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PANCYCLIC RANDOM GRAPHS

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Let $G_{n,m}$ denote a random graph chosen uniformly from the set $\mathscr{G}$ of graphs with vertex set $\{1,2,\dots,n\}$ and m edges. $G_{n,m}$ is defined to be pancyclic if for all s, $3 \leqslant s \leqslant n$, there is a cycle of size s on the edges of $G_{n,m}$. We show that the edge threshold for the pancyclic property is the same as that for minimum vertex degree at least 2, which occurs at $m = \frac{1}{2} n \log n + \frac{1}{2} n \log\log n + c_n n$.

1. Introduction

As usual let $G_{n,m}$ denote a random graph chosen uniformly from the set $\mathscr{G}$ of graphs with vertex set $V_n = \{1,\dots,n\}$ and m edges. We shall say that $G_{n,m} \in \text{Pan}$, or simply $G_{n,m}$ is pancyclic, if for all s, $3 \leqslant s \leqslant n$, there is a cycle of size s on the edges of $G_{n,m}$.

The threshold for the existence of hamiltonian cycles in $G_{n,m}$ was established by Komlos & Szemeredi [6], and is the same as the threshold for minimum vertex degree at least 2. We show that this condition is sufficient for almost every (a.e.) $G_{n,m}$ to be pancyclic. The question of the threshold for pancyclic graphs in $G_{n,m}$ was raised by Korshunov [7].

Let $m = \frac{1}{2} n \log n + \frac{1}{2} n \log\log n + c_n n$.

Theorem 1.1.

$$\lim_{n\to\infty} \Pr(G_{n,m} \in \text{Pan}) = \lim_{n\to\infty} \Pr(G_{n,m} \text{ has minimum degree} \geqslant 2) =$$

(by Erdős & Rényi [3])

$$= \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-e^{-2c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty . \end{cases} \qquad \square$$

Throughout the paper, all inequalities are implicitly assumed to be considered only for large enough n.

2. Notation

If $p = m/\binom{n}{2}$, then $G_{n,p}$ has its usual meaning, i.e. each possible edge is independently included with probability p.

If $G = G_{n,m} = (V_n, E)$, then $G[S] = (S, E_s)$ is the subgraph induced by the vertex set $S \subseteq V_n$. $d(v)$ is the degree of vertex v in $G_{n,m}$, or $G_{n,p}$ as appropriate.

For $U \subset S$, $N(U,S)$ is the disjoint neighbour set of U in $G(S)$, i.e.

$$N(U,S) = \{w \in S - U : \exists\, u \in S \text{ and } (u,w) \in E_S\}$$

and

$$d_S(v) = |N(\{v\}, V_n) \cap S|$$

is the degree of v in S.

3. Cycles of length $n/3$ or less

W.F. de la Vega's Theorem [2] on lower bounds for the length of longest paths in random graphs may be stated in the following manner (adapted from Bollobás [1, pp. 181–185]).

Theorem 3.1. *Let $0 < \theta = \theta(n) < \log n - 3 \log\log n$ and $\pi = \theta/n$. Then a.e. $G_{n,\pi}$ contains a path at least $(1 - \frac{4 \log 2}{\theta})n$.* $\square$

In what follows let $G(n,p,q)$ denote a blue-green multigraph over the vertex set V_n in which the blue and green edges are chosen independently with probabilities p and q, respectively.

Corollary 3.2. *Let $\varepsilon p = \log\log n / n$, then a.e. blue-green multigraph $G(n, \varepsilon p, (1-\varepsilon)p)$ contains a path of length at least $n(1 - \frac{4\log 2}{\log\log n})$ using only blue edges.* □

We now use the unconditioned green edges of the multigraph to look for 'triangular' cycles of length s made from two adjacent green edges and a section of the blue path of length $s-2$.

Theorem 3.3. *Almost every $G_{n,m}$ has a cycle of length s, for all s, $3 \leqslant s \leqslant n/3$.*

Proof. As cycle existence is a monotone property, we prove it for a.e. $G(n, \varepsilon p, (1-\varepsilon)p)$ and note that it extends to $G_{n,p}$ and thus $G_{n,m}$ by virtue of Lemma A1 and

$$\Pr(\text{edge in } G(n, \varepsilon p, (1-\varepsilon)p)) = 1 - (1-\varepsilon p)(1-(1-\varepsilon)p)$$

$$\leqslant p = \Pr(\text{edge in } G_{n,p}).$$

Let $l = n(1 - \frac{4\log 2}{\log\log n})$. Without loss of generality we will assume that the first l vertices on the blue path $\mathscr{B}$ in Corollary 3.2 are $A = \{1, \ldots, l\}$. For fixed $i \in A$ and fixed cycle size s, partition the vertices $i+2, \ldots, l$ on the path into $\lfloor \frac{l-i-1}{2s-4} \rfloor$ sets of size $2s-4$ running sequentially from vertex $i+2$. Thus the first such set is $\{i+2, \ldots, i+2s-3\}$. On this set we can construct $s-2$ 'triangular' cycles

$$(i, i+2)\ (i+2)\mathscr{B}(i+s)\ (i+s, i)$$

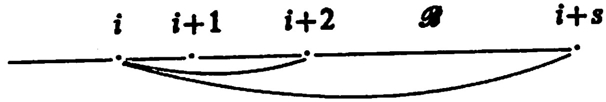
$$(i, i+3)\ (i+3)\mathscr{B}(i+s+1)\ (i+s+1, i)$$

$$\cdots$$

$$(i, i+s-1)\ (i+s-1)\mathscr{B}(i+2s-3)\ (i+2s-3, i)$$

where (e.g.) $(i+2)\mathscr{B}(i+s)$ is the section of the blue path between vertices $i+2$ and $i+s$ and $(i, i+2)$, $(i+s, i)$ are (potentially) green edges whose existence we can test for.

An example is shown in the figure below.



For fixed i, each pair of green edges is only used once, and as i runs from 1 to $l-2s+3$ no edge is ever reused for fixed s. For fixed s we examine

$$\sum_{i=1}^{l-2s+3} \left\lfloor \frac{l-i-1}{2s-4} \right\rfloor (s-2) \geqslant \frac{1}{4}(l-2s)^2$$

pairs of green edges and thus, using the independence of the green edges $\Pr(\exists s: 3 \leqslant s \leqslant n/3$ and $G_{n,m}$ contains no cycle of length $s)$

$$\leqslant \frac{n}{3}(1-((1-\varepsilon)p)^2)^{\frac{1}{4}(l-2s)^2}$$
$$\leqslant n e^{-(\frac{1}{4}(l-2s)(1-\varepsilon)p)^2} = o(1). \qquad \square$$

4. Cycles of size $n/3$ or more

The following lemma establishes certain structural properties of a.e. $G_{n,m}$. The proof of these properties is given in the Appendix.

Lemma 4.1. *The following hold in a.e.* $G_{n,m}$:

(a) (i) *If* $X = \{v \in V_n : d(v) \leqslant \frac{3}{5} np\}$, *then* $|X| > n^{22/25}$

(ii) $d_X(v) \leqslant 18$, *for all* $v \in V_n$

(b) *Assume* $n/3 \leqslant s \leqslant n - n^{22/25}$

(i) *If* $X(s) = \{v \in V_n : d_{V_s}(v) \leqslant \frac{1}{10} sp\}$, *where* $V_s = \{1, \ldots, s\}$, *then* $|X(s)| \leqslant n^{1-0.4\frac{s}{n}}$ *and thus* $|X(s)| < n - s$

(ii) $d_{X(s)}(v) \leqslant \lceil 7\frac{n}{s} \rceil$, *for all* $v \in V_n$

(c) $S \subset V_n$, $|S| \leqslant \frac{n}{375}$ *implies* $|E_S| < \frac{|S|np}{250}$

(d) *Let* $L = \{v \in V_n : d(v) < \frac{1}{10} np\}$ *then*

(i) *no cycle of length* 4 *or less contains any* $v \in L$

(ii) *no* $v' \in L$ *is within distance* 4 *of* $v \in L$

(e) *If* $U,W \subseteq V_n$, $U \cap W = \emptyset$, $|U|,|W| \geq \frac{n}{\log\log n}$ *then*

$|E_{UW}| = |\{e=(u,w) \in E : u \in U, w \in W\}| \geq \sqrt{n}$

(f) *The maximum degree of a vertex is at most* $3np$

(g) *There are at most* $n^{0.8}$ *edges incident with vertices in* L. □

Let $\mathscr{H} = \{G \in \mathscr{G}$: the conditions of Lemma 4.1 hold$\}$. The following lemma is an immediate consequence of Lemma 4.1 (c).

Lemma 4.2. *Let* $G \in \mathscr{H}$, $U \subseteq S \subset V_n$, $|U| \leq n/1500$, $F \subseteq E_s$ *and* $H=(S,F)$. *If* U *is such that the degree of* u *in* H *is at least* $np/31$ *for all* $u \in U$ *then* $|N(U,S)| \geq 3|U|$ *in* H. □

In order to prove the pancyclic condition for $s \geq n/3$ we examine two separate cases

Case 1: $n - n^{22/25} \leq s \leq n$

Case 2: $n/3 \leq s \leq n - n^{22/25}$

Working under the assumption that $G_{n,m}$ has minimum degree 2 we construct specific sets A_s (where $|A_s| = s$), such that $G[A_s]$ is hamiltonian simultaneously for all $s \geq n/3$ with probability tending to 1. In Case 1 we generate A_s by deleting vertices in degree sequence. In Case 2 we generate A_s from the set $V_s = \{1,\ldots,s\}$ by replacing vertices $v \in V_s$ of low degree in V_s by suitable vertices from $V_n - V_s$.

Assume $\delta(G_{n,m}) \geq 2$ and the conditions of Lemma 4.1 hold. We will use the following construction for the vertex set A_s of our proposed cycle of size s.

Construction

Case 1: $n - n^{22/25} \leq s \leq n$

Let $\{w_1, w_2, \ldots, w_n\}$ be the vertices of $G_{n,m}$ ordered in some ascending degree sequence.

$A_s := \{w_{n-s+1}, w_{n-s+2}, \ldots, w_n\}$

Case 2: $n/3 \leq s \leq n - n^{22/25}$

$V_s = \{1,2,\ldots,s\}$

$X(s) := \{v \in V_n : d_{V_s}(v) < \frac{1}{10} sp\}$

$Y := (V_n - V_s) - X(s)$

Y' is a random $|X(s) \cap V_s|$-subset of Y.

$A_s := (V_s - X(s)) \cup Y'$

As we propose to use the edge colouring argument of Fenner and Frieze [4] we regard our graph edges in $G_{n,m}$ as initially coloured blue, but with the option of recolouring a set R of the edges red. *À propos* of this, we say an edge set R is 'deletable' if it satisfies the following definition.

Definition 4.3.

(a) Let E be the edge set of $G_{n,m}$. $R \subset E$ is *deletable* if

(i) R is a matching

(ii) no edge of R is incident with a vertex of degree $\leqslant \frac{1}{10}np$

(iii) $|R| = \lceil n^{0.1} \rceil$

(b) If $G[A_s] = (A_s, E_{A_s})$ is the subgraph induced by A_s we refer to $G_B[A_s] = (A_s, E_{A_s} - R)$ as the blue subgraph induced by $E_{A_s} - R$.

(c) $N_B(U, A_s)$ is the disjoint neighbour set of U in $G_B[A_s]$. □

Lemma 4.4. *Let $G \in \mathscr{H}$ and let $U \subset A_s$, $|U| \leqslant \frac{n}{1500}$, then $|U \cup N_B(U, A_s)| \geqslant 3|U|$.*

Proof. Case 1: Let $L = \{v \in V_n : d(v) < \frac{1}{10}np\}$ as before.

If $U \subset A_s$, we let $U_1 = U \cap L$ and $U_2 = U - U_1$. By Lemma 4.1(d) we know that

$$|U_1 \cup N(U_1, A_s)| \geqslant 3|U_1|$$

and also

$$|(U_1 \cup N(U_1, A_s)) \cap (\{v\} \cup N(\{v\}, A_s))| \leqslant 1 \quad \text{for all } v \in U_2.$$

Delete from each $v \in U_2$ the edge (if any) responsible for this nonempty intersection with $U_1 \cup N(U_1, A_s)$. We have $d_{s'}(v) \geqslant \frac{1}{10}np - 19$, where $S' = A_s - (U_1 \cup N(U_1))$, because by Lemma 4.1(a)(ii) each $v \in U_2$ has at most 18 neighbours in $X(s)$ and at most one in $U_1 \cup N(U_1)$. By Lemma 4.2 there are at least $3|U_2|$ neighbours disjoint from $U_1 \cup N(U_1, A_s)$. The removal of $\min\{|R|, |U|\}$ deletable edges leaves $|N_B(U, A_s)| \geqslant 2|U|$.

Case 2: By the construction of A_s and Lemma 4.1(b)(ii),

$$d_{A_s}(v) \geqslant \frac{1}{10}sp - \lceil 7\frac{n}{s} \rceil \geqslant \frac{np}{31}.$$

We then use Lemma 4.2 as before. □

Lemma 4.5. *Assuming the conditions of Lemma* 4.1 *hold,* $G[A_s]$ *is connected for all* s, $n/3 \leqslant s \leqslant n$.

Proof. If $G[A_s]$ is not connected, then by Lemma 4.4 the smallest of the disconnected components cannot be less than $n/1500$. However, by Lemma 4.1(e), any two sets of vertices of size at least $n/\log\log n$ must be connected by at least $\sqrt{n}$ edges. □

It follows by well known arguments from Lemmas 4.4, 4.5 and a theorem of Pósa [8] that

Lemma 4.6. *If* $G \in \mathscr{H}$, $G[A_s]$ *is not hamiltonian and* R *is deletable, then* A_s *contains a set* $Z = \{z_1, z_2, \ldots, z_l\}$, $l \geqslant n/1500$ *and subsets* $Z_1, \ldots, Z_l$ *(not necessarily disjoint), depending only on* $G_B[A_s]$ *with* $|Z_i| \geqslant n/1500$ *for* $i = 1, 2, \ldots, l$ *such that* $w \in Z_i$ *and* $e = (z_i, w)$ *implies*

(i) *e is not an edge of* $G[A_s]$

(ii) *if e is added to* $G[A_s]$ *then the resulting graph is either hamiltonian, or the length of a longest path increases by at least one.* □

We define the set $\mathscr{F}(s)$ to be those $G \in \mathscr{H}$ for which the subgraph $G[A_s]$ is not hamiltonian.

Theorem 4.7.

$$\frac{1}{|\mathscr{G}|} \sum_{s=n/3}^{n} |\mathscr{F}(s)| = o(1).$$

Proof. Let R be a set of red edges of $G_{n,m}$ with the property $P(R)$ that

(i) R is deletable (Definition 4.3)

(ii) $\lambda(G[A_s]) = \lambda(G_B[A_s])$

where $\lambda(H)$ is the length of the longest path in the graph H.

Let $\mathscr{C}$ be the set of all red-blue colourings of $\mathscr{F}(s)$ which satisfy $P(R)$. How many specific edges do we have to avoid for a given $G \in \mathscr{F}(s)$?

(i) some longest path length $\lambda \leqslant s$,

(ii) at most $\theta = \lceil n^{0.8} \rceil$ edges which are attached to vertices of degree $\leqslant \frac{1}{10} np$, by Lemma 4.1(g).

As R is a matching we can select it in at least

$$\frac{1}{r!}(m-s-\theta)(m-s-\theta-2\Delta)\ldots(m-s-\theta-2(r-1)\Delta)$$

$$\geqslant\frac{(m-s)^r}{r!}(1-o(1))$$

ways, as $r=|R|=\lceil n^{0.1}\rceil$, and the maximum vertex degree Δ is at most $3np$. Thus

$$|\mathscr{C}|\geqslant|\mathscr{F}(s)|\frac{(m-s)^r}{r!}(1-o(1)).$$

Consider any fixed blue subgraph. We will, by Lemma 4.6, have to avoid replacing at least $\frac{1}{2}(n/1500)^2$ end point pair edges when adding back the red edges in order to construct a red-blue colouring satisfying $P(R)$. Thus

$$|\mathscr{C}|\leqslant\binom{\binom{n}{2}-(m-r)-\frac{1}{2}\left(\frac{n}{1500}\right)^2}{r}\binom{\binom{n}{2}}{m-r}$$

and thus

$$\frac{|\mathscr{F}(s)|}{|\mathscr{G}|}=O(e^{-\frac{r}{1500^2}+\frac{sr}{m}})=o(n^{-\gamma})\quad\text{for any constant }\gamma>0$$

proving the theorem and also Theorem 1.1. □

Added in proof: Recently T. Łuczak has proved independently a generalization of Theorem 1.1 for a broader range of $m=m(n)$.

Appendix

Several aspects of Lemma 4.1 are standard, and we will refer the reader to established proofs. In other cases where there are slight differences or we have had to tighten the probability estimates, we work in $G_{n,m}$ where $p=m/\binom{n}{2}$ and then use the following lemma (e.g. Bollobás [1, pp. 33–35]) to deduce the result in $G_{n,m}$.

Lemma A.1. (A1)

If a.e. $G_{n,p}\in A$ and A is a monotone property then a.e. $G_{n,m}\in A$.

$$\Pr(G_{n,m}\in A)\leqslant 3\sqrt{n}\log n \Pr(G_{n,p}\in A) \text{ for } n \text{ large.} \tag{A2}$$

Proof of Lemma 4.1. We first note (e.g. Bollobás [1, pp. 10–12]) that for $n/3\leqslant s\leqslant n$ and $h=\alpha sp$ where $\alpha\in(0,1)$ is fixed, then for sufficiently large n,

$$\frac{1}{3\sqrt{sp}}\exp\left(-\frac{\alpha^2(1+\alpha)}{2}sp\right)\leqslant\sum_{i=0}^{sp-h}\binom{s-1}{i}p^i(1-p)^{s-1-i}$$

$$\leqslant\frac{3}{\alpha\sqrt{sp}}\exp\left(-\frac{\alpha^2}{2}sp\right). \tag{A3}$$

(a)(i) By the Chebyshev inequality, $\Pr(|X|<\frac{1}{2}E(|X|))\leqslant\frac{4\mathrm{Var}(|X|)}{E(|X|)^2}$ but $\mathrm{Var}(|X|)=(1+o(1))E(|X|)$, and by (A3) $E(|X|)>2n^{22/25}$.

(b)(i)

$$\Pr(\exists s:|X(s)|>k)\leqslant\sum_{s=n/3}^{n}\binom{n}{k}\left(\sum_{i=0}^{sp/10}\binom{s-k}{i}p^i(1-p)^{s-k-i}\right)^k$$

$$\leqslant\sum_{s=n/3}^{n}\left(\frac{ne}{k}n^{-0.404s/n}\right)^k=o(1)$$

by (A3) when $\alpha=1-\frac{s}{10(s-k)}sp$ and $k=n^{1-0.4s/n}$.

(a)(ii) We compute an upper bound for the expected number of vertices Λ in V_n with at least 19 edges into X.

$$E(\Lambda)\leqslant\binom{n}{1}\binom{n-1}{1}p^{19}\left(\sum_{j=0}^{3/5np}\binom{n-20}{j}p^j(1-p)^{n-20-j}\right)^{19}$$

$$\leqslant n(np)^{19}(n^{-2/25})^{19}$$

$$=o(n^{-13/25}).$$

We now apply (A2) to deduce that the probability in $G_{n,m}$ is $o(1)$.

(b)(ii) We use the argument of (a)(ii) with $\lceil 8\frac{n}{s}\rceil$ edges, to obtain a simultaneous bound for $n/3\leqslant s\leqslant n$, and then use (A2).

(c) The number of edges occurring in an induced subgraph S of size s is a binomial random variable with parameters $\binom{s}{2}$ and p. By e.g. Bollobás [1, p. 14] we have for large deviations of binomial random variables

$$\Pr(\text{number of edges} \geqslant \alpha\binom{s}{2}p) < \left(\frac{e}{\alpha}\right)^{\alpha\binom{s}{2}p}.$$

Setting $\alpha = \frac{1}{125}\frac{n}{s}$ we see that

$$\sum_{s=1}^{n/375} \binom{n}{s}\left(\frac{e}{\alpha}\right)^{\alpha\binom{s}{2}p} = o(1).$$

(d) Bollobás [1, p. 191].

(e) Let $|U|,|W| \geqslant \frac{n}{\log\log n}$ then

$$\Pr(\exists\, U,W \subseteq V_n \text{ such that } |E_{UW}| < \sqrt{n})$$

$$\leqslant \sum_{r \geqslant n/\log\log n}\ \sum_{s \geqslant n/\log\log n} \binom{n}{r}\binom{n-r}{s} \sum_{i=0}^{\sqrt{n}} \binom{rs}{i} p^i (1-p)^{rs-i} = o\left(\frac{1}{n}\right).$$

We now use (A2).

(f) Bollobás [1, p. 191].

(g) Use Lemma 4.1 (b)(i) with $s = n$. □

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