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On the Independence and Chromatic Numbers of Random Regular Graphs

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Let G_r denote a random r-regular graph with vertex set $\{1, 2, ..., n\}$ and $\alpha(G_r)$ and $\chi(G_r)$ denote respectively its independence and chromatic numbers. We show that with probability going to 1 as $n \to \infty$ respectively

$$\left|\alpha(G_r) - \frac{2n}{r}(\log r - \log\log r + 1 - \log 2)\right| \leqslant \frac{\varepsilon n}{r}$$

and

$$\left|\chi(G_r) - \frac{r}{2\log r} - \frac{8r\log\log r}{(\log r)^2}\right| \leqslant \frac{8r\log\log r}{(\log r)^2}$$

provided $r = o(n^\theta)$, $\theta < 1/3$, $0 < \varepsilon < 1$, are constants, and $r \geqslant r_\varepsilon$, where r_ε depends on ε only.

This paper is concerned with the independence and chromatic numbers of random regular graphs. Thus let REG(n, r) denote the set of r-regular graphs with vertex set $[n] = \{1, 2, ..., n\}$. Let G_r denote a random graph sampled uniformly from REG(n, r). We use α and χ for independence and chromatic numbers, respectively.

In random graph theory, these have been studied by, inter alia, Matula [13], Grimmett and McDiarmid [9], Bollobás and Erdős [6], Shamir and Spencer [14], Bollobás [5], Frieze [8], and Łuczak [12]. The aim nof this paper is to extend the results of [8, 12] to G_r and prove

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THEOREM. (a) *Let* $0<\varepsilon<1$ *be fixed. There exists a constant* r_ε *such that if* $r \geqslant r_\varepsilon$, $r=o(n^\theta)$, $\theta<1/3$ *constant, then*

$$\left|\alpha(G_r)-\frac{2n}{r}(\log r-\log\log r+1-\log 2)\right| \leqslant \frac{\varepsilon n}{r}$$

with probability going to 1 *as* $n \to \infty$.

(b) *Moreover, for some constant* r_0 *and* $r_0<r=o(n^\theta)$, *we have*

$$\frac{r}{2\log r} \leqslant \chi(G_r) \leqslant \frac{r}{2\log r}\left(1+\frac{32\log\log r}{\log r}\right)$$

with probability going to 1 *as* $n \to \infty$.

(All logarithms are natural.)

Proof of the Theorem. We will first proceed under the assumption that r is constant. The extension to r growing will then be straightforward. We shall use the model of Bender and Canfield [2] and Bollobás [4] to study G_r. Specifically, we will adopt the configuration terminology of [4]. We let $W=[rn]$ and $W_i=\{(i-1)r+1, ..., ir\}$, $i=1, 2, ..., n$, be a partition of W into n sets of size r. For $w \in W$ we define $\psi(w)=\lceil w/r \rceil$ so that $w \in W_{\psi(w)}$ holds.

A configuration on a set Z, $|Z|=2k$, is a partition of Z into k pairs. Φ_Z denotes the set of configurations on Z. If $Z \subseteq W$ and $F \in \Phi_Z$ we let $\mu(F)$ be the multigraph with vertex set $[n]$ and k edges $\{\psi(x)\,\psi(y) : \{x, y\} \in F\}$.

We consider $\Phi=\Phi_W$ as a probability space in which each $F \in \Phi$ is equally likely. Let Q be a property of the graphs in $\mathrm{REG}(n, r)$ and let Q^* be a property of the configurations in Φ. Suppose these properties are such that for $G_r \in \mathrm{REG}(n, r)$ and $F \in \mu^{-1}(G_r)$, G_r has Q if and only if F has Q^*. All we shall need from [2, 4] is

$$Pr(G_r \in Q) \leqslant e^{r^2} Pr(F \in Q^*). \tag{0}$$

In the analysis we only claim that inequalities hold for r and n sufficiently large and ε sufficiently small.

It is well known that the binomial random variable $B(n, p)$ is sharply concentrated around its expected value np, if this is large.

We will rather loosely refer to the following as the "Chernoff bounds":

$$Pr(B(n, p) \leqslant (1-\beta)\,np) \leqslant e^{-\beta^2 np/2}$$
$$Pr(B(n, p) \geqslant (1+\beta)\,np) \leqslant e^{-\beta^2 np/3}$$

for $0 \leqslant \beta \leqslant 1$.

We will concentrate first on the independence number. The most difficult task is to bound $\alpha(F)=\alpha(\mu(F))$ from below, with high probability. To do this we will generate a random F in a somewhat complicated way. Our purpose is to use the result of [8] "halfway" through the construction.

For a multigraph H and vertex v of H we let $d(v, H)$ denote the degree of v in H, where loops count twice.

Step 1. Let $r_1 = r - \lceil r^{1/2} \log r \rceil$ and $m_1 = \lfloor r_1 n/2 \rfloor$, and let $X=(x_1, x_2, ..., x_{2m_1})$ be a random member of $[n]^{2m_1}$; i.e., $x_1, x_2, ..., x_{2m_1}$ are chosen independently at random from $[n]$. Let now G_1 denote the multigraph with vertex set $[n]$ and edge set $\{x_{2i-1}x_{2i} : 1 \leqslant i \leqslant n_1\}$.

Step 2. The next step is to delete edges from G_1 so that each vertex has degree at most r and to construct most of F.

Let $d_j = d(j, G_1)$ and $Y_1, Y_2, ..., Y_n$ be a partition of $Y=[2m_1]$ with $|Y_j| = d_j$ for $1 \leqslant j \leqslant n$. In fact let $Y_1=[d_1]$ and $Y_i = [\sum_{t=1}^{i} d_t] \backslash [\sum_{t=1}^{i-1} d_t]$ for $i \geqslant 2$. We construct a configuration F_1 on $Y = \bigcup_{j=1}^{n} Y_j$.

begin
  $F_1 := \varnothing$; $Y'_j := Y_j$ for $1 \leqslant j \leqslant n$;
  for $t := 1$ to m_1 do
  begin
    randomly choose $p_{2t-1} \in Y'_{x_{2t-1}}$; $Y'_{x_{2t-1}} := Y'_{x_{2t-1}} - \{p_{2t-1}\}$;
    randomly choose $p_{2t} \in Y'_{x_{2t}}$; $Y'_{x_{2t}} := Y'_{x_{2t}} - \{p_{2t}\}$;
    $F_1 := F_1 \cup \{\{p_{2t-1}, p_{2t}\}\}$
  end
end

CLAIM 1. *F_1 is a random configuration on Y.*

Proof. $p_1, p_2, ..., p_{2m_1}$ is a random permutation of Y since interchanging p_k, p_{k+1} yields the same distribution of permutations. Each partition arises from $2^{m_1} m_1!$ permutations. ∎

For $s \in Y$ let its rank $\rho(s) = s - \sum_{i=1}^{j-1} d_i$, where $s \in Y_j$, so that Y_j has elements of rank $1, 2, ..., d_j$. Let

$$F'_1 = \{\{p, q\} \in F_1 : \max\{\rho(p), \rho(q)\} \leqslant r\}.$$

$$F_2 = \{\{\sigma(p), \sigma(q)\} : \{p, q\} \in F'_1\},$$

where if $p \in Y_j$, $\sigma(p) = (j-1)r + \rho(p)$, and $Z = \bigcup_{e \in F_2} e$.

CLAIM 2. *F_2 is a random configuration on Z.*

Proof. F_1 is a random configuration on Y implies F'_1 is a random configuration on $\bigcup_{e \in F'_1} e$, since any such configuration has the same number of extensions to an F_1. Since F_2 is obtained by a fixed relabelling of $\bigcup_{e \in F_1'} e$, the result follows. ∎

We let G_2 be the multigraph $\mu(F_2)$.

Step 3. We now enlarge F_2 so that it "covers" the whole of W.

Suppose $Z \neq W$, $x_1, x_2 \in W - Z$, and $Z' = Z \cup \{x_1, x_2\}$. We define a function $f: \Phi_{Z'} \to \Phi_Z$ as follows: let $F' \in \Phi_{Z'}$.

(a) If $\{x_1, x_2\} \in F'$ then

$$f(F') = F' - \{\{x_1, x_2\}\},$$

otherwise

(b) suppose $\{x_1, z_1\}, \{x_2, z_2\} \in F'$, $(z_1 \neq z_2)$, then

$$f(F') = (F' \cup \{\{z_1, z_2\}\}) - \{\{x_1, z_1\}, \{x_2, z_2\}\}.$$

CLAIM 3. *If* $F \in \Phi_Z$ *then* $|f^{-1}(F)| = |Z| + 1$.

Proof. If $F' \in f^{-1}(F)$ then either

(a) $F' = F \cup \{\{x_1, x_2\}\}$ or

(b) $F' = F \cup \{\{x_1, z_1\}, \{x_2, z_2\}\} - \{\{z_1, z_2\}\}$ for some $\{z_1, z_2\} \in F$. ∎

It follows from Claim 2 and Claim 3 that the following algorithm generates a random configuration F_2' on Z':

ADD(F_2, x_1, x_2):
begin
  With probability $(|Z| + 1)^{-1}$ let $F_2' = F_2 \cup \{\{x_1, x_2\}\}$ else randomly choose
  $\{z_1, z_2\} \in F_2$ (randomly ordered z_1, z_2) and then let
  $F_2' = (F_2 \cup \{\{x_1, z_1\}, \{x_2, z_2\}) - \{\{z_1, z_2\}\}$
  Output F_2'
end

Hence if $W - Z = \{x_1, x_2, \ldots, x_{2s}\}$ the following algorithm constructs a random configuration F on W:

FINISH
begin
  $F := F_2$,
  for $i = 1$ to s do $F := \text{ADD}(F, x_{2i-1}, x_{2i})$
end

We will now show that with high probability

(i) G_1, and hence G_2, has an independent set of the required size.

(ii) Algorithm FINISH does not disturb this set too much.

To prove (i) we observe that if $G_{n,m}$ denotes the standard random graph with vertex set $[n]$ and m edges then G_{n,m_1} can be generated by adding a random number of extra random edges to the graph obtained by deleting loops and coalescing multipled edges in G_1.

Now it was shown in [8] that if $r_1 \geqslant r_\varepsilon$ then

$$Pr\left(\alpha(G_{n,m_1}) \leqslant \frac{2n}{r_1}(\log r_1 - \log\log r_1 + 1 - \log 2 - \varepsilon)\right) \leqslant \exp\left\{-\frac{An}{r_1(\log r_1)^2}\right\}$$

for some "constant" $A = A(\varepsilon)$.

It follows from this and the fact that $r_1 = r(1 - O(\log r/r^{1/2}))$ that

$$Pr\left(\alpha(G_1) \leqslant \alpha_\varepsilon = \frac{2n}{r}(\log r - \log\log r + 1 - \log 2 - \varepsilon)\right) \leqslant \exp\left\{-\frac{Bn}{r(\log r)^2}\right\}, \tag{1}$$

where $B = B(\varepsilon)$.

We now show that the transition from F_2 to F does not create too many edges contained in a given large independent set of G_1 and G_2.

Now let $d_j' = d(j, G_2)$ for $j \in [n]$ and $S_0 = \{j: d_j' \leqslant r - 3r^{1/2}\log r\}$. Our next task is to prove

$$Pr\left(|S_0| \geqslant \frac{n}{r^{1+\theta}}\right) \leqslant e^{-Cn/r^{3+2\theta}} \tag{2}$$

for some constant $C > 0$.

Now if $k \in S_0$ then either

(a) $k \in S_1 = \{j : d_j \leqslant r - 2r^{1/2}\log r\}$ or

(b) $k \in S_2 = \{j : d_j - d_j' \geqslant r^{1/2}\log r\}$.

Now

$$Pr(1 \in S_1) = Pr\left(B\left(2m_1, \frac{1}{n}\right) \leqslant r_1 - r^{1/2}\log r\right)$$

(where $B(\cdot,\cdot)$ denotes a binomial random variable)

$$\leqslant e^{-(\log r)^2/3}$$

from the Chernoff bound for the tails of the binomial. Thus

$$E(|S_1|) \leqslant ne^{-(\log r)^2/3}.$$

Now the events $i \in S_1$, $j \in S_1$ for $i \neq j$ are not independent. But on the other hand changing any x_i can only change $|S_1|$ by at most one and so, by the Hoeffding–Azuma inequality [1]

$$Pr(|S_1| \geqslant ne^{-(\log r)^2/3} + u) \leqslant \exp\left\{-\frac{2u^2}{rn}\right\}. \tag{3}$$

This inequality implies that if $\xi = \xi(X)$ is a random variable such that

$$|\xi(X) - \xi(X')| \leqslant d$$

whenever X and X' differ only in one component, then

$$Pr(\xi - E(\xi) \geqslant u) \leqslant \exp\left\{-\frac{2u^2}{2m_1 d^2}\right\}$$

(see, for example, Bollobás [7] or McDiarmid [10]).

To handle S_2 we define δ_{ij}, $i \in [m_1]$, $j \in [n]$ by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } \begin{matrix} \text{(a)} & j \in \{x_{2i-1}, x_{2i}\}, \\ \text{(b)} & \max\{d_{x_{2i-1}}, d_{x_{2i}}\} > r \end{matrix} \\ 0 & \text{otherwise.} \end{cases}$$

Then $j \in S_2$ implies $\sum_{i=1}^{m_1} \delta_{i,j} \geqslant r^{1/2} \log r$. So let $S_2' = \{j : \sum_{i=1}^{m_1} \delta_{ij} \geqslant r^{1/2} \log r\} \supseteq S_2$ and observe that S_2' depends only on X, unlike S_2 which depends on $p_1, ..., p_{2m_1}$ as well. Let now $\delta = \delta_{1,1}$. Then

$$\begin{aligned} Pr(\delta = 1) &\leqslant 2Pr(x_1 = 1 \text{ and } d_1 > r) + 2Pr(x_2 = 1 \text{ and } d_1 > r) \\ &= \frac{2}{n}\,(Pr(d_1 > r \mid x_1 = 1) + Pr(d_1 > r \mid x_1 = 2)) \\ &\leqslant \frac{4}{n}\,Pr(d_1 > r \mid x_1 = 1) \\ &= \frac{4}{n}\,Pr\left(B\left(2m_1 - 1, \frac{1}{n}\right) \geqslant r\right) \\ &\leqslant \frac{4}{n}\,e^{-(\log r)^2/4} \qquad \text{(using the Chernoff bound).} \end{aligned}$$

Hence

$$E\left(\sum_{i=1}^{m_1} \delta_{i,j}\right) \leqslant 2re^{-(\log r)^2/4} \qquad j \in [n]$$

and so

$$Pr\left(\sum_{i=1}^{m_1} \delta_{i,j} \geqslant r^{1/2} \log r\right) \leqslant 2(r^{1/2}/\log r)\, e^{-(\log r)^2/4} \qquad j \in [n]$$

and so $E(|S_2'|) \leqslant 2n(r^{1/2}/\log r)\, e^{-(\log r)^2/4}$.

Now changing any x_i can change at most 4 $\delta_{i,j}$'s and hence $|S_2'|$ by at most 4 and so by the Hoeffding–Azuma inequality

$$Pr(|S_2'| \geqslant 2n(r^{1/2}/\log r)\, e^{-(\log r)^2/4} + u) \leqslant \exp\left\{-\frac{u^2}{8rn}\right\}. \tag{4}$$

Inequality (2) follows from (3) and (4) with $u = n/3r^{1+\theta}$. So let us now assume that $\alpha(G_2) > \alpha_\varepsilon$ (see (1)) and $|S_0| < n/r^{1+\theta}$. We consider the effect of FINISH.

Let T be an independent set of vertices of G_2 of size $\lceil \alpha_\varepsilon \rceil$. Assume that $W - Z = \{x_1, x_2, ..., x_{2s}\}$ where $\psi(x_j) \in T$ iff $j \in \{1, 3, 5, 7, ..., 2\tau - 1\}$. We must estimate the number γ of *bad* edges which (i) are in $\mu(F)$ and (ii) are contained in T.

Note that an execution of the statement σ_i:

$$F := \text{ADD}(F, x_{2i-1}, x_{2i})$$

can only contribute to γ if $i \leqslant \tau$ and that

$$\begin{aligned} \tau &\leqslant r\,|S_0| + 3r^{1/2} \log r\,|T| \\ &\leqslant r\,\frac{n}{r^{1+\theta}} + 3r^{1/2} \log r \lceil \alpha_\varepsilon \rceil \\ &\leqslant \frac{2n}{r^\theta}. \end{aligned}$$

But σ_i creates a bad edge only if $\{x_{2i-1}, x_{2i}\}$ is not added to F and the pair $\{z_1, z_2\} \in F$ satisfies $\{\psi(z_1), \psi(z_2)\} \cap T \neq \emptyset$. Hence

$$\begin{aligned} Pr(\sigma_i \text{ creates a bad edge}) &\leqslant |\{z_1, z_2\} \in F : \{\psi(z_1), \psi(z_2)\} \cap T \neq \emptyset\}|/|F_2| \\ &\leqslant \frac{\lceil \alpha_\varepsilon \rceil r}{|F_2|} \leqslant 4\,\frac{\log r}{r} \end{aligned}$$

regardless of the outcome of the execution of FINISH *to this point.*

Hence γ is majorised by $B(\lceil 2n/r^{\theta}\rceil, 4\log r/r)$ and so

$$Pr\left(\gamma \geqslant \frac{16n}{r^{1+\theta}}\right) \leqslant \exp\left\{-\frac{8n}{3r^{1+\theta}}\right\} \tag{5}$$

using the Chernoff bound.

Note that $\alpha(\mu(F)) \geqslant \alpha(G_2) - 2\gamma$.

Thus (1), (2), and (5) (and a surreptitious doubling of ε) imply that

$$Pr(\alpha(F) \leqslant \alpha_{\varepsilon}) \leqslant e^{-Dn/r^{3+2\theta}} \tag{6}$$

for some $D = D(\varepsilon)$.

To bound $\alpha(F)$ from above is straightforward.

Let now $l = \lceil \alpha_{-\varepsilon} \rceil$ and Y be the random variable which counts the number of independent sets of $\mu(F)$ of size l. Then

$$\begin{aligned} P(\alpha(F) \geqslant l) &\leqslant E(Y) \\ &= \binom{n}{l} \prod_{i=1}^{rl-1} \left(1 - \frac{rl-i}{rn-2i+1}\right) \\ &\leqslant \binom{n}{l} \prod_{i=1}^{rl-1} \left(1 - \frac{rl-i}{rn}\right) \\ &\leqslant 2\binom{n}{l} \exp\left\{-\frac{rl^2}{2n}\right\} \\ &\leqslant 2\left(\frac{ne}{l} \exp\left\{-\frac{rl}{2n}\right\}\right)^l \\ &\leqslant 2e^{-\varepsilon l/2}. \end{aligned}$$

Hence, the first part of the theorem for constant r follows from (0), (6), and (7).

Now to the second part of the theorem.

The lower bound is immediate from the first part of the theorem, since $\chi(G_r) \geqslant n/\alpha(G_r)$. For the upper bound we use the fact that the main result of Łuczak [12] implies that for $r \geqslant r_0$ (= some sufficiently large constant)

$$Pr\left(\chi(G_1) \geqslant k_0 = \left\lceil \frac{r}{2\log r}\left(1 + \frac{30\,\mathrm{loglog}\, r}{\log r}\right)\right\rceil\right) = o(1),$$

G_1 as in Step 1.

Step 2 can only decrease the chromatic number and (6) shows that if G_1 has a k_0-colouring and we use it for $\mu(F)$ then with probability $1-o(1)$ $\mu(F)$ has at most

$$\frac{16n}{r^{1+\theta}} \cdot \frac{r}{2\log r}\left(1+\frac{30\log\log r}{\log r}\right) \leqslant \frac{10n}{r^{\theta}}$$

edges with both ends of the same colour. The result (for constant r) will follow if we show that with probability $1-o(1)$, all subgraphs of $\mu(F)$ with at most $s_0 = 20n/r^{\theta}$ vertices can be (re-)coloured with at most $l = r\log\log r/(\log r)^2$ colours. We prove this by showing that any subgraph H induced by $s \leqslant s_0$ vertices satisfies $\delta(H) < l$ and this is in turn implied by each such H having less than $ls/2$ edges. This latter statement is easy to prove.

$$Pr(\exists s \leqslant s_0 \text{ vertices of } \mu(F) \text{ containing } \geqslant ls \text{ edges})$$

$$\leqslant \sum_{s=l}^{s_0} \binom{n}{s}\binom{\binom{s}{2}}{ls/2}\left(\frac{r^2}{rn-rs}\right)^{ls/2}$$

$$\leqslant \sum_{s=l}^{s_0} \left(\frac{ne}{s}\right)^{s}\left(\frac{s^2e}{ls}\right)^{ls/2}\left(\frac{r}{n-s}\right)^{ls/2}$$

$$\leqslant \sum_{s=l}^{s_0} \left(\left(\frac{s}{n}\right)^{(l-2)/l}\frac{3r}{l}\right)^{ls/2} = o(1)$$

and the whole theorem has been proved, for constant r.

Let us now consider the case $r \to \infty$ but $r = o(n^{\theta})$. The above proof shows that $\mu(F)$ for a random $F \in \Omega_W$ has its independence and chromatic numbers in the right range. We have to show that this implies the same for G_r. We rely on the work of McKay and Wormald [11] for this. They give a procedure DEG which takes as input a random $F \in \Phi_W$ and tries to construct an r-regular simple graph by eliminating loops and multiple edges. The elimination of a loop or multiple edge involves the addition of at most 4 new edges. The procedure succeeds with probability $1-o(1)$ and it produces each member of REG(n, r) with the same probability. Also, it is easy to see that with probability $1-o(1)$ F has $O(r^2)$ loops and multiple edges. Thus we need only show that adding $O(r^2)$ edges to a typical F does not change α or χ by much. But now $r^2 = o(n/r)$ for $r = o(n^{\theta})$, and so part (a) requires no work. For part (b) we need to be convinced that the $O(r^2) = o(n^{2\theta})$ added edges are sufficiently random so that they usually induce the union of 4 forests which can be 8-coloured. This can be done fairly straightforwardly but requires a fair amount of detail from [11] which is inappropriate here.

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