

A RANDOM VARIANT OF THE GAME OF PLATES AND OLIVES

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ABSTRACT. The game of plates and olives was originally formulated by Nicolaescu and encodes the evolution of the topology of the sublevel sets of Morse functions. We consider a random variant of this game. The process starts with an empty table. There are four different types of moves: (1) add a new plate to the table, (2) combine two plates and their olives onto one plate, removing the second plate from the table, (3) add an olive to a plate, and (4) remove an olive from a plate. We show that with high probability the number of olives is linear as the total number of moves goes to infinity. Furthermore, we prove that the number of olives is concentrated around its expectation.

1. INTRODUCTION

The game of plates and olives is a purely combinatorial process that has an interesting application to topology and Morse theory. Morse theory involves the study of topological manifolds by considering the smooth functions on the manifolds. An *excellent Morse function* on the 2-sphere is a smooth function from $S^2 \rightarrow \mathbb{R}$ such that all the critical points are non-degenerate (i.e. the matrix of second partial derivatives is non-singular) and take distinct values.

If f is an excellent Morse function on the sphere, S^2 , with critical points x_1, \dots, x_m with $f(x_1) < \dots < f(x_m)$, a *slicing* of f is an increasing sequence a_0, \dots, a_m such that $a_0 < f(x_1) < a_1 < f(x_2) < \dots < a_{m-1} < f(x_m) < a_m$. Then two excellent Morse functions f and g , with the same number of critical points, are said to be *topologically equivalent* if for any slicing a_0, \dots, a_m of f and b_0, \dots, b_m of g , there is an orientation-preserving diffeomorphism (i.e. an isomorphism of smooth manifolds) between the sublevel sets $\{x \in S^2 \mid f(x) \leq a_i\}$ and $\{x \in S^2 \mid g(x) \leq b_i\}$ for each $0 \leq i \leq m$.

Loosely speaking, two excellent Morse functions are topologically equivalent if when their critical values are ordered as mentioned above, both functions have the same types of critical points (in terms of being local minima, maxima, or saddle points), appearing in the same order, and in a rough sense the same location relative to other critical values, which is necessary for the sublevel sets described above to diffeomorphic.

Morse functions on the sphere have exactly $2n + 2$ critical points, n of which are saddle points. It was shown in [7] that the sublevel sets $\{f(x) \leq a\}$ are topologically equivalent to either all of S^2 , or a finite (possibly empty) disjoint union of disks, each with at most a finite number of punctures (i.e. isolated “missing” points). As the value a crosses a critical point, one of the following four things will take place:

- (1) a new disk may appear,
- (2) two such disks may merge (preserving the punctures in both disks),
- (3) a new puncture may appear, or

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(4) a puncture may disappear.

Given a slicing a_0, \dots, a_m of the excellent Morse function f , we have that the first sublevel set, $\{f(x) \leq a_0\} = \emptyset$, and the second, $\{f(x) \leq a_1\}$ is a disk. The second to last sublevel set, $\{f(x) \leq a_{m-1}\}$ is also a disk, and the last sublevel set, $\{f(x) \leq a_m\}$, is the entire sphere, and this is the only sublevel set that is topologically equivalent to the sphere.

The game of plates and olives was originally formulated by Nicolaescu in [7] and encodes the evolution of the topology of the sublevel sets in a purely combinatorial process in which plates play the role of disks and olives represent punctures in the disks. The moves in the game of plates and olives are designed to resemble exactly the possible transformations that happen when a crosses a critical point:

- (1) add a plate,
- (2) combine two plates while keeping all the olives,
- (3) add an olive to a plate, or
- (4) remove an olive from a plate.

The game of plates and olives begins at an empty table and ends the first time we return to an empty table, signifying that the level sets of a Morse function start with the empty set and end with the entire sphere.

Let T_n^2 denote the number of excellent Morse functions on the 2-sphere with n saddle points, up to topological equivalence. A lower bound for T_n^2 was given by Nicolaescu in [7] by studying walks on Young's lattice. An upper bound on T_n^2 was given by Carroll and Galvin in [1] from studying the game of plates and olives directly. The bounds of these two papers give

$$(2/e)^{n+o(n)} n^n \leq T_n^2 \leq (4/e)^{n+o(n)} n^n.$$

Here we will study a random variant of the game of plates and olives.

THE MODEL

The process starts with an empty table. There are four different types of moves that can happen in the process. The four moves are as follows:

- (P^+) Add a plate; from every configuration we can add one empty plate to the table.
- (P^-) Combine two plates; assuming there are at least two plates, we can choose two of them (order does not matter), combine their olives onto one plate, and remove the other plate from the table.
- (O^+) Add an olive to a plate; we can choose any plate and add one olive to it.
- (O^-) Remove an olive from a plate; we can choose any non-empty plate and remove one olive from it.

We will describe the P^+ and P^- moves as plate moves, while the O^+ and O^- moves will be known as olive moves.

In our model, the plates are distinguishable, but the olives are not. At each time step in the process, one of the available moves will be chosen to be performed uniformly at random. It is worth noting that for the purposes of a P^- move, the order in which we choose the two plates does not matter, and does not give us two different P^- moves. To make this rigorous, we will assume that whenever a P^- move is made, the plate with less olives on it is removed, and if the two plates have the same number, the plate that was added to the table in an earlier time step is kept.

In addition to the random aspect, our model differs from the game of plates and olives only in that in our model, the plates are distinguishable, and we do not allow for the process to return to an empty table.

Our main result shows that the number of olives grows linearly with the number of steps in the process, and that the number of olives is concentrated. When we refer to an event occurring with high probability (w.h.p. for short), we mean that the probability of that event goes to 1 as the total number of moves, t , goes to infinity.

Theorem 1.1. *Let O_t be the total number of olives on the table in the preceding model at time t .*

(a) *There exist absolute constants $C > 0$ and $\frac{1}{342} \leq c_1 \leq c_2 \leq \frac{2}{3}$ such that*

$$\Pr(c_1 t \leq O_t \leq c_2 t) \geq 1 - e^{-Ct}. \quad (1)$$

(b) *Furthermore, there exists an absolute constant $A > 0$ such that for every $\delta \geq 0$ we have*

$$\Pr(|O_t - \mathbf{E}(O_t)| \geq \delta t) \leq e^{-A\delta^2 t} \quad (2)$$

(c) *Also, w.h.p. no plate, except for the first plate, has more than $B \log t$ olives at any time, for some absolute constant $B > 0$.*

We prove in Section 2 that

$$\frac{t}{342} \leq \mathbf{E}(O_t) \leq \frac{2t}{3}. \quad (3)$$

Next in Section 3 we derive the concentration result (2), which altogether will imply (1). In Section 4 we consider an auxiliary Markov chain process. It follows from our proofs that constants $1/342$ and $2/3$ are not optimal. As a matter of fact a computer simulation suggests that the number of olives O_t is concentrated around ct , where $c \approx 0.096$.

2. BOUNDS ON THE EXPECTED NUMBER OF OLIVES

2.1. Lower bound. We would like to show that the number of olives at a given time grows linearly with time t . Towards this, we will establish two facts:

- we expect to return to a single plate a linear number of times, and
- each time we return to a single plate, we expect to gain a positive number of olives.

This will give us a linear expectation.

When keeping track of the plates at any given time, our main concern is the number of olives on each plate. Due to this, in each state, we will consider the plates ordered based on how many olives the plate has on it. The first plate, or plate 1, will be the plate with the most olives, the second plate will be the plate with the next highest number of olives, and so forth. Ties between the sizes of plates can be broken arbitrarily without affecting our arguments. The most important plate to keep track of is plate 1, and while technically the plate that is labeled plate 1 can change, we will see that in practice this will not happen once enough time has passed in the process.

Now let us show that we expect to return to a single plate a linear number of times. If we have $\ell \geq 1$ plates, then the probability we do a plate move is at least

$$\frac{\binom{\ell}{2} + 1}{2\ell + \binom{\ell}{2} + 1} \geq 1/3.$$

Let t_{plate} be the random variable that counts the number of plate moves we have after t moves overall. Then

$$\mathbf{E}(t_{plate}) \geq \sum_{i=1}^t 1/3 = t/3. \quad (4)$$

Let X be the random variable which counts the number of times we transition from a state with two plates to one plate. We will consider only plate moves to get a lower bound on the random variable X .

We consider a related Markov chain. In this process, we will consider a random walk on the positive integers. We will start this walk at 1 (plate). If we are currently at 1, then we will move to 2 with probability 1. If we are currently at 2, we will move to 1 with probability 1/2 and to 3 with probability 1/2. If we are at $k \geq 3$, we will move to $k - 1$ with probability 3/4 and to $k + 1$ with probability 1/4. This Markov chain will be indexed by time t_{plate} as it only models plate moves.

Observe that in our model, for $\ell \geq 3$,

$$\Pr(P^- | \text{there are } \ell \text{ plates currently and we perform a plate move}) = \frac{\binom{\ell}{2}}{\binom{\ell}{2} + 1} \geq 3/4, \quad (5)$$

and

$$\Pr(P^+ | \text{there are } \ell \text{ plates currently and we perform a plate move}) = \frac{1}{\binom{\ell}{2} + 1} \leq 1/4.$$

Thus the Markov chain gives an underestimate for how often we transition from two plates to one plate. We present the following result about this Markov chain that will be proved in Section 4.

Theorem 2.1. *Consider a random walk on the positive integers: If we are currently at 1, then we will move to 2 with probability 1. If we are currently at 2, we will move to 1 with probability 1/2 and to 3 with probability 1/2. If we are at $k \geq 3$, we will move to $k - 1$ with probability 3/4 and to $k + 1$ with probability 1/4.*

Let $N_{1,1}(t)$ be the random variable that counts the number of times we return to state 1 when we start the walk at state 1. Then w.h.p. we have that $N_{1,1}(t) \geq t/19$.

While it will not be used until the next section, it is convenient to define here the random variable $F_{1,1}(t)$, which is the number of times the Markov chain described in Theorem 2.1 returns to 1 in the first t moves, given that the Markov chain started at 1. In section 4 (see equation (15)), we will show that

$$F_{1,1}(t) = \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} 4 \left(\frac{3}{16} \right)^{j-1} \binom{2j-3}{j-1}.$$

We will apply Theorem 2.1 with t replaced by t_{plate} since the related Markov chain only tracks plate moves. This gives us that $\mathbf{E}(N_{1,1}(t_{plate})) \geq \mathbf{E}(t_{plate})/19$. Note that $\mathbf{E}(N_{1,1}(t_{plate})) \leq \mathbf{E}(X)$, so

$$\mathbf{E}(X) \geq \mathbf{E}(t_{plate})/19 \geq t/57.$$

Now we explore what happens each time we transition from two plates to one plate. Consider a state in the process that currently has two plates. If the second plate currently has olives on it, then the probability that next time we make a plate move, plate 2 still

has olives is at least $1/2$. (We can immediately make the plate move.) If the second plate currently has no olives on it, then the probability there is at least one olive on it when we make the next plate move is at least $1/6$. ($1/3$ probability to add an olive, $1/2$ probability to perform a plate move once we have added the olive.) Thus we have at least a $1/6$ chance of adding an olive to the first plate each time we reduce the number of plates to 1. Let Y be a random variable that counts the number of times we add at least one olive to the first plate from a plate move, given that we transition from 2 plates to 1 plate X times. Then

$$\mathbf{E}(Y) \geq \mathbf{E}(X)/6.$$

Now we put everything together. We will only consider olives on the first plate. Let $O_t^{(1)}$ denote the number of olives on plate 1 at time t . Let $O_t^{(1)+}$ and $O_t^{(1)-}$ denote the total number of olives that were added to (respectively subtracted from) plate 1 from O^+ (respectively O^-) moves. Note that $\mathbf{E}(O_t^{(1)+} - O_t^{(1)-}) \geq 0$ since the probability of performing an O^+ move is always at least the probability of performing an O^- move. Finally, let O_t^{plate} denote the total number of olives added to the first plate from plate moves. Then $O_t^{plate} \geq Y$. This gives us that

$$\mathbf{E}(O_t) \geq \mathbf{E}(O_t^{(1)}) = \mathbf{E}(O_t^{(1)+} - O_t^{(1)-} + O_t^{plate}) \geq \mathbf{E}(Y) \geq \mathbf{E}(X)/6 \geq t/342. \quad (6)$$

2.2. Upper bound. Now we bound the expected value of O_t from above. Let O_t^+ and O_t^- be the random variables that count the number of O^+ moves and the number of O^- moves after t total moves, respectively. Clearly,

$$O_t = O_t^+ - O_t^- = t - t_{plate} - 2O_t^- \leq t - t_{plate}.$$

Thus, by (4) we conclude that

$$\mathbf{E}(O_t) \leq t - \mathbf{E}(t_{plate}) \leq 2t/3.$$

And this proves (3).

3. CONCENTRATION

Suppose that we transition from a state with two plates to a state with a unique plate at times t_1, t_2, \dots, t_m and recall that O_t denotes the number of olives at time t . Define $t_0 := 1$. Let $X_i = O_{t_{i+1}} - O_{t_i}$. Then the X_i are independent random variables and based on the previous section we have $\mathbf{E}(X_i) \geq 1/342$. Then $S_m := O_{t_m} = \sum_{i=0}^m X_i$. We can argue for concentration of S_m as follows.

Note that from (4) $\mathbf{E}(t_{plate}) \geq t/3$, and by the Chernoff bound (see, e.g., equation 21.24 in [8]) we have for any $0 \leq \delta \leq 1$,

$$\Pr(t_{plate} < (1 - \delta)t/3) \leq e^{-\frac{\delta^2 t}{6}}.$$

As we are not trying to optimize the constant A , we can be imprecise here and choose $\delta = 1/4$, giving

$$\Pr(t_{plate} < t/4) \leq e^{-\frac{t}{96}}. \quad (7)$$

If $t_{plate} \geq t/4$, then the probability that we start at a state with a single plate, add plates, and then return to a state with a single plate before t moves is at least $F_{1,1}(t/4)$, which is

the probability that our related Markov chain, defined in Section 2 and studied in Section 4, returns to 1 at least once in the first $t/4$ moves, assuming it started at 1. Clearly,

$$\Pr(X_i \geq k) \leq \Pr(t_{i+1} - t_i \geq k)$$

and also

$$\Pr(t_{i+1} - t_i < k) \geq F_{1,1}(k/4) - \Pr(\text{less than } k/4 \text{ plate moves happen in the first } k \text{ moves}).$$

Thus, if $k' = k/4$, then by (15) we get that

$$\begin{aligned} \Pr(X_i \geq k) &\leq 1 - F_{1,1}(k') + e^{-\frac{k}{96}} = \sum_{j=\lfloor \frac{k'}{2} \rfloor}^{\infty} 4 \left(\frac{3}{16}\right)^{j-1} \binom{2j-3}{j-1} + e^{-\frac{k}{96}} \\ &\leq 4 \sum_{j=\lfloor \frac{k'}{2} \rfloor}^{\infty} \left(\frac{3}{4}\right)^{j-1} + e^{-\frac{k}{96}} \leq C' \zeta^{k'} + e^{-\frac{k}{96}} \leq C \rho^k, \end{aligned} \quad (8)$$

where $\zeta = 3^{1/2}/2 < 1$, $\rho = \max\{\zeta^{1/4}, e^{-1/96}\}$ and $C, C' > 0$ are constants. Let $\mu_i = \mathbf{E}(X_i)$ and $\mu = \mu_1 + \dots + \mu_m$. Note that we have

$$\mu_i \leq \sum_{k=1}^{\infty} k C \rho^k = \frac{C \rho}{(\rho - 1)^2} < \infty. \quad (9)$$

Now we can easily prove a concentration result for this situation. We modify an argument from [2]. We prove

$$\Pr(|S_m - \mu| \geq \delta m) \leq e^{-A \delta^2 m} \quad (10)$$

for some constant $A > 0$. That means we have replaced (6) by a concentration inequality.

We write, for $\lambda > 0$ such that $e^\lambda < 1/\rho$,

$$\mathbf{E}(X_i^2 e^{\lambda X_i}) = \sum_{k=0}^{\infty} k^2 e^{\lambda k} \Pr(X_i = k) \leq C \sum_{k=0}^{\infty} k^2 (\rho e^\lambda)^k \leq \frac{3C}{(1 - \rho e^\lambda)^3}.$$

Now $e^x \leq 1 + x + x^2 e^x$ for $x \geq 0$, and so, using the above, we have

$$\mathbf{E}(e^{\lambda X_i}) \leq 1 + \lambda \mu_i + \lambda^2 \left(\frac{3C}{(1 - \rho e^\lambda)^3} \right) < 1 + \lambda \mu_i + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3} \right).$$

Since $\Pr(S_m \geq \mu_i m + \delta m) = \Pr(e^{\lambda S_m} \geq e^{\lambda(\mu_i m + \delta m)})$ and X_i s are independent, the Markov bound implies that

$$\begin{aligned} \Pr(S_m \geq \mu_i m + \delta m) &\leq e^{-\lambda(\mu_i m + \delta m)} \prod_{i=1}^m \mathbf{E}(e^{\lambda X_i}) \\ &\leq \exp\{-\lambda(\mu_i m + \delta m)\} \cdot \left(1 + \lambda \mu_i + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3} \right) \right)^m \\ &\leq \exp\{-\lambda(\mu_i m + \delta m)\} \cdot \exp\left\{ \left(\lambda \mu_i + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3} \right) \right) m \right\} \\ &= \exp\left\{ -\lambda \delta m + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3} \right) m \right\} \\ &\leq \exp\{-\lambda \delta m + \lambda^2 (1 + 3C \varepsilon^{-3}) m\}, \end{aligned}$$

where $\varepsilon = \varepsilon(\delta) > 0$ is a constant such that

$$e^\lambda \leq (1 - \varepsilon)/\rho. \quad (11)$$

Now choose $\lambda = \delta/(2(1 + 3C\varepsilon^{-3}))$ and ε such that (11) holds. Such a choice of ε is always possible since as $\varepsilon \rightarrow 0$, $\exp\{\delta/(2(1 + 3C\varepsilon^{-3}))\} \rightarrow 1$ and $(1 - \varepsilon)/\rho \rightarrow 1/\rho > 1$. Then

$$\Pr(S_m \geq \mu + \delta m) \leq \max_{1 \leq i \leq m} \Pr(S_m \geq \mu_i m + \delta m) \leq \exp\left\{-\frac{\delta^2 m}{4(1 + 3\varepsilon^{-3})}\right\}.$$

To bound $\Pr(S_m \leq \mu - \delta m)$, we proceed similarly. We have $e^{-x} \leq 1 - x + x^2 e^x$, so

$$\mathbf{E}(e^{-\lambda X_i}) \leq 1 - \lambda \mu_i + \lambda^2 \left(\frac{3C}{(1 - \rho e^\lambda)^3}\right) < 1 - \lambda \mu_i + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3}\right)$$

and

$$\begin{aligned} \Pr(S_m \leq \mu_i m - \delta m) &\leq e^{\lambda(\mu_i m - \delta m)} \prod_{i=1}^m \mathbf{E}(e^{-\lambda X_i}) \\ &\leq \exp\{\lambda(\mu_i m - \delta m)\} \cdot \left(1 - \lambda \mu_i + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3}\right)\right)^m \\ &\leq \exp\{\lambda(\mu_i m - \delta m)\} \cdot \exp\left\{\left(-\lambda \mu_i + \lambda^2 \left(1 + \frac{3C}{(1 - \rho e^\lambda)^3}\right)\right) m\right\} \\ &\leq \exp\{-\lambda \delta m + \lambda^2(1 + 3C\varepsilon^{-3})m\}, \end{aligned}$$

and we can proceed as before. This completes the proof of (10).

For (10) to be useful, we need to show that w.h.p. m is linear in t . We condition on performing a plate move. Let the random variables τ_i for $1 \leq i < \infty$ count how many times we have exactly i plates after t steps, where we only count when the number of plates change. So, if we are at e.g. two plates and we make three olive moves before the next plate move, we only count this as having two plates once. Then $t_{plate} = \sum_{i=1}^{\infty} \tau_i$ and $\tau_1 = m$. Note that we can express τ_1 as a sum of τ_2 indicator random variables that denotes if on the j th time we are at two plates, we then transition to one plate. Note that the probability of such a transition is $1/2$ (when we have exactly two plates, there is one way to remove a plate and one way to add a plate, giving equal probability of moving to 1 plate vs. 3 plates) and so $\mathbf{E}(\tau_1) = \tau_2/2$.

We will consider two cases based on the value of τ_2 . If $\tau_2 \geq 3t_{plate}/19$, we have from equation (7) and the Chernoff bound,

$$\begin{aligned} \Pr(\tau_1 \leq t/76 = (1/3)(1/4)(3/19)t) &\leq \Pr(\tau_1 < \tau_2/3) + \Pr(t_{plate} < t/4) \\ &\leq e^{-\tau_2/18} + e^{-t/96} \leq e^{-t_{plate}/114} + e^{-t/96} \leq e^{-t/456} + e^{-t/96}. \end{aligned}$$

Now, if $\tau_2 < 3t_{plate}/19$ and $\tau_1 < t_{plate}/19$, then $\tau_{\geq 3} := \sum_{i=3}^{\infty} \tau_i \geq 15t_{plate}/19$. Now, let $L_{\geq 3}$ be the random variable that counts how many times we have at least three plates and we remove a plate. Due to (5) we have at least a $3/4$ probability of removing a plate whenever we make a plate move, so by the Chernoff bound, we have

$$\Pr(L_{\geq 3} \leq 3\tau_{\geq 3}/5) \leq e^{-3\tau_{\geq 3}/200} \leq e^{-9t_{plate}/950} \leq e^{-9t/3800} + e^{-t/96}.$$

Thus w.h.p. when we have at least three plates, we remove plates at least $3/5$ of the time and add them at most $2/5$ of the time. This implies that we must transition from two

plates to three plates at least $\tau_{\geq 3}/5$ times to make up for the discrepancy. This implies trivially that $\tau_2 \geq \tau_{\geq 3}/5 \geq 3t_{\text{plate}}/19 > \tau_2$, a contradiction. Thus, there exists an absolute constant $D > 0$ such that

$$m = \tau_1 \geq \frac{t}{76} \text{ with probability at least } 1 - e^{-Dt}. \quad (12)$$

Now observe that (8) also implies that for $k = \log_\rho(1/Ct^2) = B \log t$ (for some constant $B > 0$) we have

$$\Pr(t_{i+1} - t_i \geq k) \leq C\rho^k = 1/t^2 \quad (13)$$

and so

$$\Pr\left(\bigcup_{1 \leq i \leq m} (t_{i+1} - t_i \geq k)\right) \leq t \cdot \frac{1}{t^2} = o(1). \quad (14)$$

Note that between time t_i and t_{i+1} the number of olives at any plate different from the first one is at most $t_{i+1} - t_i$ and so (14) implies that w.h.p. no plate, except for the first plate has more than $B \log t$ olives at any time. Part (c) of Theorem 1.1 follows directly from (14).

Now let $T = O_t - O_{t_m} = O_t - S_m$. Then, by (8) we have $\Pr(T \geq k) \leq C\rho^k$ and the triangle inequality implies

$$|O_t - \mathbf{E}(O_t)| = |T + S_m - \mathbf{E}(O_t)| \leq |S_m - \mu| + |T + \mu - \mathbf{E}(O_t)| = |S_m - \mu| + |T - \mathbf{E}(T)|.$$

Thus,

$$\Pr(|O_t - \mathbf{E}(O_t)| \geq \delta t) \leq \Pr(|S_m - \mu| \geq \delta t/2) + \Pr(|T - \mathbf{E}(T)| \geq \delta t/2).$$

Furthermore, since $T \geq 0$ and $\mathbf{E}(T) = O(1)$ (cf. (9)) we get that $\Pr(|T - \mathbf{E}(T)| \geq \delta t/2) \leq \Pr(T \geq \delta t/2)$. Hence,

$$\begin{aligned} \Pr(|O_t - \mathbf{E}(O_t)| \geq \delta t) &\leq \Pr(|S_m - \mu| \geq \delta t/2) + \Pr(T \geq \delta t/2) \\ &\leq \Pr(|S_m - \mu| \geq \delta m/2) + \Pr(T \geq \delta m/2) \\ &\leq e^{-A\delta^2 m/4} + C\rho^{\delta m/2}. \end{aligned}$$

This together with (12) proves Theorem 1.1(b) and this completes the proof of Theorem 1.1.

4. PROOF OF THEOREM 2.1

For ease of reading, we restate Theorem 2.1 here.

Theorem 2.1. *Consider a random walk on the positive integers: If we are currently at 1, then we will move to 2 with probability 1. If we are currently at 2, we will move to 1 with probability 1/2 and to 3 with probability 1/2. If we are at $k \geq 3$, we will move to $k - 1$ with probability 3/4 and to $k + 1$ with probability 1/4.*

Let $N_{1,1}(t)$ be the random variable that counts the number of times we return to state 1 when we start the walk at state 1. Then w.h.p. we have that $N_{1,1}(t) \geq t/19$.

Proof. Notice that we cannot return to 1 in an odd number of steps. Let $f_{1,1}(2t)$ be the probability that the first time we return to state 1 after $2t$ steps, given that we start at state 1 i.e. the probability that the *first return time* is $2t$. Let X_j be the location at time j . So $X_0 = X_{2t} = 1$, $X_1 = X_{2t-1} = 2$ and $X_j \neq 1$ for each $2 < j < 2t - 2$. Furthermore, we

need to control the number of steps at which $X_j = 2$. Assume that at exactly $(i + 1)$ steps we have that $X_j = 2$, where $1 \leq i \leq t - 1$. That means

$$X_1 = X_{1+2a_1} = X_{1+2a_1+2a_2} = \cdots = X_{1+2a_1+\cdots+2a_i} = 2,$$

where $1 + 2a_1 + \cdots + 2a_i = 2t - 1$ and $a_j \geq 1$. Hence,

$$\begin{aligned} f_{1,1}(2t) &= \Pr(X_{2t} = 1, X_1 \neq 1, \dots, X_{2t-1} \neq 1 \mid X_0 = 1) \\ &= \sum_{i=1}^{t-1} \sum_{a_1+\cdots+a_i=t-1} \prod_{j=1}^i C_{a_j-1} \left(\frac{1}{2}\right)^{i+1} \left(\frac{1}{4}\right)^{t-i-2} \left(\frac{3}{4}\right)^{t-1}, \end{aligned}$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ is the *Catalan number*. Now the Catalan i -fold convolution formula (see, e.g., [6]) gives that

$$\sum_{a_1+\cdots+a_i=t-1} \prod_{j=1}^i C_{a_j-1} = \frac{i}{2t-i-2} \binom{2t-i-2}{t-1}.$$

Thus,

$$f_{1,1}(2t) = \sum_{i=1}^{t-1} \frac{i}{2t-i-2} \binom{2t-i-2}{t-1} \left(\frac{1}{2}\right)^{i+1} \left(\frac{1}{4}\right)^{t-i-2} \left(\frac{3}{4}\right)^{t-1}$$

and equivalently by replacing i by $k = t - i - 1$ we get

$$f_{1,1}(2t) = 4 \left(\frac{3}{16}\right)^{t-1} \cdot \sum_{k=0}^{t-2} \binom{(t-1)+k}{k} \frac{(t-1)-k}{(t-1)+k} 2^{(t-2)-k}.$$

Now we apply the following identity (see, e.g., (1.12) in [4])

$$\sum_{k=0}^n \binom{x+k}{k} \frac{x-k}{x+k} 2^{n-k} = \binom{x+n}{n}$$

with $n = t - 2$ and $x = t - 1$ to conclude that

$$f_{1,1}(2t) = 4 \left(\frac{3}{16}\right)^{t-1} \binom{2t-3}{t-1}.$$

Consequently, the probability $F_{1,1}(t)$, given $X_0 = 1$, that we return to 1 at some point in the first t steps is given by

$$F_{1,1}(t) = \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} f_{1,1}(2j) = \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} 4 \left(\frac{3}{16}\right)^{j-1} \binom{2j-3}{j-1}. \quad (15)$$

By (5.7) in [5], we can calculate the mean time $\bar{T}_{1,1}$, to return to state 1 after starting at state 1 by

$$\bar{T}_{1,1} = \sum_{t=0}^{\infty} \Pr(T_{1,1} \geq t) = 1 + \sum_{t=1}^{\infty} (1 - F_{1,1}(t)) = 1 + \sum_{t=1}^{\infty} \sum_{j=\lfloor \frac{t}{2} \rfloor + 1}^{\infty} 4 \left(\frac{3}{16}\right)^{j-1} \binom{2j-3}{j-1}.$$

Now we will evaluate the latter double sum. First observe that

$$\sum_{t=1}^{\infty} \sum_{j=\lfloor \frac{t}{2} \rfloor + 1}^{\infty} a_j = 2 \sum_{j=1}^{\infty} j a_j - \sum_{j=1}^{\infty} a_j = 2 \sum_{j=1}^{\infty} (j-1) a_j + \sum_{j=1}^{\infty} a_j$$

assuming that all a_j s are nonnegative. Thus,

$$\begin{aligned}\bar{T}_{1,1} &= 1 + 8 \sum_{j=1}^{\infty} (j-1) \left(\frac{3}{16}\right)^{j-1} \binom{2j-3}{j-1} + 4 \sum_{j=1}^{\infty} \left(\frac{3}{16}\right)^{j-1} \binom{2j-3}{j-1} \\ &= 1 + 8 \sum_{k=1}^{\infty} k \left(\frac{3}{16}\right)^k \binom{2k-1}{k} + 4 \sum_{k=0}^{\infty} \left(\frac{3}{16}\right)^k \binom{2k-1}{k}.\end{aligned}$$

Now we will use the following identities that can be obtained by an application of Newton's binomial theorem (see, e.g., (1.30) and (1.3) in [3]):

$$\sum_{k=1}^{\infty} k \binom{2k}{k} \left(\frac{x}{4}\right)^k = \frac{x}{2(1-x)^{3/2}} \quad (16)$$

and

$$\sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{x}{4}\right)^k = \frac{1}{\sqrt{1-x}} \quad (17)$$

for $|x| < 1$. Applying (16) with $x = \frac{3}{4}$ and using $\binom{2k-1}{k} = \frac{1}{2} \binom{2k}{k}$ for $k \geq 1$ yields

$$8 \sum_{k=1}^{\infty} k \left(\frac{3}{16}\right)^k \binom{2k-1}{k} = 4 \sum_{k=1}^{\infty} k \left(\frac{3}{16}\right)^k \binom{2k}{k} = 4 \cdot \frac{3/4}{2(1-3/4)^{3/2}} = 12.$$

Similarly, we apply (17) to get

$$4 \sum_{k=0}^{\infty} \left(\frac{3}{16}\right)^k \binom{2k-1}{k} = 4 + 2 \sum_{k=1}^{\infty} \left(\frac{3}{16}\right)^k \binom{2k}{k} = 4 + 2 \left(\frac{1}{\sqrt{1-3/4}} - 1 \right) = 6.$$

Consequently,

$$\bar{T}_{1,1} = 1 + 12 + 6 = 19.$$

Let $N_{1,1}(t)$ be the number of times that, given starting at state 1, we return to state 1 in the first t moves. By (5.8) in [5], we have that w.h.p.

$$\lim_{t \rightarrow \infty} N_{1,1}(t)/t = 1/\bar{T}_{1,1} = 1/19.$$

□

5. DIFFERENCES BETWEEN OUR MODEL AND THE ORIGINAL GAME OF PLATES AND OLIVES

Aside from the fact that moves are chosen randomly in our model, the two ways in which our model differs from the original game of plates and olives discussed in [7] and [1] involve the distinguishability of plates and the possibility of returning to an empty table. This section briefly explores these adaptations.

First, we choose to distinguish plates in our model in order to reduce the parameters that need to be tracked to understand the process. If plates are not distinguishable, then the number of plate moves at any given time does not depend on how many plates are on the table, but instead depends on how many different numbers there are such that there exists a plate with that many olives on it, and also how many of those numbers have at least two plates with that many olives on it. Furthermore, when the plates are indistinguishable,

performing a plate move may not actually change the total number of plate moves possible, so a much more careful analysis is required.

In addition to the distinguishability of plates, our model also does not allow for the return to an empty table. In this case, we do not lose much since as we can see, the number of olives grows off to infinity, so we do not expect to get to a state where we could return to an empty table even if it was allowed. A question originally posed by Nicolaescu in [9] asks about if when the game of plates and olives is played randomly it would return to an empty table in finite time. Considering that in our model the number of olives tends towards infinity, it is the belief of the authors that if our work was adapted to allow for indistinguishable plates, it would answer the question of Nicolaescu in the negative. To see this more easily though, one needs only notice that the total number of olives on the table at a given time can be modeled by a random walk on the non-negative integers similarly to how we model the plate moves in this paper. The expected time for this random walk to return to 0 is bounded below by the expected time for a symmetric random walk on the integers to return to 0 since in any state there are always at least as many ways to add an olive as there are to remove one. Symmetric random walks on the integers are well-known to take infinite time to return to 0 in expectation, and so one does not expect the game of plates and olives to end in finite time when played randomly.

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