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ON MATCHINGS AND HAMILTONIAN CYCLES IN RANDOM GRAPHS

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Let $m=\frac{1}{2}n\log n+\frac{1}{2}n\log\log n+c_n$. Let Γ denote the set of graphs with vertices $\{1,2,\dots,n\}$, m edges and minimum degree 1. We show that if a random graph G is chosen uniformly from Γ then

$$\lim_{n\to\infty}\Pr(G\text{ has a perfect matching})=\begin{cases}0 & \text{if } c_n\to-\infty,\text{ sufficiently slowly,}\\ e^{-e^{-4c}/8} & \text{if } c_n\to c,\\ 1 & \text{if } c_n\to+\infty.\end{cases}$$

We also show that if a random graph G with vertices $\{1,2,\dots,n\}$ is constructed by randomly adding edges one at a time then, almost surely, as soon as G has degree k, G has $\lfloor k/2\rfloor$ disjoint hamiltonian cycles plus a disjoint perfect matching if k is odd, where k is a fixed positive integer.

1.

Let $G_{n,m}$ denote a random graph with vertices $\{1,2,\dots,n\}$ and m edges where each of the $\left(\binom{n}{2}\atop m\right)$ possible graphs is equally likely to be chosen.

Erdös and Rényi [5] showed that if $m=\frac{1}{2}n\log n+c_n n$ then

$$\lim_{n\to\infty}\Pr(\mu(G_{n,m})=\lfloor n/2\rfloor)=\begin{cases}0 & \text{if } c_n\to-\infty,\\ e^{-e^{-2c}} & \text{if } c_n\to c,\\ 1 & \text{if } c_n\to+\infty,\end{cases}\tag{1.1}$$

where $\mu(G)$ denotes the maximum cardinality of a matching in a graph G.

The probabilities in (1.1) are the limiting probabilities for $\delta(G_{n,m})\geqslant 1$, where $\delta(G)$ denotes the minimum vertex degree of a graph G. Thus Erdös and Rényi proved (1.1) by showing

$$\lim_{n\to\infty}\Pr(\mu(G_{n,m}^{(1)})=\lfloor n/2\rfloor)=1\,, \tag{1.2}$$

where $G_{n,m}^{(1)}$ denotes a random graph chosen uniformly from the set of graphs with vertices $\{1,2,\ldots,n\}$, m edges and minimum degree 1.

The first result of this paper is to tighten (1.2) and prove

Theorem 1.1. *Let* $m=\frac{1}{4}n\log n+\frac{1}{2}n\log\log n+c_n n$, *then*

$$\lim_{n\to\infty}\Pr(\mu(G_{n,m}^{(1)})=\lfloor n/2\rfloor)=\begin{cases}0 & \textit{if}\quad c_n\to-\infty\,,\ \textit{sufficiently slowly,}\\ e^{-e^{-4c}/8} & \textit{if}\quad c_n\to c,\\ 1 & \textit{if}\quad c_n\to+\infty.\end{cases}$$

There is at present, an unfortunate restriction $|c_n|=o(\log\log n)$ for $c_n\to-\infty$. We cannot at present relax this because of the difficulty of dealing with the conditioning of $\delta(G_{n,m})\geqslant 1$. Note that some restriction must be placed on the growth rate of $|c_n|$ when $c_n\to-\infty$ as

$$\Pr(\mu(G_{n,\lceil n/2\rceil}^{(1)})=\lfloor n/2\rfloor)=1\,.$$

Our second result is a generalization of one stated by Komlós and Szemerédi [13]. To state this we need to define the following: a *graph process* $\tilde{G}_n=(G_0, G_1, \ldots, G_m, \ldots)$ is a Markov process in which G_m is a graph with vertices $V_n=\{1, 2, \ldots, n\}$ and edges E_m, where $|E_m|=m$. G_m is obtained from G_{m-1} by choosing $e\in V_n^{(2)}-E_{m-1}$ uniformly at random and putting $E_m=E_{m-1}\cup\{e\}$. Note that G_m above is distributed exactly as $G_{n,m}$.

For a graph property Π (usually monotone) and graph process $\tilde{G}_n$ let

$$\tau(\Gamma,\Pi)=\min(m: G_m\in\Pi)\,.$$

In particular let

Π_k='The minimum degree of G is at least k'

and

$\tilde{\Pi}_k$='G has $\lfloor k/2\rfloor$ disjoint hamiltonian cycles plus a disjoint matching if k is odd.'

Our second result is

Theorem 1.2. *If k is a fixed positive integer then*

$$\lim_{n\to\infty} \Pr(\tau(\Gamma, \Pi_k)=\tau(\Gamma, \tilde{\Pi}_k))=1.$$

Komlós and Szemerédi stated this result for $k=2$. Note that Theorem 1.2 is most clearly stated as: if we randomly add edges one by one then when the graph constructed has minimum degree k then it a.s. has $\lfloor k/2 \rfloor$ disjoint hamiltonian cycles plus a disjoint matching if k is odd.

For other results on matchings and hamiltonian cycles in random graphs see Bollobás [2], Bollobás, Fenner and Frieze [4], Fenner and Frieze [7], [8], Frieze [10], [11], [12], Richmond, Robinson and Wormald [14], Richmond and Wormald [15], Robinson and Wormald [16], Shamir [17], and Shamir and Upfal [18], [19].

Notation

For a graph G we let $V(G)$ denote its set of vertices and $E(G)$ denote its set of edges.

For $v \in V(G)$, $d_G(v)$ is the degree of v, and for $S \subseteq V(G)$, $N_G(S)=\{v \notin S$: there exists $w \in S$ such that $\{v, w\} \in E(G)\}$.

For non-negative x, $V_x(G)=\{v \in V(G): d_G(v) \geqslant x\}$. For $S \subseteq V(G)$, $G[S]=(S, E_S)$, where $E_S=\{e \in E(G): e \subseteq S\}$.

Let $D_1=D_1(G)$ be the set of vertices of degree 1 in G and let $\psi(G)=G[V_2(G) - N_G(D_1)]$.

For $e \in E(G)$ we let $G-e=(V(G), E(G)-\{e\})$ and for $e \notin E(G)$ we let $G+e =(V(G), E(G) \cup \{e\})$.

For $0 \leqslant p \leqslant 1$, $G_{n,p}$ denotes a random graph with vertices $\{1,2, \ldots, n\}$ in which each of the $\binom{n}{2}$ possible edges is chosen with probability p and not chosen with probability $1-p$.

2.

Throughout this section $m=n \log n/4+n \log\log n/2+c_n n$ where for the moment we assume $|c_n| \nrightarrow -\infty$. The proof of Theorem 1.1. is obtained by a sequence of lemmas.

Lemma 2.1. *Let $G=G_{n,m}$, $LARGE=V_{\log n/100}(G)$ and $SMALL=V(G)-LARGE$. Consider the following conditions:*

$$\textit{No cycle of length 3 contains 2 small vertices;} \qquad (2.1a)$$

No path of length 2 *contains* 3 *small vertices*; (2.1b)

$S \subseteq V(G)$, $4 \leqslant |S| \leqslant 7$, $|S \cap SMALL| \geqslant 3$ *implies* $G[S]$ *is not connected*; (2.1c)

$|SMALL| \leqslant n^{.55}$; (2.1d)

$\varphi \neq S \subseteq LARGE$, $|S| \leqslant n/\log n$ *implies* $|N_G(S)| \geqslant (\log n/1000)|S|$; (2.1e)

No vertex has degree exceeding $5\log n$. (2.1f)

Then for n large

$$\Pr(G_{n,m} \text{ fails to satisfy } (2.1)) \leqslant n^{-.35}. \tag{2.2}$$

Proof (Outline). To estimate the probabilities for (2.1a), (2.1b), (2.1c), (2.1f) we simply compute the expected number of triangles containing 2 small vertices, etc. This is tedious but straightforward.

To deal with (2.1d), (2.1e) we let $p = (\log n/2 + \log\log n + 2c_n)/n$ and consider the random graph $G_{n,p}$.

As $|E(G_{n,p})|$ is a binomial random variable with parameters $\binom{n}{2}$ and p, it is easy to verify that

$$\Pr(|E(G_{n,p})| = m) \geqslant \tfrac{1}{2}(n\log n)^{-\frac{1}{2}} \text{ for } n \text{ large}. \tag{2.3}$$

Also

$$G_{n,p} \text{ conditional on } |E(G_{n,p})| = m \text{ is distributed exactly as } G_{n,m}. \tag{2.4}$$

Thus for any property Π

$$\Pr(G_{n,m} \text{ has } \Pi) \leqslant 2(n\log n)^{\frac{1}{2}} \Pr(G_{n,p} \text{ has } \Pi). \tag{2.5}$$

We show next that

$$\Pr(G_{n,p} \text{ violates } (2.1\text{d})) = O(n^{-\varepsilon n^{.55}}) \text{ for some } \varepsilon > 0 \tag{2.6}$$

and

$$\Pr(G_{n,p} \text{ violates } (2.1\text{e})) = O(n^{-26}). \tag{2.7}$$

Lemma 2.1 is completed using (2.5), (2.6) and (2.7).

Proof of (2.6). $\Pr(G_{n,p}$ violates (2.1d)$) \leqslant \Pr($there exists S, $s = |S| = \lceil n^{.55} \rceil$ and each $v \in S$ is adjacent to fewer than $\log n/100$ vertices in $V(G) - S)$

$$\leqslant \binom{n}{s} \Big(\sum_{k=0}^{\log n/100} \binom{n-s}{k} p^k (1-p)^{n-s-k} \Big)^s = O(n^{-\varepsilon s}).$$

Proof of (2.7). We first consider the case $1 \leqslant |S| \leqslant n/(\log n)^3$ and note that if (2.1e) fails then, where $s = |S|$, $G[S \cup N_G(S)]$ has at most $(\log n/1000+1)s$ vertices and at least $(\log n/200)s$ edges. The probability of this happening is, for large n, no more than

$$\sum_{r=1}^{n/(\log n)^3} \binom{n}{r} \sum_{k=4r}^{\binom{r}{2}} \binom{\binom{r}{2}}{k} p^k (1-p)^{\binom{r}{2}-k} = \mathrm{O}(n^{-26}).$$

For $s > n/(\log n)^3$ we need not restrict $S \subseteq LARGE$ and then the probability that (2.1e) fails is no more than

$$\sum_{s=n/(\log n)^3}^{n/\log n} \binom{n}{s} \sum_{k=0}^{(\log n/1000)s} \binom{n-s}{k} (1-(1-p)^s)^k (1-p)^{s(n-s-k)}$$
$$= \mathrm{O}(n^{-\varepsilon n/(\log n)^3}). \quad \square$$

Let $\mathscr{G}_0 = \mathscr{G}_0(n)$ denote the set of graphs with vertices $\{1, 2, \ldots, n\}$ and m edges. Let $\mathscr{G}_1 = \mathscr{G}_1(n)$ denote the set of graphs in $\mathscr{G}_0$ that satisfy (2.1). We prove the following lemma on the neighborhoods of sets of vertices.

Lemma 2.2. *Let $G \in \mathscr{G}_1$ and $X \subseteq E(G)$ be a matching of G that does not meet any small vertex. Let $H = \psi((V(G), E(G)-X))$. Then for n large we have*

$$0 \neq S \subseteq V(H), \quad |S| \leqslant n/8000 \ \textit{implies}\ |N_H(S)| \geqslant |S|. \tag{2.8}$$

Proof. Let $T = N_G(D_1)$ and let $S_1 = S \cap SMALL$ and $S_2 = S - S_1$. We note first that (2.1) implies that no large vertex is adjacent to 3 small vertices and no large vertex is adjacent to 3 members of T. Hence

$$|N_H(S)| \geqslant |N_H(S_1)| - |S_2| + |N_G(S_2)| - 3|S_2| - \min(|S_1|, 2|S_2|), \tag{2.9}$$

where the factor 3 in (2.9) accounts also for the deletion of X. We must now prove that

$$|N_H(S_1)| \geqslant |S_1|. \tag{2.10}$$

Note next that (2.1b) implies $H[S_1]$ consists of isolated vertices and edges. So let $\{u, v\}$ be any edge of $H[S_1]$. Then (2.1c) implies

neither u nor v have a neighbor in common with any other vertex of S_1; (2.11a)

neither u nor v have a neighbor in T. (2.11b)

Also (2.1a) implies that

$$u \text{ and } v \text{ have no common neighbor.} \tag{2.11c}$$

Now consider the components of the graph induced by the isolated vertices I of $H[S_1]$ and their neighbors in G. Let C be the set of vertices of such a component.

$$|C \cap I| = 1 \text{ implies, by (2.1c), that } |C \cap T| \leqslant 1. \tag{2.11d}$$

To deal with the case $|C \cap I| \geqslant 2$ we note that if $u, v \in I$ then by (2.1c)

$$|N_G(\{u\}) \cap T| \leqslant 1 \tag{2.11e}$$

$$N_G(\{u\}) \cap N_G(\{v\}) \neq \emptyset \text{ implies } N_G(\{u\}) \cap T = \emptyset. \tag{2.11f}$$

Using (2.11) plus the fact that $S_1 \subseteq V_2(G)$ yields (2.10). We now use this in (2.9).

Case 1. $|S_1| \geqslant 2|S_2|$.

From (2.9) and (2.10) and (2.1d) and (2.1e) we obtain

$$\begin{aligned} |N_H(S)| &\geqslant |S_1| - |S_2| + ((\log n/1000) - 5)\,|S_2| \\ &= |S| + ((\log n/1000) - 7)\,|S_2|. \end{aligned}$$

Case 2. $|S_1| < 2|S_2| \leqslant 2n/\log n$.

From (2.1), (2.9) and (2.10) we have

$$\begin{aligned} |N_H(S)| &\geqslant |S_1| - |S_2| + ((\log n/1000) - 3)\,|S_2| - |S_1| \\ &= |S| + ((\log n/1000) - 5)\,|S_2| - |S_1|). \end{aligned}$$

Case 3. $|S_1| < 2|S_2|$, $n/\log n < |S_2| \leqslant n/8000$.

Choose $S_3 \subseteq S_2$ such that $|S_3| = n/\log n$, then $|N_H(S_2)| \geqslant |N_H(S_3)| - |S_2| \geqslant 7n/8000$ using (2.1e).

Then from (2.10) and (2.11) we obtain

$$\begin{aligned} |N_H(S)| &\geqslant |S_1| - |S_2| + 7n/8000 - 3\,|S_2| - |S_1| \\ &\geqslant |S| + (7n/8000 - 7\,|S_2|). \end{aligned}$$

We deduce from these 3 cases that the conclusion of the lemma holds. □

Next let $\mathscr{A}$ be the set of graphs which contain 2 vertices of degree 1, with a common neighbor. Clearly no graph belonging to $\mathscr{A}$ has a perfect or near perfect matching. Our aim is to show that the main obstruction to a graph of minimum degree at least one having a perfect or near perfect matching is that the graph belongs to $\mathscr{A}$.

Lemma 2.3. *Suppose* $G \in \mathscr{G}_2 = \{G \in \mathscr{G}_1 \backslash \mathscr{A} : \mu(G) < \lfloor |V_1(G)|/2 \rfloor\}$ *and we remove a set of edges* X *as in the statement of* Lemma 2.2 *to obtain a graph* G_1. *Let* $\mathscr{M}$ *be the set of maximum cardinality matchings of* G_1 *which cover every vertex of degree* 1. *Let* Z *be the set of vertices which are left uncovered by at least one member* M *of* $\mathscr{M}$, *i.e. not incident with any edge of* M. *For* $v \in Z$ *let* $Z(v)$ *be the set of vertices* w *for which there exists* $M \in \mathscr{M}$ *such that both* v *and* w *are uncovered by* M. *Then*

$$\text{if } w \in Z(v) \text{ then } w \text{ is not adjacent to } v, \tag{2.12a}$$

$$|Z| \geqslant n/8000 \text{ and } |Z(v)| \geqslant n/8000 \text{ for } v \in Z. \tag{2.12b}$$

Proof. If (2.12a) is false, then we have the contradiction that $\{v, w\}$ can be added to any $M \in \mathscr{M}$ leaving v and w uncovered.

To prove (2.12b) we note that $Z(v) \subseteq Z$ and so it suffices to prove $|Z(v)| \geqslant n/8000$ for $v \in Z$. Note first that $H = \psi(G_1)$ satisfies $\delta(H) \geqslant 1$ and that as $G \notin \mathscr{A}$ we have $|V(H)| - 2\mu(H) = |V_1(G_1)| - 2\mu(G_1) \geqslant 2$.

Let $v \in Z$ and $M \in \mathscr{M}$ leave v uncovered and let $S \neq \varphi$ be the other vertices left uncovered by M. If $M' = M \cap E(H)$ then $\{v\} \cup S \subseteq V(H)$ and M' is a maximum cardinality matching of H. Let S_1 be the set of vertices reachable from S by an even length alternating path with respect to M', $S \subseteq S_1$ here. Then $Z(v) \subseteq S_1$ $(= S_1$ actually) and we prove the lemma by showing

$$|N_H(S_1)| < |S_1| \tag{2.13}$$

and applying Lemma 2.2.

If $x \in N_H(S_1)$ then $x \notin S$ and so there exists y_1 such that $\{x, y_1\} \in M'$. We show $y_1 \in S_1$ which will prove (2.13). Now there exists $y_2 \in S_1$ such that $\{x, y_2\} \in E(H)$. Let P be an even length alternating path from some $s \in S$ terminating at y_2. If P contains $\{x, y_1\}$ we can truncate it to terminate with $\{x, y_1\}$, otherwise we can extend it using edges $\{y_2, x\}$ and $\{x, y_1\}$. □

We can now prove that, excluding isolated vertices, if $G_{n,m} \notin \mathscr{A}$ then it a.s. has a perfect or near perfect matching. We use a coloring argument introduced by Fenner and Frieze [7].

Lemma 2.4. *For n large*

$$\Pr(\mu(G_{n,m}) < \lfloor |V_1(G_{n,m})|/2 \rfloor \mid G_{n,m} \notin \mathscr{A}) \leqslant n^{-.35} \quad (2.14$$

Proof. Let $a=64\times 10^6$ and $\omega=\lfloor a\log n\rfloor$. We show that for n large

$$|\mathcal{G}_2|/|\mathcal{G}_0|\leqslant 2(1-a^{-1})^{\omega} \tag{2.15}$$

which in conjunction with Lemma 2.1 proves (2.12).

For each $G\in\mathcal{G}_0$ consider the $\binom{m}{\omega}$ ways of coloring ω edges green and $m-\omega$ edges blue. For a given coloring we let G^b denote the subgraph spanned by all vertices of G and the blue edges only. Let Δ denote the number of blue-green colorings which satisfy

$$\mu(G^b)=\mu(G)<\lfloor |V_1(G)|/2\rfloor, \tag{2.16a}$$

$$\text{(2.12b) holds for } H=\psi(G^b). \tag{2.16b}$$

We show that

$$\Delta\geqslant|\mathcal{G}_2|\binom{m}{\omega}(1-\varepsilon(n))^{\omega}, \tag{2.17a}$$

where $\varepsilon(n)=O((\log n)^2/n)$ and that

$$\Delta\leqslant|\mathcal{G}_0|\binom{m}{\omega}(1-a^{-1})^{\omega} \tag{2.17b}$$

which will imply (2.15).

Proof of (2.17a). If $G\in\mathcal{G}_2$, let M be a fixed maximum cardinality matching of G. Now there are $(1-\varepsilon(n))^{\omega}\binom{m}{\omega}$ ways of choosing ω green edges X such that (i) $X\cap M=\emptyset$, (ii) X does not meet any small vertices, and (iii) X is itself a matching (this is the only place that we need (2.1f)). For such colorings (2.16) must hold, which proves (2.17a).

Proof of (2.17b). Consider a fixed blue subgraph G^b and count the number of ways $\beta=\beta(G^b)$ of adding ω green edges so that (2.16) holds. If (2.16b) does not hold then $\beta=0$. If (2.16b) holds then in order for (2.16a) to hold we must avoid adding an edge $\{v, w\}$, where $w\in Z(v)$ as in Lemma 2.3. But there are at most $(1-a^{-1})^{\omega}\binom{\binom{n}{2}-m+\omega}{\omega}$ ways of doing this and (2.17b) follows. □

To study the behavior of $G_{n,m}^{(1)}$ we use the following.

Lemma 2.5. *Let H be the graph obtained from $G_{n,m}$ by deleting isolated vertices and re-labelling the remaining vertices $i_1 < i_2 < \ldots < i_h$ as $1, 2, \ldots, h$ respectively. Then for a* fixed *value of h, H is distributed as $G_{h,m}^{(1)}$.*

Proof. Each such H is obtained from the same number of $G_{n,m}$. □

The following lemma will enable us to pass, via Lemma 2.5, from results concerning $G[V_1(G_{n,m})]$ to results concerning $G_{n,m}^{(1)}$.

Lemma 2.6. *Let $t = \lceil e^{-2c} n^{1/2}/\log n \rceil$, then for large n*

$$\Pr(|V_1(G_{n+t,m})| = n) \geq n^{-.25}. \tag{2.18}$$

Proof. Let $p = (\log n/2 + \log\log n + 2c_n)/n$. We show first that for n large

$$A_1 = \Pr(|V_1(G_{n,p})| = n) \geq (\log n)^{1/3} n^{-.25}. \tag{2.19}$$

Now $A_1 = \binom{n+t}{t} \Pr(A) \Pr(B|A)$ where

$$A = \text{'vertices } n+1, \ldots, n+t \text{ are all isolated,'}$$

and

$$B = \text{'vertices } 1, 2, \ldots, n \text{ are all non-isolated.'}$$

For n large

$$\Pr(A) = (1-p)^{\binom{t}{2} + tn} \geq (t/n)^t (1 - o(1))$$

and

$$\Pr(B|A) = \Pr(\delta(G_{n,p}) \geq 1) \geq \Pr(d_{G_{n,p}}(1) \geq 1)^n.$$

The latter inequality is a consequence of

$$\Pr(d_{G_{n,p}}(k+1) \geq 1 | d_{G_{n,p}}(i) \geq 1,\ i = 1, 2, \ldots, k) \geq \Pr(d_{G_{n,p}}(k+1) \geq 1)$$

which follows from the FKG inequality [9].

Thus,

$$\Pr(B|A) \geqslant (1-(1-p)^{n-1})^n \geqslant (1-t/n)^n(1-o(1)).$$

Thus, $A_1 \geqslant \binom{n+t}{t}(t/n)^t(1-t/n)^n(1-o(1))$ and (2.19) follows on using Stirling's inequalities.

Although (2.19) does not give (2.18) directly it does show

$$\begin{aligned}&\text{there exists } m_1,\ |m-m_1| \geqslant 2n^{\frac{1}{2}}\log n \text{ such that}\\ &\Pr(|V_1(G_{n+t,m_1})|=n) \geqslant (\log n)^{\frac{1}{2}}n^{-\frac{1}{2}}.\end{aligned} \tag{2.20}$$

This is because $\Pr(||E(G_{n+t,p})|-m| > 2n^{1/2}\log n) \leqslant 1/n$, which follows from the Chernoff bound.

To obtain (2.18) from (2.20) we define

$$\mathscr{G}(m') = \{G:\ V(G)=\{1,2,\dots,n+t\},\quad |V_1(G)|=n \text{ and } |E(G)|=m'\},$$

where we assume throughout that $|m'-m| \leqslant 2n^{1/2}\log n$.

For $G \in \mathscr{G}(m')$ let $a(G) = |\{e \in E(G) : G-e \in \mathscr{G}(m'-1)\}|$.

We note

$$m' \geqslant a(G) \geqslant m' - |D_1(G)|. \tag{2.21}$$

Also

$$\sum_{G \in \mathscr{G}(m')} a(G) = \left(\binom{n}{2} - m' + 1\right)|\mathscr{G}(m'-1)| \tag{2.22}$$

as both sides of (2.22) count the number of pairs (G, e), where $G \in \mathscr{G}(m'-1)$, $e \notin E(G)$ and $G+e \in \mathscr{G}(m')$.

Now (2.21) implies

$$m'\,|\mathscr{G}(m')| \geqslant \sum_{G \in \mathscr{G}(m')} a(G) \geqslant (m' - \bar{n}_1(m'))|\mathscr{G}(m')|, \tag{2.23}$$

where $\bar{n}_1(m')$ is the expected number of vertices of degree 1 in a random graph chosen uniformly from $\mathscr{G}(m')$.

Next let

$$\lambda_{m'} = \Pr(|V_1(G_{n+t,m'})|=n) = |\mathscr{G}(m')| \Big/ \binom{\binom{n+t}{2}}{m'}.$$

It follows from (2.22) and (2.23) that

$$\rho_{m'}/m' \leqslant \lambda_{m'}/\lambda_{m'-1} \leqslant \rho_{m'}/(m' - \bar{n}_1(m')), \tag{2.24}$$

where

$$\rho_{m'} = m'\left(\binom{n}{2} - m' + 1\right) \Big/ \left(\binom{n+t}{2} - m' + 1\right).$$

In order to apply (2.24) to "close the gap" between m and m_1 in (2.20) we must estimate $\bar{n}_1(m')$.

We show first that if $\alpha(c) = (e^{1-2c}/2)(1+o(1))$ then, where $p = (\log n/2 + \log\log n + 2c_n')/n$, $c_n' \to c$,

$$\Pr\left(|D_1(G_{n+t,p})| \geqslant \beta n^{1/2}\right) \leqslant (\beta/\alpha(c))^{-\beta n^{1/2}}. \tag{2.25}$$

The above probability is no more than the probability that there exists $s = \lceil \beta\alpha(c) n^{1/2} \rceil$ vertices, each of which is adjacent to at most one of the other $n-s$ vertices.

This latter probability is

$$\leqslant \binom{n}{s}\left((1-p)^{n-s} + (n-s)\,p\,(1-p)^{n-s-1}\right)^s \leqslant (\beta/\alpha(c))^{-\beta n^{1/2}}$$

which implies (2.25).

We next prove the very crude lower bound

$$\Pr\left(|V_1(G_{n+t,m'})| = n\right) > e^{-n^{1/2}} \quad \text{for } n \text{ large}. \tag{2.26}$$

To do this, we proceed as in the proof of (2.19), using $G_{n+t,m'}$ in place of $G_{n+t,p}$, and define events A and B. Now $\Pr(A) \geqslant (t/n)^t(1-o(1))$ as before but we cannot use the FKG inequality to bound $\Pr(B \mid A)$ which is $\Pr(\delta(G_{n,m'}) \geqslant 1)$.

Instead, let now $p = \log n/2n$ and then

$$\Pr(\delta(G_{n,p}) \geqslant 1) \leqslant \Pr(\delta(G_{n,m'}) \geqslant 1) + \Pr\left(|E(G_{n,p})| > m'\right). \tag{2.27}$$

We then use the FKG inequality as before to get a lower bound

$$\Pr(\delta(G_{n,p}) \geqslant 1) \geqslant (1-o(1))\,e^{-n^{1/2}} \quad \text{for } n \text{ large}.$$

The Chernoff bound gives

$$\Pr\left(|E(G_{n,p})| > m'\right) \leqslant e^{-n(\log\log n)^2/4\log n}$$

for n large. Using these inequalities in (2.27) gives

$$\Pr(\delta(G_{n,m'})\geqslant 1)\geqslant e^{-n^{\frac{1}{2}}}/4 \quad \text{for } n \text{ large}.$$

This is easily good enough to prove (2.26). Now (2.5), (2.25) and (2.26) together imply

$$\Pr(|D_1(G_{n+t,m'})|\geqslant \beta n^{1/2} \,|\, |V_1(G_{n+t,m'})|=n)$$
$$\leqslant 2(n\log n)^{1/2}\, e^{n^{1/2}}(\beta/\alpha(c))^{-\beta n^{1/2}}. \tag{2.27}$$

Putting $\beta=\max(2, \alpha(c)\,e)$ in (2.27) we easily obtain

$$\bar{n}_1(m')\leqslant 2\beta n^{1/2} \quad \text{for } n \text{ large}. \tag{2.28}$$

Using (2.28) in (2.24) we see that for large n

$$|\lambda_{m'}/\lambda_{m'-1}-1|\leqslant \theta/(n^{1/2}\log n), \tag{2.29}$$

where θ depends only on c.
(2.20) and (2.29) together imply the lemma. □

For the remainder of this section t is as in Lemma 2.6. Now let

$$X=\text{`}\mu(G_{n+t,m})=\lfloor |V_1(G_{n+t,m})|/2\rfloor,\text{'}$$
$$Y=\text{`}|V_1(G_{n+t,m})|=n,\text{'}$$
$$Z=\text{`}G_{n+t,m}\in\mathscr{A}.\text{'}$$

Now Lemma 2.5 implies

$$\Pr(\mu(G^{(1)}_{n,m})=\lfloor n/2\rfloor)=\Pr(X|Y).$$

Now

$$\Pr(X|Y)=\Pr(X\cap Z|Y)+\Pr(X\cap \bar{Z}|Y)$$
$$=(\Pr(X\cap Y\cap Z)+\Pr(Y\cap\bar{Z})-\Pr(\bar{X}\cap Y\cap\bar{Z}))/\Pr(Y).$$

However, it follows from Lemma 2.4 (with $n+t$ in place of n) and Lemma 2.6 that

$$\Pr(\bar{X}\cap Y\cap\bar{Z})/\Pr(Y)\leqslant\Pr(\bar{X}\cap\bar{Z})/\Pr(Y)\leqslant n^{-.1}.$$

Similarly

$$\Pr(X \cap Y \cap Z)/\Pr(Y) \leqslant \Pr((2.1b))/\Pr(Y) \leqslant n^{-.1}$$

and so we have

$$\lim_{n\to\infty} \Pr\left(\mu(G_{n,m}^{(1)}) = \lfloor n/2 \rfloor\right)$$

$$= \lim_{n\to\infty} \Pr(G_{n+t,m} \notin \mathscr{A} \,\big|\, |V_1(G_{n+t,m})| = n). \tag{2.30}$$

Lemma 2.7.

$$\lim_{n\to\infty} \Pr(G_{n+t,m} \in \mathscr{A} \,\big|\, |V_1(G_{n+t,m})| = n) = 1 - e^{-e^{-4c/8}}.$$

Proof. Note that although it is very easy to prove that

$$\lim_{n\to\infty} \Pr(G_{n+t,m} \in \mathscr{A}) = 1 - e^{-e^{-4c/8}}$$

the conditional result seems to require more work. We shall in fact first prove the equivalent result for the random *multigraph* $MG_{n+t,m}$ defined as follows: Let $X = \{1, 2, \ldots, n+t\}$ and let $x \in X^{2m}$ be chosen at random so that each of the $(n+t)^{2m}$ vectors is equally likely to be chosen. Let $MG(x)$ be the multigraph with edges $\{x_{2i-1}, x_{2i}\}$ for $i = 1, 2, \ldots, m$. We use $MG_{n+t,m}$ to denote a random $MG(x)$ chosen as above. Furthermore, the random graph $RG_{n+t,m}$ is obtained by taking $MG_{n+t,m}$, deleting loops and replacing multiple edges by single copies.

We note first that

$$\text{Exp}\,(\text{number of isolated loops in } MG_{n+t,m}) = o(n^{-1/2})$$

and hence

$$\Pr(|V_1(MG_{n+t,m})| \neq |V_1(RG_{n+t,m})|) = O(n^{-1/2}). \tag{2.31}$$

Also

$$\Pr(MG_{n+t,m} \text{ has more than } 2\log n \text{ loops}) = O(n^{-1/2}) \tag{2.32a}$$

(the number of loops in $MG_{n+t,m}$ is a binomial random variable with parameters m and $2/(n-1)$). 14

$$\Pr(MG_{n+t,m} \text{ has more than } (\log n)^2 \text{ edge repetitions}) = O(n^{-1/2}) \quad (2.32b)$$

(the number of edge repetitions in $MG_{n+t,m}$ is dominated probabilistically by a binomial random variable with parameters m and $m/\binom{n+1}{2}$),

and so

$$\Pr(|E(RG_{n+t,m})| < m - 2(\log n)^2) = O(n^{-1/2}). \quad (2.33)$$

We note that

if $m' = |E(RG_{n+t,m})|$ then, for fixed m', $RG_{n+t,m}$ is distributed as $G_{n+t,m'}$. (2.34)

We now estimate

$$\Pr(|V_1(RG_{n+t,m})| = n) = \sum_{m'} \Pr(|V_1(G_{n+t,m'})| = n) \Pr(|E(RG_{n+t,m})| = m')$$

by (2.34)

$$\geqslant 1/2n^{.25} \quad \text{for } n \text{ large, by (2.33) and Lemma 2.6.}$$

It follows from (2.31) that

$$\Pr(|V_1(MG_{n+t,m})| = n) > 1/3n^{.25} \text{ for large } n. \quad (2.35)$$

Now it is easy to show that Pr(there exists vertex adjacent to 3 vertices of degree 1 in $MG_{n+t,m}) = O(n^{-1/2})$.

Thus if we define $\mathscr{A}'$ = "there exists a vertex with precisely 2 neighbors of degree 1" then

$$\lim_{n\to\infty} \Pr(MG_{n+t,m} \in \mathscr{A} \,|\, |V_1(MG_{n+t,m})| = n)$$
$$= \lim \Pr(MG_{n+t,m} \in \mathscr{A}' \,|\, |V_1(MG_{n+t,m})| = n). \quad (2.36)$$

We now write

$$\Pr(MG_{n+t,m} \in \mathscr{A}' \,|\, |V_1(MG_{n+t,m})| = n) \quad (2.37)$$
$$= \sum_{d\in\Omega} \Pr(MG_{n+t,m} \in \mathscr{A}' | MG_{n+t,m} \in \mathscr{MG}(d)) \Pr(MG_{n+t,m} \in \mathscr{MG}(d)),$$

where

$$\Omega=\{\boldsymbol{d}\in Z^{n+t}|0\leqslant d_1\leqslant d_2\leqslant\ldots\leqslant d_{n+t},$$

$$\sum_{i=1}^{n+t} d_i=2m \quad \text{and} \quad |\{i:d_i\geqslant 1\}|=n\},$$

and $\mathcal{MG}(\boldsymbol{d})$ is the set of multigraphs with vertices $\{1, 2, \ldots, n+t\}$, m edges and degree sequence $\boldsymbol{d}$.

Let now

$$\Omega_0=\{\boldsymbol{d}\in\Omega \ : \text{(a)} \ ||\{i:d_i=1\}|-e^{-2c}n^{1/2}/2|\leqslant e^{-c}n^{5/12},$$

$$\text{(b)} \ |\{i:|d_i-2m/n|>2m/n\log\log n\}|<2n/\log\log n,$$

$$\text{(c)} \ d_{n+t}\leqslant 5\log n\}. \tag{2.38}$$

We show that

$$\lim_{n\to\infty}\big(\sum_{\boldsymbol{d}\in\Omega_0}\Pr(MG_{n+t,m}\in\mathcal{MG}(\boldsymbol{d}))/\sum_{\boldsymbol{d}\in\Omega}\Pr(MG_{n+t,m}\in\mathcal{MG}(\boldsymbol{d}))=1, \tag{2.39a}$$

$$\lim_{n\to\infty}\Pr(MG_{n+t,m}\in\mathcal{A}'|MG_{n+t,m}\in\mathcal{MG}(\boldsymbol{d}))=1-e^{-e^{-4c}/8} \text{ for } \boldsymbol{d}\in\Omega_0. \tag{2.39b}$$

We can then deduce, using (2.36) and (2.37), that

$$\lim_{n\to\infty}\Pr(MG_{n+t,m}\in\mathcal{A}|\,|V_1(MG_{n+t,m})|=n)=1-e^{-e^{-4c}/8}. \tag{2.40}$$

Proof of (2.39a). In view of (2.35) we need only show that the probability that $MG_{n+t,m}$ fails to satisfy any of the conditions in (2.38) is $o(n^{-1/4})$.

(i) (2.38c) Here we simply verify that the expected number of vertices of degree exceeding $5\log n$ is $o(n^{-.25})$.

(ii) (2.38a). Here we simply verify that if D_1 is the set of vertices of degree 1 in $MG_{n+t,m}$ then

$$\operatorname{Exp}(|D_1|)\sim\operatorname{Var}(|D_1|)\sim n^{1/2}e^{-2c}/2$$

and then use the Chebyshev inequality.

(iii) (2.38b). Let $\varepsilon=1/\log\log n$ and $a=\lceil 2(1+\varepsilon)m/n\rceil$. Now one can easily see, by conditioning on vertex degrees, that for $1\leqslant k\leqslant n/\log\log n$ and $G=MG_{n+t,m}$

$$\Pr(d_G(k+1)\geqslant a \mid d_G(i)\geqslant a,\ 1\leqslant i\leqslant k)$$
$$\leqslant \Pr(d_G(k+1)\geqslant a \mid d_G(i)=a,\ 1\leqslant i\leqslant k)$$
$$=\sum_{r\geqslant a}\binom{2m-ka}{r}(1/(n+t-k))^r(1-1/(n+t-k))^{2m-ka-r}$$
$$\leqslant e^{-(2m-ka)^2\,\varepsilon/3\,(n+t-k)}$$
$$\leqslant e^{-\varepsilon^2\log n/13} \text{ for } n \text{ large}.$$

Thus Pr(there exist more than $s=n/\log\log n$ vertices of degree exceeding a)

$$\leqslant\binom{n+t}{s}e^{-\varepsilon^2 s\log n/13}=O(n^{-\gamma}) \quad \text{or any } \gamma>0.$$

A similar argument deals with vertices of degree less than $2(1-\varepsilon)\,m/n$.

Proof of (2.39b). To prove (2.39b) we need to be able to generate a random $G\in\mathcal{MG}(\boldsymbol{d})$ with probability

$$\Pr(MG_{n+t,m}=G)/\Pr(MG_{n+t,m}\in\mathcal{MG}(\boldsymbol{d}))$$

(note that this is not the same for all $G\in\mathcal{MG}(\boldsymbol{d})$).

We modify the method of Bollobás [1]. Thus, let $\boldsymbol{d}\in\Omega_0$ be fixed and let W_1 $W_2,\dots,W_{n+t}$ be disjoint sets with $|W_i|=d_i$ for $i=1,2,\dots,n+t$. Let $W=\bigcup_{i=1}^{n+t}W_i$ and let the members of W be denoted as *points*. A configuration F is a partition of W into m pairs of points called the *edges* of F. Let ζ be the set of possible configurations and note that $|\zeta|=N(m)=(2m!)/m!2^m$. For $p\in W_i$ let $\varphi(p)=i$, for $i=1,2,\dots,n+t$ and for $F\in\zeta$ let $\varphi(F)$ be the multigraph $(\{1,2,\dots,n+t\},\ \{\{\varphi(p),\varphi(q)\}:\{p,q\}\in F\})$. Note that $\varphi(\zeta)=\mathcal{MG}(\boldsymbol{d})$.

We turn ζ into a probability space by giving each $F\in\zeta$ the same probability. This induces the required probability space on $\varphi(\zeta)$. (Think of generating $MG_{n+t,m}$ conditional on $MG_{n+t,m}\in\mathcal{MG}(\boldsymbol{d})$ by taking d_i copies of integer i for $i=1,2,\dots,n+t$ and then randomly permuting these $2m$ integers and picking up edges from this sequence as usual. Note that this is essentially how $\varphi(F)$ is generated — the k-th copy of integer i corresponds to the k-th element of W_i.)

To prove (2.39b) we define a random variable

$$X(i,j,k)=\begin{cases}1 & \text{if } i<j,\ d_G(i)=d_G(j)=1 \text{ and } \{i,k\},\ \{j,k\}\in E(G) \text{ and no other vertex of degree 1 is adjacent to } k \text{ in } G, \text{ where } G=\varphi(F),\\ 0 & \text{otherwise.}\end{cases}$$

We shall use inclusion-exclusion to show that

$$\lim_{n\to\infty} \Pr\Big(\sum_{i,j,k} X(i,j,k)>0\Big)=1-e^{-e^{-4c}/8} \tag{2.41}$$

which proves (2.39b).

Let $N_3=\{1,2,\dots,n+t\}^3$ and for $S\subseteq N_3$ let $\Pi_S=\Pr(X(i,j,k)=1 \text{ for } (i,j,k)\in S)$. The definition of $X(i,j,k)$ implies

$$\Pi_S=0 \text{ unless } S \text{ is of the form } \{(i_1,j_1,k_1),\dots,(i_t,j_t,k_t)\}, \tag{2.42}$$

where $i_1,\dots,i_t,j_1,\dots,j_t,k_1,\dots,k_t$ are all different.

Let

$$p_t=\sum_{\substack{S\subseteq N_3\\ |S|=t}} \Pi_S .$$

(2.41) will follow from the Bonferroni Inequalities (e.g. Feller [6]) if we show that for *fixed r*

$$\lim_{n\to\infty} p_r=(e^{-4c}/8)^r/r! \tag{2.43}$$

Let $s=|\{i: d_i=1\}|$, $D_2=\{i: d_i\geqslant 2\}$ then, in view of (2.42), we have

$$p_r=\frac{s!}{(s-2r)!\,2^r}\Big(\sum_{\substack{R\subseteq D_2\\ |R|=r}}\prod_{i\in R} d_i(d_i-1)\Big)N(m-2r)/N(m).$$

Using $\boldsymbol{d}\in\Omega_0$ and r fixed gives (2.43) without difficulty, and so (2.40) is proved.

Now simple estimations, using expectations, show

$$\Pr(\text{there exists } v \text{ such that } d_{MG_{n+t,m}}(v)>1=d_{RG_{n+t,m}}(v))$$
$$=O(\log n/n^{1/2})$$

and hence

$$\Pr(MG_{n+t,m}\notin\mathscr{A} \text{ and } RG_{n+t,m}\in\mathscr{A})=O(\log n/n^{1/2})$$

and so (2.31), (2.35) and (2.40) give

$$\lim_{n\to\infty}\Pr(RG_{n+t,m}\in\mathscr{A}\,|\,|V_1(RG_{n+t,m})|=n)=1-e^{-e^{-4c}/8}.$$

Thus, where

$$\sigma_{m'}=\Pr(RG_{n+t,m}\in\mathscr{A}\,|\,|E(RG_{n+t,m})|=m',\ |V_1(RG_{n+t,m})|=n)$$

we have

$$\lim_{n\to\infty}\sum_{m'}\sigma_{m'}\Pr(|E(RG_{n+t,m})|=m'\,|\,|V_1(RG_{n+t,m})|=n)=1-e^{-e^{-4c}/8}. \tag{2.44}$$

Now in view of (2.34) we can write

$$\sigma_{m'}=\Pr(G_{n+t,m'}\in\mathscr{A}\,|\,|V_1(G_{n+t,m'})|=n). \tag{2.45}$$

We can deduce our lemma from (2.33), (2.35), (2.44), (2.45) and

$$|\sigma_{m'}/\sigma_{m'-1}-1|=O(n^{-1/2})\ \text{for}\ m\geqslant m'\geqslant m-2(\log n)^2. \blacksquare \tag{2.46}$$

To prove (2.46) let

$$\mathscr{G}_A(m')=\mathscr{G}(m')\cap\mathscr{A},$$

where $\mathscr{G}(m')$ is as defined in Lemma 2.6. For $G\in\mathscr{G}_A(m')$ let $a(G)=|\{e\in E(G): G-e\in\mathscr{G}_A(m'-1)\}|\geqslant m'-1-|D_1(G)|$ and for $G\in\mathscr{G}_A(m'-1)$ let

$$b(G)=|\{e\notin E(G): G+e\in\mathscr{G}_A(m')\}|\geqslant\binom{n}{2}-m'+1-n|D_1(G)|.$$

Arguing as for (2.22) we have

$$\sum_{G\in\mathscr{G}_A(m')}a(G)=\sum_{G\in\mathscr{G}_A(m'-1)}b(G)$$

and so arguing as for (2.23) we obtain

$$\left(\binom{n}{2}-m'+1-n\tilde{n}_1(m'-1)\right)/m'\leqslant|\mathscr{G}_A(m')|/|\mathscr{G}_A(m'-1)|$$
$$\leqslant\left(\binom{n}{2}-m'+1\right)/(m'-\tilde{n}_1(m')), \tag{2.47}$$

where $\tilde{n}_1(m')$ denotes the expected number of vertices of degree 1 in a random graph chosen uniformly from $\mathscr{G}_A(m')$.

We deduce from (2.27) that $\tilde{n}_1(m')\leqslant 2\beta n^{1/2}$, where β is as in (2.28). Now $\sigma_{m'}=|\mathcal{G}_A(m')|/|\mathcal{G}(m')|$ and so (2.46) now follows from (2.22), (2.23), (2.28) and (2.47). □

The reader familiar with [1] will realize that we had to work with multigraphs and proceed in this way because the probability that a graph in $\Omega(\boldsymbol{d})$ has no loops or multiple edges is too small.

Proof of Theorem 1.1. The case $c_n\to c$ follows immediately from Lemma 2.5, Lemma 2.7 and (2.30).

For $c_n\to+\infty$, $c_n\leqslant\log n$ we simply repeat the arguments almost unchanged. For $c_n>\log n$ we have no conditioning problems as $\delta(G_{n,m})\geqslant 1$ a.s. in this case.

For $c_n\to-\infty$, $-c_n=o(\log\log n)$ we can again repeat the argument for $c_n\to c$ without much change. □

If $c_n\to-\infty$ rather fast then we are unable to prove Lemma 2.6. The reader will observe that we only just managed to close the gap in (2.24).

3.

We now turn to the proof of Theorem 1.2. We first define a random *edge colored* graph $G(n, m, k)$ as follows:

> Start with $G_{n,m}$ and all its edges painted blue;
> **while** $\delta(G)<k$ **do**
> **begin**
> choose a vertex v with degree $<k$, uniformly at random;
> Let $X-\{e\in V_n^{(2)}-E(G): v\in e\}$;
> choose $e\in X$ uniformly at random and paint it *red*;
> $E(G):=E(G)\cup\{e\}$
> **end**

The following lemma is taken from Bollobás [3] and is given here for completeness.

Lemma 3.1. *Let Π be a monotone graph property such that $G\in\Pi$ implies $\delta(G)\geqslant k$. Let $m=\frac{1}{2}n\log n+\frac{1}{2}(k-1)n\log\log n-nw$, where $w=w(n)\to\infty$ and $w(n)\leqslant\log\log\log n$. Then*

$$G(n, m, k)\in\Pi \text{ a.s.} \to \tau(\Gamma, \Pi)=\tau(\Gamma, \Pi_k) \text{ a.s.}$$

Proof. Consider an instance of $\tilde{G}$. Color edges $e_1, e_2, \ldots, e_m$ blue. For $i>m$ paint e_i red if e_i is incident with a vertex of degree $\leq k-1$ in G_{i-1}. Let $m'=\tau(\Gamma, \Pi_k)$. The blue-red subgraph of $G_{m'}$ is distributed exactly as $G(n, m, k)$ and so $G_{m'} \in \Pi$ a.s. as Π is monotone. Furthermore $G_{m'-1} \notin \Pi_k$ as $\delta(G_{m'-1})<k$. □

In view of this we can prove Theorem 1.2 if we can prove that $G(n, m, k) \in \tilde{\Pi}_k$ a.s. where m is as defined in Lemma 3.1. We shall use this value for m throughout this section.

We state the following lemma which can easily be verified.

Lemma 3.2. *Let $G=G_{n,m}$ and let $SMALL=\{v \in V_n : d_G(v) \leq \log n/10\}$ and $LARGE = V_n - SMALL$. The following properties hold a.s.*

$$\delta(G)=k-1, \tag{3.1a}$$

$$|\{v \in V_n : d_G(v)=k-1\}| \leq \log n, \tag{3.1b}$$

$$|SMALL| \leq n^{1/2}, \tag{3.1c}$$

no pair of small vertices are adjacent or share a common neighbor, (3.1d)

$$\emptyset \neq S \subseteq LARGE,\ |S| \leq n/\log n \ \textit{implies}\ |N_G(S)| \geq |S| \log n/100, \tag{3.1e}$$

$$d_G(v) \leq 5 \log n \ \textit{for}\ v \in V_n. \tag{3.1f}$$

From this we easily derive

Lemma 3.3. *Let $G=G(n, m, k)$ and let SMALL, LARGE be as in* Lemma 3.2. *The G has the following properties a.s.*

If $\{v, w\}$ is a red edge then $d_G(v)=k$ and $w \in LARGE$, assuming $d_G(v) \leq d_G(w)$. (3.2)

Let X be a matching of G that is only incident with large vertices and let $H=G-X$. Then there exist real constants α_k, $\beta_k>0$ such

$$\emptyset \neq S \subseteq V_n,\ |S| \leq \alpha_k n \ \textit{implies}\ |N_H(S)| \geq k|S|, \tag{3.3a}$$

$$|S| > \alpha_k n \ \textit{implies}\ |\{v, w\} \in E(G) : v \in S, w \notin S\}| \geq \beta_k n \log n. \tag{3.3b}$$

Proof. (3.2) follows from (3.1b) and (3.1c). (3.3a) is proved in a similar way to Lemma 2.2, and we can take $\beta_k=\alpha_k(1-\alpha_k)/2$ in (3.3b). □

For non-negative integer h, if graph G contains h disjoint hamiltonian cycles $H_1\ H_2, \ldots, H_h$ let $G-\bigcup_{i=1}^{h} H_i$ be called an h-subgraph of G.

Let $\varphi(G)=(h, p)$, where

$$h = \text{maximum number of disjoint hamiltonian cycles in } G,$$

$$p=\begin{cases} 0 & \text{if } k=2h, \\ \text{maximum cardinality of a matching} & \\ \text{in any } h\text{-subgraph of } G & \text{if } k=2h+1, \\ \text{maximum length of a path} & \\ \text{in any } h\text{-subgraph of } G & \text{if } k\geqslant 2h+2. \end{cases}$$

Thus $G \in \tilde{\Pi}_k$ if and only if $\varphi(G)=\theta(k, n)=(\lfloor k/2\rfloor, \lfloor n/2\rfloor (k-2\lfloor k/2\rfloor))$.

If $\varphi(G)=(h, p)$ we define a *φ-subgraph* of G to be any h-subgraph of G containing either a matching of size p or a path of length p as the case may be.

Lemma 3.4. *Suppose $G=G(n, m, k)$ satisfies the conditions* (a) *and* (b) *of* Lemma 3.2 *and let X be as in* (b) *there. Let $s=\lceil \alpha_k n\rceil$, then for n large*

> *there exists a φ-subgraph $\tilde{H}$ of $H=G-X$, $A=\{a_1, a_2, \ldots, a_t\}$, $A_1, A_2, \ldots, A_t \subseteq V_n$, $t\geqslant s$, such that for $i=1, 2, \ldots, t$, $|A_i|\geqslant t$, $a_i \notin A$; and if $a \in A_i$ then $e=\{a, a_i\} \notin E(H)$ and $\varphi(H+e)\neq\varphi(H)$.* (3.4)

Proof. Let $\tilde{H}$ be any φ-subgraph of H.

Suppose first that $k=2h+1$ and $p<\lfloor n/2\rfloor$. Let $A=\{a : a$ is left exposed by some maximum cardinality matching of $\tilde{H}\}=\{a_1, a_2, \ldots, a_t\}$. Let $A_i=\{a : a$ and a_i are left exposed by some maximum cardinality matching of $\tilde{H}\}\subseteq A$. Then we deduce as in Lemma 2.3 that $|N_{\tilde{H}}(A_i)|<|A_i|$ and hence that $|N_H(A_i)|<k|A_i|$ and hence that $|A_i|\geqslant s$.

If $k>2h+1$ let P be a path of length p in $\tilde{H}$ and let a_1 be one endpoint of P. Pósa [20] shows that there exists a set A_1 such that $|N_H(A_1)|<2|A_1|$ and each $b \in A_1$ is an endpoint of a path of length p joining a_1 and b. We see by reasoning as above that $|A_1|\geqslant s$. We must show $a_1 \notin N_{\tilde{H}}(A_1)$. Now (3.3) can be used to show that H is connected for n large and so if $a_1 \in N_{\tilde{H}}(A_1)$, P is not a longest path of $\tilde{H}$ or H contains $h+1$ disjoint hamiltonian cycles. We take $A=\{a_1\} \cup A_1$ and repeat the argument for $a \in A_1$ with any path of length p with a as endpoint. □

We now use the coloring argument (as in Lemma 2.4) to prove

Lemma 3.5.

$$\lim_{n\to\infty} \Pr(G(n, m, k)\in\tilde{\Pi}_k)=1.$$

Proof. Let $\mathscr{G}_t=\{G=G(n,m,k):(3.2)$ holds and G has exactly t red edges$\}$. Note that each $G\in\mathscr{G}_t$ has the same probability of being chosen. Next let $\hat{\mathscr{G}}_t=\{G\in\mathscr{G}_t$: (3.3b) holds and $\varphi(G)\neq\theta(k,n)\}$.

In view of (3.1b) and Lemma 3.2, this lemma will follow if we prove

$$\lim_{n\to\infty}|\hat{\mathscr{G}}_t|/|\mathscr{G}_t|=0 \qquad 0\leqslant t\leqslant|\log n|. \tag{3.5}$$

Let $w=\lceil\log n\rceil$ and for $G\in\mathscr{G}_t$ let $E^b(G)$, $E^r(G)$ denote the blue and red edges respectively. Consider now all the $\binom{m}{w}$ ways of choosing w blue edges and recoloring them green.

For $G\in\mathscr{G}_t$ and $X\subseteq E^b(G)$, $|X|=w$, define

$$a(G,X)=\begin{cases}1 & \text{if (a) } \varphi(H)=\varphi(G),\text{ where } H=G-X,\\ & \quad\text{(b) } H \text{ satisfies (3.3)},\\ & \quad\text{(c) where } H^b=(V_n,E^b(G)-X),\ \delta(H^b)=k-1 \text{ and}\\ & \qquad H^b \text{ has exactly } t \text{ vertices of degree } k-1,\\ 0 & \text{otherwise}.\end{cases}$$

Let Ω be the set of blue-red edge-colored graphs obtainable by deleting w blue edges from a graph G in $\mathscr{G}_t$.

For $H\in\Omega$ let $X_H=\{S\subseteq V_n^{(2)}-E(H):$ there exists $G=G(H,S)\in\mathscr{G}_t$ with $E^b(G)=E^b(H)\cup S$ and $E^r(G)=E^r(S)\}$ and let $\Delta_H=|\{S\in X_H:a(G(H,S),S)=1\}|$.

We prove (3.5) by showing

$$G\in\hat{\mathscr{G}}_t \text{ implies } \sum_{\substack{X\subseteq E^b(G)\\ |X|=w}} a(G,X)\geqslant(1-o(1))\binom{m}{w}\left(1-\frac{k+3}{\log n}\right)^w \tag{3.6a}$$

$$\Delta_H\leqslant(1-\alpha_k^2)|X_H|(1+o(1)), \tag{3.6b}$$

for then

$$S=\sum_{G\in\mathscr{G}_t}\sum_{\substack{X\subseteq E^b(G)\\ |X|=w}} a(G,X)\geqslant(1-o(1))\binom{m}{w}\left(1-\frac{k+3}{\log n}\right)^w|\hat{\mathscr{G}}_t|$$

and

$$S\leqslant\sum_{H\in\Omega}\Delta_H\leqslant(1-\alpha_k^2)^w\sum_{H\in\Omega}|X_H|(1+o(1))$$

$$=(1-\alpha_k^2)\binom{m}{w}|\mathscr{G}_t|(1+o(1))$$

and (3.5) follows.

Proof of (3.6a). Given $G \in \hat{\mathscr{G}}_t$ with $\varphi(G)=(h,p)$ choose h disjoint hamiltonian cycles $H_1, H_2, \ldots, H_h$ plus a path or matching A of size p as necessary. Now there are at least $(1-o(1))\binom{m}{w}(1-(k+3)/\log n)^w$ ways of choosing a matching X that only meets small vertices of G and does not meet $A \cup \bigcup_{i=1}^{h} H_i$. For each such X, $a(G,X)=1$, on using Lemma 3.3.

Proof of (3.6b). Let $H \in \Omega$. If H does not satisfy (3.4) or H^b does not have t vertices of degree $k-1$ then $\Delta_H=0$. So assume these conditions hold. It follows that $S \in X_H$ if and only if $S \subseteq V_n^{(2)} - E(H)$ and S does not meet any vertices of degree $k-1$ in H^b. (We included the last condition in (3.4) in order to give such a simple description of X_H.) Let $\tilde{H}$ be the φ-subgraph guaranteed by (3.3).

According to (3.3) we can only have $a(G(H,X),X)=1$ if no edge of S joins $a_i \in A$ to A_i. But there are at least $(\alpha_k^2 n^2 - kn)/2$ possibilities for choosing such an edge (we subtract $kn/2$ to account for those that may occur in $E(H)-E(\tilde{H})$). (3.6b) follows along with the lemma. □

Proof of Theorem 1.2. Just use Lemma 3.1 and Lemma 3.4. □

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