LOCALIZATION GAME FOR RANDOM GRAPHS

ANDRZEJ DUDEK, SEAN ENGLISH, ALAN FRIEZE, CALUM MACRURY,
AND PAWEŁ PRAŁAT

Abstract. We consider the localization game played on graphs in which a cop tries to determine the exact location of an invisible robber by exploiting distance probes. The corresponding graph parameter $\zeta(G)$ for a given graph G is called the localization number. In this paper, we improve the bounds for dense random graphs determining an asymptotic behaviour of $\zeta(G)$. Moreover, we extend the argument to sparse graphs.

1. Introduction

Graph searching focuses on the analysis of games and graph processes that model some form of intrusion in a network and efforts to eliminate or contain that intrusion. One of the best known examples of graph searching is the game of Cops and Robbers, wherein a robber is loose on the network and a set of cops attempts to capture the robber. For a book on graph searching see [4].

In this paper we consider the *Localization Game* that is related to the well studied Cops and Robbers game. A *robber* is located at a vertex v of a graph G. In each round, a *cop* can ask for the graph distance between v and vertices $W = \{w_1, w_2, \ldots, w_k\}$, where a new set of vertices W can be chosen at the start of each round. The cops win immediately if the W-signature of v, that is, the vector of distances $(d_1, d_2, \ldots, d_k)$, where $d_i = d(s_i, v)$ is the distance between s_i and v_i, is sufficient to determine v. Otherwise, the robber will move to a neighbour of v and the cop will try again with a (possibly) different test set W.

Given G, the localization number, written $\zeta(G)$, is the minimum k so that the cop can eventually locate the robber using sets W of size k. The localization game was introduced for one probe ($k = 1$) in [11, 12] and was further studied in [6, 7, 5, 9, 3]. The localization number is related to the metric dimension of a graph, in a way that is analogous to how the cop number is related to the domination number. The metric dimension of a graph G, written $\beta(G)$, is the minimum number of probes needed in the localization game so that the cop can win in one round. It follows that $\zeta(G) \leq \beta(G)$, but in many cases this inequality is far from tight.

In this paper we present results obtained for the *binomial random graph* $\mathcal{G}(n,p)$. More precisely, $\mathcal{G}(n,p)$ is a distribution over the class of graphs with vertex set $[n]$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity. We say

The first author is supported in part by Simons Foundation Grant #522400.
The third author is supported in part by NSF grant DMS1362785.
The fifth author is supported in part by NSERC Discovery Grant.

that $\mathcal{G}(n,p)$ has some property *asymptotically almost surely* or *a.a.s.* if the probability that $\mathcal{G}(n,p)$ has this property tends to 1 as n goes to infinity.

The localization number for dense random graphs (diameter two case) was studied in [8]. The results obtained in [8] can be summarized as follows. (See Section 3.1 for asymptotic notation that we use below.) If $d := pn = n^{x+o(1)}$ for some $x \in (1/2, 1)$, then the following holds a.a.s. for $G \in \mathcal{G}(n,p)$:

$$(1+o(1))(2x-1)\frac{n\log n}{d} \le \zeta(G) \le (1+o(1))f(x)\frac{n\log n}{d},$$

where

$$f(x) := \begin{cases} x & \text{if } 2/3 < x < 1 \\ 1-x/2 & \text{otherwise.} \end{cases}$$

Hence, the order of $\zeta(G)$ is determined for this range of d. If $d = pn = n^{1+o(1)}$ and $p \le 1 - 3\log\log n/\log n$, then the following holds a.a.s. for $G \in \mathcal{G}(n,p)$:

$$\zeta(G) \sim \frac{2\log n}{\log(1/\rho)},$$

where

$$\rho := p^2 + (1-p)^2.$$

The asymptotic behaviour of $\zeta(G)$ is determined for such dense graphs.

In this paper, we improve the bounds for dense graphs showing that if $d := pn = n^{x+o(1)}$ for some $x \in (1/2, 1)$, then a.a.s. $\zeta(\mathcal{G}(n,p)) \sim xn\log n/d$. Our proofs can be easily generalized so we extend our results to cover sparser graphs. The main results are stated in Section 2. Notation and some auxiliary observations are presented in Section 3. Section 4 provides a convenient reformulation of the game so that it can be viewed as a combinatorial game. Finally, lower and upper bounds are proved in Section 5 and, respectively, Section 6.

2. Results

Recall that the asymptotic behaviour of the localization number is already determined for very dense graphs and so we may concentrate on $d = o(n)$. Our results are slightly stronger than what is stated below but our goal is to summarize the most important consequences. The reader is directed to Sections 5 and 6 for more details. The first theorem below concentrates on random graphs with diameter $i+1$ and the average degree not too close to the threshold where the diameter drops to i. This result follows immediately from Theorem 5.1 and Theorem 6.1.

Theorem 2.1. *Suppose that $d := pn$ is such that $\log n \ll d \ll n$. Suppose that $i = i(n) \in \mathbb{N}$ is such that $d^i \ll n$ and $d^{i+1}/n - 2\log n \to \infty$. Then, the following holds a.a.s. for $G \in \mathcal{G}(n,p)$:*

$$(\log d - 3\log\log n)\,\frac{n}{d^i} \le \zeta(G) \le (1+o(1))\,(\log d + 2\log\log n)\,\frac{n}{d^i}\,.$$

As a result, if $d \geq (\log n)^{\omega}$ *for some* $\omega = \omega(n) \to \infty$ *as* $n \to \infty$*, then*

$$\zeta(G) \sim \frac{n \log d}{d^i}.$$

In particular, if there exists $i \in \mathbb{N}$ *such that* $d = n^{x+o(1)}$ *for some* $x \in (\frac{1}{i+1}, \frac{1}{i})$*, then*

$$\zeta(G) \sim \frac{xn \log n}{d^i}.$$

Before we move to our next result, let us mention about the relationship between $\zeta(G)$ and $\beta(G)$. The bounds for $\beta(G)$ obtained in [2] are quite technical but for the range of d covered by Theorem 2.1 we see that the following holds a.a.s. for $G \in \mathcal{G}(n,p)$:

$$(1+o(1))\frac{n \log(d^i)}{d^i} \leq \beta(G) \leq (1+o(1))\frac{n \log n}{d^i}.$$

An upper bound for $\zeta(G)$ obtained in [8] when $d = n^{x+o(1)}$ for some $x \in (2/3, 1)$ was matching a lower bound for $\beta(G)$ (and their lower bound for $\zeta(G)$ was much smaller). Hence, they conjectured that for this range of d we have $\zeta(G) < \beta(G)$. Of course, it is still plausible but we showed that their upper bound for $\zeta(G)$ was correct and so the two graph parameters are not separated. On the other hand, for sparser graphs (of diameter at least 3; $i \geq 2$) they are clearly separated; it follows that $\zeta(G) < \beta(G)$. In fact, for very sparse graphs, say for $d = \log^6 n$, a.a.s. $\zeta(G) = \Theta(n \log\log n / d^i)$ whereas $\beta(G) = \Theta(n \log n / d^i)$. The ratio $\zeta(G)/\beta(G) = \Theta(\log\log n / \log n) \to 0$ as $n \to \infty$.

We are less precise once we get closer to the threshold where the diameter drops from $i+1$ to i. If $c = c(n) := d^i/n = \Theta(1)$, then we only determine the order of $\zeta(G)$. When $c \to \infty$ as $n \to \infty$, then the upper bound for $\zeta(G)$ does not match the corresponding lower bound. Thus, determining the behaviour of the localization number when $c = \Omega(1)$ remains an open problem. Below, we state the result for $c = \Theta(1)$ and we direct the reader for more details on the case when $c \to \infty$ to Sections 5 and 6. This result follows immediately from Theorem 5.1 and Theorem 6.2.

Theorem 2.2. *Suppose that* $d := pn$ *is such that* $\log^3 n \ll d \ll n$*. Suppose that* $i = i(n) \in \mathbb{N}$ *is such that* $c = c(n) := d^i/n \to A \in \mathbb{R}_+$*. Then, the following holds a.a.s. for* $G \in \mathcal{G}(n,p)$*:*

$$(\log d - 3\log\log n)\frac{1}{A} \leq \zeta(G) \leq (1+o(1))(\log d + 2\log\log n)\frac{e^A}{1-e^{-A}}.$$

As a result, if $d \geq (\log n)^{3+\epsilon}$ *for some* $\epsilon > 0$*, then*

$$\zeta(G) = \Theta\left(\frac{n \log d}{d^i}\right).$$

3. Notation and Probabilistic Preliminaries

In this section we give a few preliminary results that will be useful for the proof of our main result. First, we introduce standard asymptotic notation, then we state a specific instance of Chernoff's bound that we will find useful. Finally, we mention some specific expansion properties that $\mathcal{G}(n,p)$ has and state the well-known result about the diameter of $\mathcal{G}(n,p)$.

3.1. **Notation and Convention.** Given two functions $f = f(n)$ and $g = g(n)$, we will write $f = O(g)$ if there exists an absolute constant c such that $f \le cg$ for all n, $f = \Omega(g)$ if $g = O(f)$, $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$, and we write $f = o(g)$ or $f \ll g$ if the limit $\lim_{n\to\infty} f/g = 0$. In addition, we write $f = \omega(g)$ or $f \gg g$ if $g = o(f)$, and unless otherwise specified, ω will denote an arbitrary function that is $\omega(1)$, assumed to grow slowly. We also will write $f \sim g$ if $f = (1 + o(1))g$.

For a vertex $v \in V$ of some graph $G = (V, E)$, let $\mathcal{S}(v, i)$ and $\mathcal{N}(v, i)$ denote the set of vertices at distance i from v and the set of vertices at distance at most i from v, respectively. For any $V' \subseteq V$, let $\mathcal{S}(V', i) = \bigcup_{v\in V'} \mathcal{S}(v, i)$ and $\mathcal{N}(V', i) = \bigcup_{v\in V'} \mathcal{N}(v, i)$.

Through the paper, all logarithms with no subscript denoting the base will be taken to be natural. Finally, as typical in the field of random graphs, for expressions that clearly have to be an integer, we round up or down but do not specify which: the choice of which does not affect the argument.

3.2. **Concentration inequalities.** Throughout the paper, we will be using the following concentration inequality. Let $X \in \mathrm{Bin}(n, p)$ be a random variable with the binomial distribution with parameters n and p. Then, a consequence of Chernoff's bound (see e.g. [10, Corollary 2.3]) is that

$$\mathbb{P}(|X - \mathbb{E}X| \ge \varepsilon \mathbb{E}X)) \le 2\exp\left(-\frac{\varepsilon^2 \mathbb{E}X}{3}\right) \tag{1}$$

for $0 < \varepsilon < 3/2$.

3.3. **Expansion properties.** In this paper, we focus on dense random graphs, that is, graphs with average degree asymptotic to $d := pn \gg \log n$. Such dense random graphs will have some useful expansion properties that hold a.a.s. We will use the following two technical lemmas. The first one is proved in [2] but we include the proof for completeness.

Lemma 3.1 ([2])**.** *Let $\omega = \omega(n)$ be a function tending to infinity with n such that $\omega \le (\log n)^4(\log\log n)^2$. Then the following properties hold a.a.s. for $G = (V, E) \in \mathcal{G}(n, p)$. Suppose that $\omega \log n \le d := pn = o(n)$. Let $V' \subseteq V$ with $|V'| \le 2$ and let $i = i(n) \in \mathbb{N}$ be such that $d^i = o(n)$. Then,*

$$|\mathcal{S}(V', i)| = \left(1 + O\left(\frac{1}{\sqrt{\omega}}\right) + O\left(\frac{d^i}{n}\right)\right) d^i |V'|.$$

In particular, for every $x, y \in V$ $(x \ne y)$ we have

$$|\mathcal{S}(x, i) \setminus \mathcal{S}(y, i)| = \left(1 + O\left(\frac{1}{\sqrt{\omega}}\right) + O\left(\frac{d^i}{n}\right)\right) d^i.$$

For the next lemma we need to assume that our random graph is slightly denser, namely, that $d := pn \gg \log^3 n$.

Lemma 3.2. *Let $\omega' = \omega'(n)$ be a function tending to infinity with n such that $\omega' \le (\log n)^2(\log\log n)^2$. Then the following properties hold a.a.s. for $G = (V, E) \in \mathcal{G}(n, p)$. Suppose that $\omega' \log^3 n \le d := pn = o(n)$. Suppose that $i = i(n) \in \mathbb{N}$ is such that $c = c(n) := d^i/n = \Omega(1)$ and $c \le 3\log n$. Then, for every $x, y \in V$ $(x \ne y)$ we have*

$$|\mathcal{S}(x,i) \setminus \mathcal{S}(y,i)| = \left(1 + O\left(\frac{1}{\sqrt{\omega'}}\right)\right) n(1-e^{-c})e^{-c},$$

provided that $c \le \log n - 4\log\log n$. For $\log n - 4\log\log n \le c \le 3\log n$, we have

$$|\mathcal{S}(x,i) \setminus \mathcal{S}(y,i)| = O\left(\log^4 n\right).$$

Proof of Lemma 3.1. We will show that a.a.s. for every $V' \subseteq V$ with $|V'| \le 2$ and $i \in \mathbb{N}$ we have the desired concentration for $|\mathcal{S}(V', i)|$, provided that $d^i = o(n)$. The statement for any pair of vertices x, y will follow immediately (deterministically) from this.

In order to investigate the expansion property of neighbourhoods, let $Z \subseteq V$, $z = |Z| = o(n/d)$, and consider the random variable $X = X(Z) = |\mathcal{N}(Z, 1)|$. We will bound X in a stochastic sense. There are two things that need to be estimated: the expected value of X, and the concentration of X around its expectation.

Since for $x = o(1)$ we have $(1-x)^z = e^{-xz(1+O(x))}$ and also $e^{-x} = 1 - x + O(x^2)$, it is clear that

$$\begin{aligned}
\mathbb{E}[X] &= n - \left(1 - \frac{d}{n}\right)^z (n-z) \\
&= n - \exp\left(-\frac{dz}{n}(1 + O(d/n))\right)(n-z) \\
&= dz(1 + O(dz/n)), \qquad (2)
\end{aligned}$$

provided $dz = o(n)$. It follows from Chernoff's bound (1), applied with $\varepsilon = 2/\sqrt{\omega}$, that the expected number of sets V' with $|V'| \le 2$ satisfying

$$\big||\mathcal{N}(V',1)| - \mathbb{E}\left[\,|\mathcal{N}(V',1)|\,\right]\big| > \varepsilon d|V'|$$

is at most

$$\sum_{z\in\{1,2\}} 2n^z \exp\left(-\frac{\varepsilon^2 zd}{3+o(1)}\right) \le \sum_{z\in\{1,2\}} 2n^z \exp\left(-\frac{\varepsilon^2 z\omega\log n}{3+o(1)}\right) = o(1),$$

since $d \ge \omega\log n$. Hence the statement holds for $i = 1$ a.a.s.

Now, we will estimate the cardinalities of $\mathcal{N}(V', i)$ up to the i'th iterated neighbourhood, provided $d^i = o(n)$ and thus $i = O(\log n/\log\log n)$. It follows from (2) and (1) (with $\varepsilon = 4(\omega|Z|)^{-1/2}$) that in the case $\omega\log n/2 \le |Z| = o(n/d)$ with probability at least $1 - n^{-3}$

$$|\mathcal{N}(Z,1)| = d|Z|\left(1 + O\left(d|Z|/n\right) + O\left((\omega|Z|)^{-1/2}\right)\right),$$

where the bounds in $O()$ are uniform. As we want a result that holds a.a.s., we may assume this statement holds deterministically, since there are only $O(n^2\log n)$ choices for V' and i. Given this assumption, we have good bounds on the ratios of the cardinalities of $\mathcal{N}(V',1)$, $\mathcal{N}(\mathcal{N}(V',1),1) = \mathcal{N}(V',2)$, and so on. Since $i = O(\log n/\log\log n)$

and $\sqrt{\omega} \le (\log n)^2(\log\log n)$, the cumulative multiplicative error term is

$$
\begin{aligned}
&(1+O(d/n)+O(1/\sqrt{\omega}))\prod_{j=2}^{i}\left(1+O\left(d^j/n\right)+O\left(\omega^{-1/2}d^{-(j-1)/2}\right)\right)\\
&\quad=(1+O(1/\sqrt{\omega})+O(d^i/n))\prod_{j=7}^{i-3}\left(1+O\left(\log^{-3}n\right)\right)=(1+O(1/\sqrt{\omega})+O(d^i/n)),
\end{aligned}
$$

and the proof is complete. □

Proof of Lemma 3.2. Fix any $x, y \in V$ $(x \neq y)$. Since $d = o(n)$ and $d^i = \Omega(n)$, it follows that $i \ge 2$. We expose edges around vertices x and y to get $\mathcal{N}(\{x,y\}, i-1)$. Note that $d^{i-1} = d^i/d = cn/d = O(n\log n/d) = O(n/(\omega'\log^2 n)) = o(n)$. Hence, by Lemma 3.1 applied with $\omega = \omega'\log^2 n$, we may assume that

$$
\begin{aligned}
|\mathcal{S}(x,i-1)\setminus\mathcal{S}(y,i-1)| &= \left(1+O\left(\frac{1}{\sqrt{\omega}}\right)+O\left(\frac{d^{i-1}}{n}\right)\right)d^{i-1}\\
&= \left(1+O\left(\frac{1}{\sqrt{\omega'}\log n}\right)+O\left(\frac{1}{\omega'\log n}\right)\right)d^{i-1}\\
&= \left(1+O\left(\frac{1}{\sqrt{\omega'}\log n}\right)\right)d^{i-1}.
\end{aligned}
$$

Similarly,

$$
|\mathcal{S}(y,i-1)| = \left(1+O\left(\frac{1}{\sqrt{\omega'}\log n}\right)\right)d^{i-1}.
$$

Let $X = X(x,y) = |\mathcal{S}(x,i)\setminus\mathcal{S}(y,i)|$. It is clear that $v \in V\setminus\mathcal{N}(\{x,y\},i-1)$ belongs to $\mathcal{S}(x,i)\setminus\mathcal{S}(y,i)$ if and only if v has a neighbour in $\mathcal{S}(x,i-1)\setminus\mathcal{S}(y,i-1)$ but has no neighbour in $\mathcal{S}(y,i-1)$. It follows that

$$
\mathbb{E}[X] = \Big(n-|\mathcal{N}(\{x,y\},i-1)|\Big)\left(1-(1-p)^{|\mathcal{S}(x,i-1)\setminus\mathcal{S}(y,i-1)|}\right)(1-p)^{|\mathcal{S}(y,i-1)|}.
$$

Since

$$
\begin{aligned}
(1-p)^{\left(1+O\left(\frac{1}{\sqrt{\omega'}\log n}\right)\right)d^{i-1}} &= \exp\left(-\left(1+O\left(\frac{1}{\sqrt{\omega'}\log n}\right)\right)\frac{d^i}{n}\right)\\
&= \exp\left(-c+O\left(\frac{c}{\sqrt{\omega'}\log n}\right)\right)\\
&= e^{-c}\exp\left(O\left(\frac{1}{\sqrt{\omega'}}\right)\right)\\
&= e^{-c}\left(1+O\left(\frac{1}{\sqrt{\omega'}}\right)\right),
\end{aligned}
$$

we get that

$$\begin{aligned}\mathbb{E}[X] &= \left(1+O\left(\frac{d^{i-1}}{n}\right)\right)n(1-e^{-c})e^{-c}\left(1+O\left(\frac{1}{\sqrt{\omega'}}\right)\right)\\ &= \left(1+O\left(\frac{1}{\sqrt{\omega'}}\right)\right)n(1-e^{-c})e^{-c}.\end{aligned}$$

Suppose first that $c \le \log n - 4\log\log n$ so that $\mathbb{E}[X] \ge (1+o(1))\log^4 n$. It follows from Chernoff's bound (1), applied with $\varepsilon = 1/\sqrt{\omega'} \ge (\log n)^{-1}(\log\log n)^{-1}$, that

$$X = \left(1+O\left(\frac{1}{\sqrt{\omega'}}\right)\right)n(1-e^{-c})e^{-c}$$

with probability at least

$$1-\exp\Big(-\Theta(\varepsilon^2\mathbb{E}[X])\Big) = 1-\exp\Big(-\Omega((\log n)^2/(\log\log n)^2)\Big) = 1-o(n^{-2}).$$

The desired property holds by the union bound taken over all pairs x, y.

For $\log n - 4\log\log n < c \le 3\log n$, we have $\mathbb{E}[X] \le (1+o(1))\log^4 n$. We may couple the binomial random variable X with another random variable $Y \ge X$ with expectation equal to $(1+o(1))\log^4 n$. Then, we may use Chernoff's bound (1) with, say, $\varepsilon = 1$ to get that with the desired probability $X \le Y \le (2+o(1))\log^4 n$. (Alternatively, one could use a more general version of Chernoff's bound that allows $\varepsilon \ge 3/2$.) The desired bound for $X = X(x,y)$ holds a.a.s. for all pairs of x, y. $\square$

Remark 3.3. *Let us mention how we are going to apply Lemma 3.1 (or other properties that hold a.a.s.) in the paper. This is a standard technique in the theory of random graphs but it is quite delicate. We wish to use the expansion properties guaranteed a.a.s. by Lemma 3.1, but we also wish to avoid working in a conditional probability space. Thus, we will use an unconditioned probability space, but we will provide an argument that assumes we have the expansion properties of Lemma 3.1. Since these properties hold a.a.s., the measure of the set of outcomes in which our argument does not apply to is $o(1)$, and thus can be safely excised at the end of the argument.*

3.4. **Diameter of $\mathcal{G}(n,p)$.** We will use the following well-known result.

Lemma 3.4 ([1], Corollary 10.12)**.** *Suppose that $d := pn \gg \log n$ and*

$$d^{i+1}/n - 2\log n \to \infty \qquad \text{and} \qquad d^i/n - 2\log n \to -\infty.$$

Then the diameter of $G \in \mathcal{G}(n,p)$ is equal to $i+1$ a.a.s.

4. Reformulation of the Game

The localization game we investigate in this paper involves an invisible robber but it can be reformulated so that it becomes a complete information combinatorial game. After such reformulation, it will be easier to analyze the game.

Let $G = (V,E)$ be a connected graph. Given a set $S \subseteq V$ of size k, $S = \{s_1, s_2, \dots, s_k\}$, and a vertex $v \in V$, we say the S-*signature* of v is the vector $\mathbf{d} = \mathbf{d}(S,v) = (d_1, d_2, \dots, d_k)$ where $d_i = d(s_i, v)$ for each $1 \le i \le k$ is the distance from s_i to v. Then the *localization game with k sensors* is a game played with two players, the *cops* and the *robber*. In the first round, the cops choose a set $S_1 \subseteq V$, $|S_1| = k$ (called the *sensor locations*), the robber chooses any vertex $v_1 \in V$, and then the cops receive the S_1-signature of v_1, say $\mathbf{d}_1$. If the S_1-signature of v_1 is sufficient to determine the location of the robber, the cops win, otherwise the game continues to the next round. Then, in round i, the cops choose a new set $S_i \subseteq V$, and the robber chooses a vertex $v_i \in \mathcal{N}(v_{i-1}, 1)$ as her new location, and the cops learn the S_i-signature of v_i, say $\mathbf{d}_i$.

We call the sequence $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_i)$ the *info trail of the walk* $(v_1, v_2, \ldots, v_i)$ *with respect to sensor locations* $(S_1, S_2, \ldots, S_i)$. Then the cops win in round i if the info trail of the robber is sufficient to determine the location of the robber, and otherwise the game proceeds to round $i+1$. More precisely, the cops win in round i if for every two walks $W = (w_1, w_2, \ldots, w_i), X = (x_1, x_2, \ldots, x_i)$, both with info trail $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_i)$ with respect to $(S_1, S_2, \ldots, S_i)$, we have $w_i = x_i$.

The *localization number* of the graph G, denoted $\zeta(G)$ is defined to be the least integer k such that the cops can win the localization game with k sensors in finite time, regardless of the strategy of the robber.

Since the definition of localization number requires the cops to be able to win in finite time regardless of the strategy of the robber, we can view this problem equivalently as follows: when the cops choose S_1, we partition the vertex set V into $R_{1,1} \cup R_{1,2} \cup \ldots \cup R_{1,\ell_1}$ such that the sets $R_{1,j}$ are the equivalence classes of vertices in V that have the same S_1-signature for $1 \le j \le \ell_1$. Then, instead of choosing a specific location, the robber can choose some equivalence class R_{1,j_1}. Then once the cops choose S_2, we partition the set $\mathcal{N}(R_{1,j_1}, 1)$ into equivalence classes $R_{2,1} \cup R_{2,2} \cup \ldots \cup R_{2,\ell_2}$ so that every vertex in $R_{2,j}$ has the same S_2-signature. Then the robber chooses a set R_{2,j_2}. Iteratively, in round i, once the cops choose S_i, this gives the partition $\mathcal{N}(R_{i-1,j_{i-1}}, 1) = R_{i,1} \cup R_{i,2} \cup \ldots \cup R_{i,\ell_i}$ with every vertex in $R_{i,j}$ having the same S_i signature, then the robber chooses some R_{i,j_i}. In this version of the game, the cops win in round i if the robber is forced to choose a set R_{i,j_i} with only one vertex, that is, $|R_{i,j_i}| = 1$.

It can be seen that these two formulations of the localization game are equivalent in the sense that if the robber performs the walk $(v_1, v_2, \ldots, v_i)$ in response to sensor locations $(S_1, S_2, \ldots, S_i)$, this is equivalent to the robber choosing sets $(R_{1,j_1}, R_{2,j_2}, \ldots, R_{i,j_i})$, and if there is enough information to determine that the robber is at v_i at time i, it must be because $R_{i,j_i} = \{v_i\}$ has only one element. Conversely, if the robber chooses sets $(R_{1,j_1}, R_{2,j_2}, \ldots, R_{i,j_i})$ in response to the cop choosing sensor locations $(S_1, S_2, \ldots, S_i)$, then there exists at least one walk $(v_1, v_2, \ldots, v_i)$ with $v_k \in R_{k,j_k}$ for each $1 \le k \le i$, and if $|R_{i,j_i}| = 1$, we must have $R_{i,j_i} = \{v_i\}$ and every walk that shares an info trail with $(v_1, v_2, \ldots, v_i)$ must have terminal vertex v_i, so the cops locate the robber. Thus, the two formulations are equivalent.

5. Lower Bound

In this section, we will prove a lower bound. One can easily relax an upper bound of $o(n \log\log n)$ for d^i but for $d^i = \Omega(n)$ we do not get an asymptotic behaviour of $\zeta(G)$ and so we do not make an effort and claim a weaker bound.

Theorem 5.1. *Suppose that* $d := pn$ *is such that* $\log n \ll d \ll n$. *Let* $i = i(n) \in \mathbb{N}$ *be the largest integer* i *such that* $d^i \ll n \log\log n$. *Suppose that* $d^{i+1}/n - 2\log n \to \infty$.

Then the following holds a.a.s. for $G \in \mathcal{G}(n,p)$:

$$\zeta(G) \ge (\log d - 3\log\log n)\,\frac{n}{d^i}\,.$$

We start with bounding the number of vertices that are diametrically opposed to all the vertices in a set of sensors. Recall that by Lemma 3.4, G has diameter at least $i+1$ a.a.s. but the lemma below is stronger.

Lemma 5.2. *Suppose that $d := pn$ is such that $\log n \ll d \ll n$. Let $i = i(n) \in \mathbb{N}$ be the largest integer i such that $d^i \ll n \log\log n$. Let*

$$s := \Big(\log d - 3\log\log n\Big)\frac{n}{d^i} \qquad \text{and} \qquad r := \frac{n\log^3 n}{d}.$$

Then the following holds a.a.s. for $G = (V,E) \in \mathcal{G}(n,p)$: for every set $S \subseteq V$ with $|S| = s$, we have

$$|V \setminus \mathcal{N}(S,i)| = n - |\mathcal{N}(S,i)| \ge r/2.$$

Proof. Let $S \subseteq V$ be any set of size s. We will expose edges around set S to determine $\mathcal{N}(S,i-1)$. Note that for each $v \in S$, we have $\mathcal{N}(v,i-1) = \{v\} \cup \bigcup_{j=1}^{i-1} \mathcal{S}(v,j)$, so by Lemma 3.1, we may assume that

$$\begin{aligned}
|\mathcal{N}(v,i-1)| &= 1 + \sum_{j=1}^{i-1} |\mathcal{S}(v,j)| \\
&= 1 + \sum_{j=1}^{i-1} \left(1 + o\left(\frac{1}{\log n}\right)\right) d^j \\
&= \left(1 + o\left(\frac{1}{\log n}\right)\right) d^{i-1},
\end{aligned}$$

where the last equality follows from the fact that $d^j = o(d^{j+1}/\log n)$ for all $1 \le j < i-1$. (See Remark 3.3 to see how we apply the lemma.) Thus

$$|\mathcal{N}(S,i-1)| = \left|\bigcup_{v\in S} \mathcal{N}(v,i-1)\right| \le \left(1 + o\left(\frac{1}{\log n}\right)\right) d^{i-1}s.$$

Our goal is to determine the size of set $R = R(S) := V \setminus \mathcal{N}(S,i)$, the set of vertices that are at distance at least $i+1$ from every vertex of S. Note that at this point edges within $\mathcal{N}(S,i-1)$ as well as edges between $\mathcal{N}(S,i-2)$ and $V \setminus \mathcal{N}(S,i-1)$ are exposed. However, more importantly, *no* edge between $\mathcal{N}(S,i-1) \setminus \mathcal{N}(S,i-2)$ and $V \setminus \mathcal{N}(S,i-1)$ is exposed yet. Thus, R is exactly the set of vertices in $V \setminus \mathcal{N}(S,i-1)$

that are *not* adjacent to any vertex in $\mathcal{N}(S,i-1)\setminus\mathcal{N}(S,i-2)$. It follows

$$\begin{aligned}
\mathbb{E}\Big[|R|\Big] &= |V\setminus\mathcal{N}(S,i-1)|\cdot(1-p)^{|\mathcal{N}(S,i-1)\setminus\mathcal{N}(S,i-2)|}\\
&\geq (n-2d^{i-1}s)\cdot\left(1-\frac{d}{n}\right)^{(1+o(1/\log n))d^{i-1}s}\\
&\sim n\exp\Big(-(1+o(1/\log n))\,d^is/n\Big)\\
&\sim n\exp\left(-\log d+3\log\log n\right)=n\log^3 n/d=r.
\end{aligned}$$

Then by Chernoff's bound, we have

$$\Pr(|R|\leq r/2)\leq\exp(-\Theta(r))=\exp(-\Theta(n\log^3 n/d)).$$

We wish to say that a.a.s. for any set S, we have $R(S)\geq r/2$, so we will consider the union bound over all sets S. The probability that some set $R(S)$ does *not* satisfy the desired bound for its size is at most

$$\begin{aligned}
\binom{n}{s}\exp(-\Theta(n\log^3 n/d)) &\leq \left(\frac{ne}{s}\right)^s\exp(-\Theta(n\log^3 n/d))\\
&\leq \exp(s\log n-\Theta(n\log^3 n/d))\\
&= \exp(\Theta(n\log^2 n/d^i)-\Theta(n\log^3 n/d))\\
&= \exp(\Theta(-\Theta(n\log^3 n/d))=o(1).
\end{aligned}$$

It follows that, indeed, a.a.s. $R(S)\geq r/2$ for all sets $S\in\binom{V}{s}$. ∎

The next lemma will allow us to bound the number of vertices that are not reachable by the robber.

Lemma 5.3. *Suppose that $d:=pn$ is such that $\log n\ll d\ll n$. Let*

$$r:=\frac{n\log^3 n}{d}.$$

Then the following holds a.a.s. for $G=(V,E)\in\mathcal{G}(n,p)$: for every set $R\subseteq V$ with $|R|=r/4$, we have

$$|V\setminus\mathcal{N}(R,1)|\leq r/4.$$

Proof. Fix any set $R\in\binom{V}{r/4}$. Our goal is to estimate the size of set $U=U(R):=V\setminus\mathcal{N}(R,1)$, that is, the set of vertices of $V\setminus R$ that are not adjacent to any vertex in

R. Clearly,

$$\begin{aligned}
\Pr(|U| \geq r/4) &\leq \binom{|V \setminus R|}{r/4} \left((1-p)^{|R|}\right)^{r/4} \\
&\leq \binom{n}{r/4} \left((1-p)^{r/4}\right)^{r/4} \\
&\leq \left(\frac{4ne}{r}\right)^{r/4} \cdot \exp\left(-\frac{d}{n} \cdot \frac{r}{4} \cdot \frac{r}{4}\right) \\
&\leq \exp\left(\frac{r}{4}\log n - \frac{r}{16}\log^3 n\right) \\
&= \exp\left(-\Theta(r\log^3 n)\right).
\end{aligned}$$

Hence, by the union bound, the probability that some set $U(R)$ does *not* satisfy the desired bound for its size is at most

$$\binom{n}{r/4} \exp\left(-\Theta(r\log^3 n)\right) \leq \exp\left(\Theta(r\log n) - \Theta(r\log^3 n)\right) = \exp\left(-\Theta(r\log^3 n)\right) = o(1).$$

It follows that a.a.s. $|U(R)| \leq r/4$ for all sets R of size $r/4$. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Since we aim for a statement that holds a.a.s. we may assume that we have a deterministic graph G that satisfies the properties in the conclusions of Lemmas 5.2, 5.3 and 3.4. The strategy for the robber is simple; he always stays in the equivalence class of vertices whose S_j-signature is $(i+1, i+1, \ldots, i+1)$.

Let $r := n\log^3 n/d$. Assume the cops first choose set S_1 of size s as the sensor locations. Combining Lemma 5.2 and Lemma 3.4 we get that the equivalence class of vertices with S_1-signature $(i+1, i+1, \ldots, i+1)$, call it $X_1 = V_1$ is at least of size $r/2 \geq r/4$. Indeed, Lemma 5.2 provides a bound for the size of *all* equivalence classes with at least one value at most i in their signatures. Lemma 3.4 guarantees that the only equivalence class left is the class with signature $(i+1, i+1, \ldots, i+1)$. The robber will choose this equivalence class. We can continue iteratively: for $j \in \mathbb{N}$, assume that the robber has chosen a set V_j of size at least $r/4$, and the cops respond with sensors at set S_{j+1}. Then let X_{j+1} be the set of all vertices with S_{j+1}-signature $(i+1, i+1, \ldots, i+1)$. By Lemma 5.2 and Lemma 3.4, $|X_{j+1}| \geq r/2$, and by Lemma 5.3, $V_{j+1} := N(V_j, 1) \cap X_{j+1}$ is of size at least $r/4$. Thus the robber can always choose the equivalence class of vertices with signature $(i+1, i+1, \ldots, i+1)$, and this class will always be of size at least $r/4$. This shows that the cops will never be able to locate the robber. □

6. Upper Bound

In this section, we will prove two upper bounds. The first one will apply to random graphs with $pn \gg \log n$, the diameter equal to $i+1$, and when $d^i = o(n)$. The second bound will cover the remaining cases, provided that $pn \gg \log^3 n$.

Theorem 6.1. *Suppose that $d := pn$ is such that $\log n \ll d \ll n$. Let $i = i(n) \in \mathbb{N}$ be the largest integer i such that $d^i \ll n$. If $d^{i+1}/n - 2\log n \to \infty$, then the following holds a.a.s. for $G \in \mathcal{G}(n,p)$:*

$$\zeta(G) \le (1+o(1))\,(\log d + 2\log\log n)\,\frac{n}{d^i}\,.$$

Proof. In fact, we will prove something slightly stronger. Let

$$\omega = \omega(n) := \min\left\{\frac{d}{\log n}, \frac{n}{d^i}, (\log n)^4(\log\log n)^2\right\}.$$

Since $d \gg \log$ and $d^i \ll n$, we get that $\omega \to \infty$ as $n \to \infty$. Suppose that $G_n = (V_n, E_n)$ is a family of graphs that satisfies the following properties: for each $n \in \mathbb{N}$

(a) $|V_n| = n$,
(b) for every $x, y \in V_n$ ($x \ne y$) and $j \in \mathbb{N}$ such that $1 \le j \le i$ we have
$$|\mathcal{S}(x,j) \setminus \mathcal{S}(y,j)| = (1 + O(1/\sqrt{\omega}))d^j,$$
(c) the diameter of G_n is $i+1$,
(d) the maximum degree of G_n is $(1+o(1))d$.

Then, there exists some $N \in \mathbb{N}$ (that depends only on the bounds in (b) and (d), and not on the family G_n) such that for all $n \ge N$, (deterministically!)

$$\zeta(G_n) \le k := \left(1 + \frac{1}{\omega^{1/3}}\right)(\log d + 2\log\log n)\frac{n}{d^i} \sim (\log d + 2\log\log n)\frac{n}{d^i}.$$

The result will follow from Lemma 3.1 (that shows that $\mathcal{G}(n,p)$ satisfies property (b) and (d) a.a.s. with a uniform choice of error function) and Lemma 3.4 (that shows that property (c) is satisfied a.a.s.). Indeed, Lemma 3.1 can be applied as $d \ge \omega \log n$, $d^i/n \le 1/\omega = O(1/\sqrt{\omega})$, and $\omega \le (\log n)^4(\log\log n)^2$—see the definition of ω. Lemma 3.4 can be applied as $d^i/n - 2\log n = o(1) - 2\log n \to -\infty$ and, by assumption, $d^{i+1}/n - 2\log n \to \infty$.

Let us then concentrate on a deterministic family of graphs $G_n = (V_n, E_n)$ satisfying (a)-(d). Recall that in Section 4 we reformulated the game so that it can be viewed as a perfect information game, and so we may assume that the moves of the robber are guided by a perfect player that has a fixed strategy for a given deterministic graph G_n. In particular, the robber chooses sets $(R_{1,j_1}, R_{2,j_2}, \ldots, R_{i,j_i})$ in response to the cop choosing sensor locations $(S_1, S_2, \ldots, S_i)$. Such responses are predetermined before the game actually starts. See Section 4 for more details.

On the other hand, to get an upper bound for the localization number, the cops are going to use a simple strategy, namely, at each round t of the game, the cops choose a random set $S_t \subseteq V_n$ of cardinality k for the sensor locations (regardless of anything that happened during the game thus far). Clearly, this is a sub-optimal strategy but, perhaps surprisingly, it turns out that it works very well.

Trivially, $|\mathcal{N}(R_{1,j_1}, 1)| \le n$. Our goal is to show that with high probability, for each round t, we have

$$|\mathcal{N}(R_{t+1,j_{t+1}}, 1)| \le |\mathcal{N}(R_{t,j_t}, 1)|/\log n.$$

As a result, this bound will hold a.a.s. for $1 \le t \le t_F := \log n / \log\log n$, and so $|\mathcal{N}(R_{t+1,j_{t+1}}, 1)| \le n/\log^t n$. In particular, we will get that $|\mathcal{N}(R_{t_F+1,j_{t_F+1}}, 1)| \le 1$ and so the cops win before the end of round $t_F + 1$.

Suppose that at some round t, the robber "occupies" set R_{t,j_t} in response to the cop choosing sensor locations $(S_1, S_2, \ldots, S_t)$. As mentioned above, the cops choose set S_{t+1} at random. It would be convenient to generate this random set as follows: select k vertices to form S_{t+1} one by one, each time choose a random vertex with uniform probability from the set of vertices not selected yet. Once S_{t+1} is fixed, the set $\mathcal{N}(R_{t,j_t}, 1)$ is partitioned into sets having the same S_{t+1}-signatures. The robber then has to pick $R_{t+1,j_{t+1}}$, one of the equivalence classes. We will show that, regardless of her choice, $|\mathcal{N}(R_{t+1,j_{t+1}}, 1)| \le |\mathcal{N}(R_{t,j_t}, 1)|/\log n$ will hold with high probability.

There are

$$\binom{|\mathcal{N}(R_{t.j_t}, 1)|}{2} \le |\mathcal{N}(R_{t,j_t}, 1)|^2$$

pairs of vertices. Let us focus on one such pair, x, y, and suppose that the cops put a sensor on some vertex $v \in V_n$. Note that this pair is distinguished by v *if and only if* v belongs to the set

$$\begin{aligned} D(x,y) &:= \bigcup_{j\ge 0} \Big(\mathcal{S}(x,j) \setminus \mathcal{S}(y,j)\Big) \cup \Big(\mathcal{S}(y,j) \setminus \mathcal{S}(x,j)\Big) \\ &= \bigcup_{j=0}^{i} \Big(\mathcal{S}(x,j) \setminus \mathcal{S}(y,j)\Big) \cup \Big(\mathcal{S}(y,j) \setminus \mathcal{S}(x,j)\Big). \end{aligned}$$

Indeed, if $v \in \mathcal{S}(x,j) \setminus \mathcal{S}(y,j)$, then the distance between v and x is j but the distance between v and y is not. Moreover, since the diameter of G_n is $i+1$ (property (c)), in order to distinguish the pair x, y, the distance from v to at least one of x, y has to be at most i. This justifies the equality above. By property (b), we may estimate the size of the distinguishing set as follows:

$$|D(x,y)| = \sum_{j=0}^{i} (1 + O(1/\sqrt{\omega}))2d^j = (1 + O(1/\sqrt{\omega}))2d^i.$$

The probability that the pair cannot be distinguished by any of the sensors in S_{t+1} is at most

$$\begin{aligned} \Big(1 - |D(x,y)|/n\Big)^k &= \Big(1 - (1 + O(1/\sqrt{\omega}))2d^i/n\Big)^k \\ &= \exp\Big(-(1 + O(1/\sqrt{\omega}))2d^i k/n\Big) \\ &= \exp\Big(-(1 + 1/\omega^{1/3})(1 + O(1/\sqrt{\omega}))2(\log d + 2\log\log n)\Big) \\ &\le \exp\Big(-2(\log d + 2\log\log n)\Big) = \frac{1}{d^2 \log^4 n}. \end{aligned}$$

Let X_{t+1} be the number of pairs in $\mathcal{N}(R_{t.j_t}, 1)$ with the same signature in S_{t+1}. Since $\mathbb{E}[X_{t+1}] \le |\mathcal{N}(R_{t,j_t}, 1)|^2 d^{-2} \log^{-4} n$, it follows immediately from Markov's inequality that $X_{t+1} \le |\mathcal{N}(R_{t,j_t}, 1)|^2 d^{-2} \log^{-3} n$ with probability at least $1 - 1/\log n$. If this bound holds, then we say the round is *good*. If this is the case, then, regardless which equivalence class of the partition of $\mathcal{N}(R_{t,j_t},1) = R_{t+1,1} \cup R_{t+1,2} \cup \ldots \cup R_{t+1,\ell_{t+1}}$ the robber selects as her response, the selected set $R_{t+1,j_{t+1}}$ is of size at most $2\sqrt{X_{t+1}} \leq 2|\mathcal{N}(R_{t,j_t},1)| d^{-1} \log^{-3/2} n$. Indeed, note that

$$X_{t+1} = \sum_{j=1}^{\ell_{t+1}} \binom{|R_{t+1,j}|}{2} \geq \binom{|R_{t+1,j_{t+1}}|}{2} \geq |R_{t+1,j_{t+1}}|^2/4.$$

Finally, since the maximum degree of G_n is asymptotic to d (property (d)), the closed neighbourhood of $R_{t+1,j_{t+1}}$ has the size at most

$$(2+o(1))|\mathcal{N}(R_{t,j_t},1)| \log^{-3/2} n \leq |\mathcal{N}(R_{t,j_t},1)| \log^{-1} n,$$

as required.

It remains to show that a.a.s. the first $t_F = \log n/\log\log n$ rounds are good. Since each round is *not* good with probability at most $1/\log n$, the probability that some round is not good is at most $t_F/\log n = o(1)$, and the proof is finished. We get that this randomized strategy for k cops works a.a.s. and so the probability it works is larger than, say, $1/2$ for n sufficiently large. It follows that the cops have a winning strategy and so the claimed bound for $\zeta(G_n)$ holds deterministically. $\square$

Before we move to the upper bound that covers the remaining cases, let us briefly discuss why the bound changes. The size of the set $D(x,y)$ defined in the proof above that distinguishes the pair of vertices (x,y) plays an important role in the proof—the larger the set, the smaller the upper bound we get. We noticed that

$$s = s(n) := |D(x,y)| = \sum_{j\geq 0} s_j,$$

where

$$s_j := \Big|\Big(\mathcal{S}(x,j) \setminus \mathcal{S}(y,j)\Big) \cup \Big(\mathcal{S}(y,j) \setminus \mathcal{S}(x,j)\Big)\Big|.$$

Suppose that $d \gg \log^3 n$. Let $i = i(n) \in \mathbb{N}$ be the largest integer i such that $d^i \leq 3\log n$, and let $c = c(n) = d^i/n$. The previous bound, Theorem 6.1, applies to the case when $c = o(1)$; in particular, the diameter is equal to $i+1$ a.a.s. For this case, s_i is the dominating term in the sum: $s \sim s_i \sim 2d^i$. If $c \to A \in (0,\infty)$, then $s \sim s_i \sim 2n(1-e^{-A})e^{-A}$; in particular, s increases when $A \in (0,\log 2)$ reaching its maximum at $(1/2+o(1))n$ but then it starts decreasing when $A \in (\log 2,\infty)$. When $c \to \infty$ but

$$c - (\log d - \log\log d) \to B \in \mathbb{R},$$

then s is dominated by two terms: $s_{i-1} \sim 2d^{i-1}$, and

$$s_i \sim 2ne^{-c} \sim \frac{2n(\log d)e^{-B}}{d} \sim \frac{2nce^{-B}}{d} = 2d^{i-1}e^{-B}.$$

It follows that $s \sim s_{i-1} + s_i \sim 2d^{i-1}(1+e^{-B}) \sim 2ne^{-c}(e^B+1)$. In particular, $s \sim 2ne^{-c}$ when $B \to -\infty$ and $s \sim 2d^{i-1}$ when $B \to \infty$. Here is the summary of our observations:

$$s \sim \begin{cases} 2d^i & \text{if } c = o(1) \\ 2n(1-e^{-A})e^{-A} & \text{if } c \to A \in \mathbb{R}_+ \\ 2ne^{-c} & \text{if } c \to \infty \text{ and } c - (\log d - \log\log d) \to -\infty \\ 2d^{i-1}(1+e^{-B}) = 2ne^{-c}(e^B+1) & \text{if } c - (\log d - \log\log d) \to B \in \mathbb{R} \\ 2d^{i-1} & \text{if } c - (\log d - \log\log d) \to \infty \text{ and } c \le 3\log n. \end{cases}$$

We are now ready to cover the remaining cases that Theorem 6.1 did not cover, and finalize the upper bound.

Theorem 6.2. *Suppose that $d := pn$ is such that $\log^3 n \ll d \ll n$. Let $i = i(n) \in \mathbb{N}$ be the largest integer i such that $d^i \le 3\log n$, and $c = c(n) = d^i/n$. Then, the following holds a.a.s. for $G \in \mathcal{G}(n,p)$.*

(i) *if $c \to A \in \mathbb{R}_+$, then*

$$\zeta(G) \le (1+o(1))\left(\log d + 2\log\log n\right)\frac{e^A}{1-e^{-A}}.$$

(ii) *if $c \to \infty$ and $c - (\log d - \log\log d) \to -\infty$, then*

$$\zeta(G) \le (1+o(1))\left(\log d + 2\log\log n\right)e^c.$$

(iii) *if $c - (\log d - \log\log d) \to B \in \mathbb{R}$, then*

$$\begin{aligned} \zeta(G) &\le (1+o(1))\left(\log d + 2\log\log n\right)\frac{e^c}{e^B+1} \\ &\sim \left(\log d + 2\log\log n\right)\frac{n}{d^{i-1}(1+e^{-B})}. \end{aligned}$$

(iv) *if $c - (\log d - \log\log d) \to \infty$ and $c \le 3\log n$, then*

$$\zeta(G) \le (1+o(1))\left(\log d + 2\log\log n\right)\frac{n}{d^{i-1}}.$$

Proof. The proof of this theorem is almost identical to the one of Theorem 6.1, and so we will only highlight differences. We will use the definitions of s and s_j that we introduced right before the statement of this theorem. We used Lemma 3.1 to estimate s in Theorem 6.1 but this time we will also need Lemma 3.2. As the asymptotic behaviour of s changes, we will need to adjust k accordingly. However, in each case, $k \sim 2n(\log d + 2\log\log n)/s$. We will deal with each case independently.

For part (i), after setting

$$\omega' = \omega'(n) = \min\left\{\frac{d}{\log^3 n}, \frac{\log^2 n}{(\log\log n)^2}\right\},$$

we get that

$$\begin{aligned} s &= \sum_{j=0}^{i} s_j = \sum_{j=0}^{i-1} s_j + s_i = (1+o(1))2d^{i-1} + (1+O(1/\sqrt{\omega'}))2n(1-e^{-A})e^{-A} \\ &= (1+O(1/\sqrt{\omega'}))2n(1-e^{-A})e^{-A}, \end{aligned}$$

and so the upper bound has to be adjusted to

$$k := \left(1 + \frac{1}{\omega'^{1/3}}\right)(\log d + 2\log\log n)\frac{e^A}{1-e^{-A}}.$$

For part (iii), we set

$$\omega' = \omega'(n) = \min\left\{\frac{d}{\log^3 n}, \log\log n\right\},$$

and observe that

$$\begin{aligned} s_i &= (1+O(1/\sqrt{\omega'}))2n(1-e^{-c})e^{-c} \\ &= (1+O(1/\sqrt{\omega'}))2n(1+O(1/c))e^{-c} \\ &= (1+O(1/\sqrt{\omega'}))2ne^{-c} \\ &= (1+O(1/\sqrt{\omega'}))2nd^{-1}(\log d)e^{-B} \\ &= (1+O(1/\sqrt{\omega'}))2nd^{-1}c(1+O(\log\log d/\log d))e^{-B} \\ &= (1+O(1/\sqrt{\omega'}))2cnd^{-1}e^{-B} \\ &= (1+O(1/\sqrt{\omega'}))2d^{i-1}e^{-B}. \end{aligned}$$

On the other hand,

$$\sum_{j=0}^{i-1} s_j = (1+O(1/d))s_{i-1} = (1+O(1/(\sqrt{\omega'}\log n)))2d^{i-1} = (1+O(1/\sqrt{\omega'}))2d^{i-1}.$$

It follows that $s = (1+O(1/\sqrt{\omega'}))2d^{i-1}(1+e^{-B})$, and so the upper bound has to be adjusted to

$$k := \left(1 + \frac{1}{\omega'^{1/3}}\right)(\log d + 2\log\log n)\frac{n}{d^{i-1}(1-e^{-B})}.$$

For part (ii), we observe that $s_i/s_{i-1} = e^{-B} \to \infty$. One can apply a trivial bound $s \ge s_i = (1+O(1/\sqrt{\omega'}))2d^{i-1}e^{-B}$ and adjust the definition of k to get the desired bound. Similarly, for part (iv), we observe that $s_i/s_{i-1} = e^{-B} \to 0$. (In fact, if $c - 2\log n \to \infty$, then a.a.s. the diameter of a random graph is i and so there is no need to consider s_i anymore.) This time we use the fact that $s \ge s_{i-1} = (1+O(1/\sqrt{\omega'}))2d^{i-1}$. The claimed bound holds and the proof is finished. □

References

[1] B. Bollobás, *Random Graphs*, Cambridge University Press, Cambridge, 2001.
[2] B. Bollobás, D. Mitsche, and P. Prałat, Metric dimension for random graphs, Electronic Journal of Combinatorics 20(4) (2013), #P1.
[3] A. Bonato and W.B. Kinnersley, Bounds on the localization number, preprint.

[4] A. Bonato and P. Prałat, Graph Searching Games and Probabilistic Methods, CRC Press, 2017.
[5] B. Bosek, P. Gordinowicz, J. Grytczuk, N. Nisse, J. Sokół, M. Śleszyńska-Nowak, Localization game on geometric and planar graphs, Discrete Applied Mathematics 251 (2018) 30–39.
[6] A. Brandt, J. Diemunsch, C. Erbes, J. LeGrand, C. Moffatt, A robber locating strategy for trees, Discrete Applied Mathematics 232 (2017) 99–106.
[7] J. Carraher, I. Choi, M. Delcourt, L.H. Erickson, D.B. West, Locating a robber on a graph via distance queries, Theoretical Computer Science 463 (2012) 54–61.
[8] A. Dudek, A. Frieze, and W. Pegden, A note on the localization number of random graphs: diameter two case, Discrete Applied Mathematics 254, 107–112.
[9] J. Haslegrave, R.A.B. Johnson, S. Koch, Locating a robber with multiple probes, Discrete Mathematics 341 (2018) 184–193.
[10] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley, New York, 2000.
[11] S. Seager, Locating a robber on a graph, Discrete Mathematics 312 (2012) 3265–3269.
[12] S. Seager, Locating a backtracking robber on a tree, Theoretical Computer Science 539 (2014) 28–37.

Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA
E-mail address: andrzej.dudek@wmich.edu

Department of Mathematics, University of Illinois Urbana-Champaign, Champaign, IL, USA
E-mail address: SEnglish@Illinois.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA
E-mail address: alan@random.math.cmu.edu

Department of Computer Science, University of Toronto, Toronto, ON, Canada
E-mail address: cmacrury@cs.toronto.edu

Department of Mathematics, Ryerson University, Toronto, ON, Canada
E-mail address: pralat@ryerson.ca