N(X_1)| \quad \text{for } i \in I.$$

To see this note that $T_i - S_i \subseteq X_1 \cup N(X_0)$ at this stage. Now use (3.1a) and (3.1e) and $X_0 \subseteq X_1$.

A set $K \subseteq E$ is said to be *good* if
(i) K is a matching,
(ii) no $e \in K$ is incident with a vertex of $X_0' \cup X_1'$,
(iii) $|K| = \lceil \log n \rceil$.

Lemma 4.1. *Let $G = G_{n,m} \in \Gamma_0$, $\delta(G) \geqslant 2$ and K be good. Let $H_i = (T_i, E_T - K)$ for $i = 1, 2, \ldots, r$. Then, for large n,*
(a) *$S \subseteq T_i$, $|S| \leqslant (\alpha_r/9r)n$ implies $|N_i(S)| \geqslant 2|S|$ for $i = 1, 2, \ldots, r$, where $N_i(S) = \{v \in T_i - S : \exists w \in S \text{ such that } (v, w) \in E - K\}$.*
(b) *H_i is connected for $i = 1, 2, \ldots, r$.*

Proof. (a) We note first that $(T_i - S_i) \cup (S_i - T_i) \subseteq X_1 \cup N(X_1) \cup Y$ and so (3.1) implies that

$$|N(v, T_i)| \geqslant |N(v, S_i)| - r, \quad \text{for } v \in T_i. \tag{4.1}$$

Now let $Z \subseteq T_i$, $|Z| \leqslant (\alpha_r/9r)n$ and let $Z_0 = Z \cap X_0$, $Z_1 = Z \cap (X_1 - X_0)$ and $Z_2 = Z - X_1$.

Case 1. $|Z_2| \leqslant n/\log n$. Now

$$\begin{aligned} N_1(Z)| \geqslant{}& |N(Z_0, T_i)| + |N(Z_1, T_i)| + |N(Z_2, T_i)| - |N(Z_0, T_i) \cap Z_2| \\ &- |N(Z_1, T_i) \cap Z_2| - |N(Z_2, T_i) \cap (Z_0 \cup Z_1)| \\ &- |N(Z_0, T_i) \cap N(Z_2, T_i)| - |N(Z_1, T_i) \cap N(Z_2, T_i)| - |Z_2|. \end{aligned} \tag{4.2}$$

(The term $-|Z_2|$ allows for the deletion of K.) Now, using (3.1)

$$|N(Z_0, T_i)| \geqslant 2|Z_0|, \tag{4.3a}$$

$$|N(Z_0, T_i) \cap Z_2| + |N(Z_1, T_i) \cap Z_2| \leqslant 2|Z_2|, \tag{4.3b}$$

$$\begin{aligned} &|N(Z_2, T_i) \cap (Z_0 \cup Z_1)| + |N(Z_0, T_i) \cap N(Z_2, T_i| \\ &\qquad + |N(Z_1, T_i) \cap N(Z_2, T_i)| \leqslant (r + 1 + r)|Z_2|. \end{aligned} \tag{4.3c}$$

$$|N(Z_1, T_i)| \geqslant ((\alpha_r/r^2)\log n)|Z_1| \tag{4.3d}$$

as $v \in Z_1$ implies $|N(v, T_i)| \geq (\alpha_r/r)\log n$ and no vertex can be adjacent to more than r members of Z_1. We deduce from (3.1) and (4.1) that

$$|N(Z_2, T_i)| \geq ((\alpha_r/3r)\log n - r)\,|Z_2|. \tag{4.4}$$

(4.2)–(4.4) then imply

$$\begin{aligned}|N_i(Z)| &\geq 2\,|Z_0| + ((\alpha_r/r^2)\log n)\,|Z_1| + ((\alpha_r/3r)\log n - (3r+4))\,|Z_2| \\ &\geq 2\,|Z|.\end{aligned} \tag{4.5}$$

Case 2. $n/\log n \leq |Z_2| \leq (a_r/9r)n$. Choose $Z_2' \subseteq Z_2$ with $|Z_2| = \lfloor n/\log n \rfloor$ and we obtain

$$\begin{aligned}|N_i(Z)| &\geq |N(Z_2, S_i)| - |S_i - T_i| - |Z_0 \cup Z_1| - |K| \\ &\geq |N(Z_2', S_i) - (|Z_2| - |Z_2'|) - o(n/\log n) \\ &\geq ((\alpha_r/3r)\log n + 1)\lfloor n/\log n \rfloor - |Z_2| - o(n/\log n) \\ &\geq 2\,|Z|.\end{aligned}$$

(b) If H_i is not connected then T_i can be partitioned into 2 non-empty sets Z_1, Z_2 with $|Z_1| \leq |Z_2|$ such that $N_i(Z_1) = \emptyset$. Part (a) shows that $|Z_1| \geq (\alpha_r/9r)n$. But then (3.1g) implies that there are at at least $\sqrt{n} - \lceil \log n \rceil$ edges joining Z_1 and Z_2: contradiction. □

Lemma 4.2. *Let $G = G_{n,m} \in \Gamma_0$ and $\delta(G) \geq 2$. Let H_i be as in Lemma 4.1 for $i = 1, 2, \ldots, r$. If H_i is not Hamiltonian, then T_i contains a set $Z = \{z_1, z_2, \ldots, z_p\}$, $p \geq (\alpha_r/9r)n$ and subsets $Z_1, Z_2, \ldots, Z_p$ with $|Z_i| \geq (\alpha_r/9r)n$ for $i = 1, 2, \ldots, p$ such that*

$$w \in Z_i \quad \text{and} \quad e = (z_i, w) \quad \text{implies} \tag{4.6}$$

(a) *$e \notin E(H_i)$,*
(b) *if $H_i' = (T_i, E(H_i) \cup \{e\})$, then either H_i' is Hamiltonian or $\lambda(H_i') = \lambda(H_i) + 1$.*

Here $\lambda(H)$ = the length of a longest path of a graph H.

Proof. Posa [14] shows that if a graph H is non-hamiltonian then there exists a set $Z = \{z_1, z_2, \ldots, z_p\}$ of vertices and sets of vertices $Z_1, Z_2, \ldots, Z_p$ such that H contains longest paths with endpoints z_i, w for each $w \in Z_i$, $i = 1, 2, \ldots, r$. Furthermore

$$|N_H(Z)| < 2\,|Z| \quad \text{and} \quad |N_H(Z_i)| < 2\,|Z_i|, \quad i = 1, 2, \ldots, r,$$

where for $S \subseteq V(H)$,

$$N_H(S) = \{w \in V(H) - S \colon \exists v \in S \text{ such that } (v, w) \in E(H)\}.$$

The lower bounds for the sizes of $Z, Z_1, Z_2, \ldots, Z_p$ follow from Lemma 4.1. To see (4.6) let P be a longest path of H with endpoints z_1, $w \in Z_1$. If P is a Hamilton path, then (a), (b) are immediate, otherwise by connectivity H contains $x \notin P$ which is a neighbour of a vertex of P and so adding e creates a path with endpoint x which is longer than P.

5. Finishing the proof

To finish this proof we use the edge colouring argument of Fenner and Frieze [5].

Let $\omega = \lceil \log n \rceil$, $N = \binom{n}{2}$ and assume once again that $c_n \to c$. Let $\Gamma_i = \{G \in \Gamma_0$: $\delta(G) \geq 2$ but G_i is not Hamiltonian$\}$ for $i = 1, 2, \ldots, r$. Since

$$\Pr(G_{n,m} \notin A_r \text{ and } \delta(G) \geq 2) \leq \Pr(G_{n,m} \notin \Gamma_0) + \sum_{i=1}^{r} |\Gamma_i| \Big/ \binom{N}{m}$$

it suffices to show that

$$|\Gamma_i| = o\left(\binom{N}{m}\right) \quad \text{for } i = 1, 2, \ldots, r. \tag{5.1}$$

Let now i be fixed and for $G \in \Gamma_i$, $K \subseteq E(G)$ with $|K| = \omega$ let $G_K = (V_n, E(G) - K)$, $H_{iK} = (T_i, E(H_i) - K)$ and

$$a(G, K) = \begin{cases} 1, & \text{if (a) } K \text{ is good, with respect to } G, \\ & \quad \text{(b) } \lambda(H_{iK}) = \lambda(H_i), \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$S = \sum_{G \in \Gamma_i} \sum_{K \subseteq E(G)} a(G, K).$$

We prove 2 inequalities.

$$S \geq |\Gamma_i| \binom{m}{\omega} \left(1 - \frac{2}{\log n}\right)^{\omega}, \quad \text{for } n \text{ large}, \tag{5.2a}$$

$$S \leq \binom{N}{m - \omega} \binom{N - m + \omega}{\omega} (1 - \beta)^{\omega}, \quad \text{for } n \text{ large}, \tag{5.2b}$$

where $\beta = \alpha_r^2/163r^2$.

(5.1) follows immediately from (5.2), and we have the theorem for $c_n \to c$. For $c_n \to -\infty$ there are almost always vertices of degree ≤ 1 and for $c_n \to \infty$ we can use the monotonicity of A_r.

Proof of (5.2). (a) It suffices to prove that $G \in \Gamma_i$ implies

$$\sum_{K \subseteq E(G)} a(G, K) \geq \binom{m}{\omega} \left(1 - \frac{2}{\log n}\right)^{\omega}, \quad \text{for } n \text{ large}. \tag{5.3}$$

It is however, easily verified, that for n large the right-hand side of (5.3) is a lower bound for the number of 'good' choices of K that avoid a particular longest path P of H_i (to make a $(G, K)=1$ we need only choose a matching that avoids $|P|+0(|X_i'|\log n)$ edges).

(b) Let Δ be a graph with vertex set V_n and $m-\omega$ edges. Let

$$\Omega_\Delta=\{G\colon \exists K \text{ such that } G_K=\Delta \text{ and } a(G,K)=1\}.$$

Clearly $S=\sum|\Omega|$ and (5.2) will follow from

$$|\Omega_\Delta|\leq\binom{N-m+\omega}{\omega}(1-\beta)^\omega. \tag{5.4}$$

Fix Δ such that $\Omega_\Delta\neq\emptyset$ and let $G\in\Omega_\Delta$. If we apply the partitioning algorithm of Section 4 to Δ, then we obtain exactly the same partition $T_1, T_2, \ldots, T_r$ of V_n as for G, using the definitions of X_0, X_1 as applied to Δ. Let $\Delta_i=\Delta[T_i]$. $G\in\Gamma_0$ implies that the conclusions of Lemma 4.2 hold for Δ_i, where $Z, Z_1, Z_2, \ldots, Z_p$ are determined by Δ and not G. Since $\lambda(\Delta_i)=\lambda(H_i)$ we deduce that $\Omega_\Delta=\emptyset$ or

$$\Omega_\Delta\subseteq\{\Delta+K\colon |K|=\omega \text{ and } K\cap(E(\Delta)\cup\{(z_i,z)\colon 1\leq i\leq p \text{ and } z\in Z_i\})=\emptyset\}$$

and (5.4) follows. (Here $\Delta+K$ denotes the graph obtained by adding the edges K to Δ.) To obtain a constructive proof we would have to show that the algorithm HAM of [3] a.s. succeeds on each G_i. Such a proof can easily be constructed from that given in [3].

Appendix

Let $p=m/\binom{n}{2}$. It is not difficult to see that for any graph property A:

$$\Pr(G_{n,p}\in A)=\sum_{m'}\Pr(G_{n,m'}\in A)\Pr(|E(G_{n,p})|=m'),$$

as $G_{n,p}$, given m' edges is $G_{n,m'}$. We deduce from this that

If $G_{n,p}\in A$ a.s. and A is monotone, then $G_{n,m}\in A$ a.s. (A1)

If $G_{n,p}\in A$ a.s., then $\exists m'$, $m-\sqrt{n}\log n\leq m'\leq m$ such that $G_{n,m'}\in A$ a.s. (A2)

$\Pr(G_{n,m}\in A)\leq 3\sqrt{n}\log n\Pr(G_{n,p}\in A)$ for n large. (A3)

(a) $E_p(|X_0'|)=nS(n-1,\alpha_r\log n+1)=o(n^{1/2r})$, where E_p is the expectation in the $G_{n,p}$ model and

$$S(N,u)=\sum_{k=1}^{\lfloor u\rfloor}\binom{N}{k}p^k(1-p)^{N-k}.$$

Now use the Markov inequality plus (A1). Similarly

$$E_p(|X_1'|)\leqslant rnS(\lceil n/r\rceil,(\alpha_r\log n)/r+1) = \mathrm{o}(n^{1-1/r+2/(3r^3)}).$$

(b)

$$\Pr((3.1\mathrm{b})\text{ in }G_{n,p})\leqslant\sum_{s=1}^{\lambda}(n)_{s+1}\beta p^sS(n-2,\alpha_r\log n) = \mathrm{o}(1),$$

where $\lambda=\lfloor\log\log n\rfloor$ and $\beta=r(S(\lceil n/r\rceil,(\alpha_r/r)\log n)+n_2/n)$, by treating Y as a random rn_2-subset of V_n in these calculations. Thus, using (A2), $\exists m'$ such that

$$\Pr((3.1\mathrm{b})\text{ in }G_{n,m'})=\mathrm{o}(1).$$

Given (3.1a), and assuming (3.1e), we see that the addition of $m-m'$ random edges to a 'typical' $G_{n,m'}$ satisfying (3.1b) is 'unlikely' to upset (3.1b).

(c) If (3.1c) fails, then there is a tree with $t\leqslant t_0=\lceil(r+1)\log\log n+1\rceil$ vertices containing $\geqslant r+1$ members of $X_1\cup Y$. Thus

$$\Pr((3.1\mathrm{e})\text{ in }G_{n,p})\leqslant\sum_{t=r+1}^{t_0}\binom{n}{t}t^{t-2}p^{t-1}\gamma^{r+1} = \mathrm{o}(1),$$

where $\gamma=r(S(\lceil n/r\rceil-t_0,(\alpha_r/r)\log n)+n_2/n)$. Now proceed as in (b)

(d) $$\Pr((3.1\mathrm{d})\text{ in }G_{n,p})\leqslant\sum_{s=3}^{\lambda}\sum_{t=0}^{\lambda}\binom{n}{s}\binom{n}{t}\tfrac{1}{2}(s-1)!\,t!\,p^{s+t}s\beta = \mathrm{o}(1).$$

(e) $$\Pr(|N(1)|>4\log n\text{ in }G_{n,p})=1-S(n-1,4\log n) = \mathrm{o}(n^{-1.54}).$$

Hence $\Pr((3.1\mathrm{e})\text{ in }G_{n,p})=\mathrm{o}(n^{-0.54})$. Now use (A3).

(f) Failure of (3.1f) for $|X|\leqslant n/(\log n)^5$ implies the existence of a set S $(=X\cup N(X,S_i))$ of size s, $(\alpha_r/r)\log n=s_0\leqslant s\leqslant s_1=n/(\log n)^4$ containing at least $3s/2$ edges. The probability of this in $G_{n,p}$ is no more that

$$\sum_{s=s_0}^{s_1}\binom{n}{s}\binom{\binom{s}{2}}{3s/2}p^{3s/2}=\mathrm{o}(n^{-\text{any constant}}).$$

For $|X| \geq n/(\log n)^5$, the probability of (3.1f) in $G_{n,p}$ is no more than

$$r \sum_{s=s_1}^{n/\log n} \binom{\lceil n/r \rceil}{s} \sum_{t=0}^{(\alpha_r s/3r)\log n} \binom{\lceil n/r \rceil - s}{t} (1-(1-p)^s)^t (1-p)^{s(\lceil n/r \rceil - s - t)}$$
$$= o(n^{-\text{any constant}}).$$

(An almost identical calculation is done in [3]—see [3, Lemma 3.1(d)].) Now use (A3).

(g) $$\Pr((3.1\text{g}) \text{ in } G_{n,p}) \leq \sum_{r \geq n/\log\log n} \sum_{s \geq n/\log\log n} \frac{n!}{r!\, s!\, (n-r-s)!} S(rs, \sqrt{n})$$
$$= o(n^{-\text{any constant}}).$$

Now use (A3).

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