Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B

Vol. 37, No. 2, October 1984
Printed in Belgium

Hamiltonian Cycles in Random Regular Graphs

T. I. FENNER

Department of Computer Science, Birkbeck College, London WC1E 7HX, England

AND

A. M. FRIEZE

Department of Computer Science & Statistics, Queen Mary College, London E1 4NS, England

Communicated by the Managing Editors

The existence of Hamiltonian cycles in random vertex-labelled regular graphs is investigated. It is proved that there exists $r_0 \leqslant 796$ such that for $r \geqslant r_0$ almost all vertex-labelled r-regular graphs with n vertices have Hamiltonian cycles as $n \to \infty$.

1. INTRODUCTION

The study of the properties of random graphs was initiated by Erdös and Rényi [2]. There they considered the asymptotic properties of random graphs with n labelled vertices and $M(n)$ edges as $n \to \infty$. One problem they left open was the number of edges needed to ensure the almost certain existence of a Hamiltonian cycle. This was later partially solved by Pósa [8] and then completely solved by Korshunov [6] and Komlós and Szemerédi [5]. The corresponding result for digraphs follows from a theorem of McDiarmid [7].

An interesting feature of this result, highlighted in [5], is that Hamiltonian cycles "appear" when $M(n)$ is large enough to "ensure" that the graph has no vertices of degree one or zero. It is therefore of interest to study classes of graphs in which the minimal vertex degree exceeds some prescribed value.

In [4] we studied random graphs obtained by taking a random digraph in which each vertex has outdegree m and then ignoring the orientation of the arcs. We showed that there exists $m_0 \leqslant 23$ such that, if $m \geqslant m_0$ is kept fixed and $n \to \infty$, almost all such graphs have Hamiltonian cycles. We conjecture that the minimum value for m_0 is 3.

In this paper we consider random vertex-labelled regular graphs of degree r and prove a similar result, viz that there exists $r_0 \leqslant 796$ such that, if $r \geqslant r_0$

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and $n \to \infty$, almost all r-regular graphs with n labelled vertices have Hamiltonian cycles.

In the next section, after introducing some definitions and notation, we show how to generate random vertex-labelled regular graphs; the last section is devoted to proving our main result.

2. Definitions and Notation

For ease of reference we define most of our terms in this section.

A *multigraph* $G=(V, E, \psi)$ has a vertex set V, an edge set E, and an incidence function ψ, where for each edge e in E, $\psi(e)=\{x, y\}$ for some pair of (not necessarily distinct) vertices x and y in V. For any vertex v in V, $\deg_G(v)$ denotes the *degree* of v in G, where loops count twice towards the degree of a vertex. For any subset of vertices $S \subseteq V$, we define

$$\delta_G(S)=\{w \in V-S \mid \exists e \in E \text{ such that } w \in \psi(e) \text{ and } \psi(e) \cap S \neq \varnothing\}.$$

Configurations

For positive integers n and r, let $A=A(n, r)=\{1, 2, ..., rn\}$, where we assume throughout that rn is even. The elements of A are referred to as *points*. Let $A_i=A_i(n, r)=\{(i-1)r+1, (i-1)r+2, ..., ir\}$ for $1, 2, ..., n$ and for $S \subseteq V(n)=\{1, 2, ..., n\}$ let $A_S=\bigcup_{i \in S} A_i$.

Let $\Omega(n, r)$ be the set of partitions of A into $rn/2$ disjoint two-element subsets. The elements of $\Omega(n, r)$ are referred to as *configurations* by Bollobás [1]. The $rn/2$ two-element subsets of A are called the *edges* of the configuration.

For F in $\Omega(n, r)$, let $G(F)=(V(n), F, \psi_r)$ be the vertex-labelled multigraph with edge set F, where for all $\{p, q\} \in F$, $\psi_r(\{p, q\})=\{\lceil p/r \rceil, \lceil q/r \rceil\}$.

The set of *simple* configurations is defined by $\Omega^*(n, r)=\{F \in \Omega(n, r) \mid G(F)$ is a *simple* graph$\}$. (A *simple* graph has no loops or repeated edges; for simple graphs we identify e with $\psi(e)$ for each $e \in E$.)

Let $RG(n, r)$ be the set of vertex-labelled r-regular *simple* graphs with vertex set $V(n)$; thus

$$RG(n, r)=\{G(F) \mid F \in \Omega^*(n, r)\}.$$

Bollobás [1] has shown that

$$\lim_{n \to \infty} |\Omega^*(n, r)|/|\Omega(n, r)|=e^{-(r^2-1)/4}, \tag{2.1}$$

for each G in $RG(n, r)$ there are exactly $(r!)^n$ simple configurations F such that $G(F)=G$. (2.2)

We turn $RG(n, r)$ into a *probability space* by assigning each member the same probability; we also do the same for $\Omega(n, r)$. For any subset $P(n, r)$ of $RG(n, r)$ we say that *almost all* G in $RG(n, r)$ are in $P(n, r)$ if $\lim_{n\to\infty} |P(n, r)|/|RG(n, r)| = 1$, and similarly for a subset $Q(n, r)$ of $\Omega(n, r)$.

The following remark, which is a straightforward consequence of (2.1) and (2.2), shows why Bollobás's construction is so useful.

Remark 2.1. If almost all F in $\Omega(n, r)$ are in $Q(n, r)$ and $F \in \Omega^*(n, r) \cap Q(n, r)$ implies $G(F) \in P(n, r)$, then almost all G in $RG(n, r)$ are in $P(n, r)$.

Coloured Configurations

For any positive integer $s \leqslant r$, we define an *s-colouring* σ of A as an n-tuple $(X_1, X_2, ..., X_n)$, where $X_i \subseteq A_i$ and $|X_i| = s$ for $i = 1, 2, ..., n$; we let $X(\sigma) = \bigcup_{i=1}^{n} X_i$. Now let $K(n, r, s) = \{\sigma_1, \sigma_2, \sigma_3, ...\}$ denote the set of s-colourings of A. We shall assume always that

$$\sigma_1 = \sigma_1(n, r, s) = (\{1, 2, ..., s\}, \{r+1, r+2, ..., r+s\}, ...,$$
$$\{nr - r + 1, nr - r + 2, ..., nr - r + s\}).$$

We say that $\sigma' = (X'_1, X'_2, ..., X'_n) \in K(n, r, s-1)$ is a *subcolouring* of σ if $X'_i \subseteq X_i$ for $i = 1, 2, ..., n$. We define $K_1(n, r, s)$ to be the subset of $K(n, r, s)$ containing those s-colourings which have $\sigma_1(n, r, s-1)$ as a subcolouring.

Suppose we first colour the points in $X(\sigma)$ *blue* and the points in $A - X(\sigma)$ *green*; now let us colour *blue* those edges having at least one blue point and colour *green* the remaining edges, those having two green points. Then, for each $F \in \Omega(n, r)$ and $\sigma \in K(n, r, s)$, we define H, the *blue subgraph* of $G(F)$ by

$$H = H(F, \sigma) = (V(n), F_\sigma, \psi_r),$$

where $F_\sigma = \{e \in F \mid e \cap X(\sigma) \neq \varnothing\}$, i.e., H is the subgraph of $G(F)$ containing just the *blue* edges.

3. Main Result

THEOREM 3.1. *There exists r_0 such that if $r \geqslant r_0$ then almost all G in $RG(n, r)$ are Hamiltonian.*

Proof. Bollobás [1] has shown that for $r \geqslant 3$ almost all G in $RG(n, r)$ are connected. In Corollary 3.7 we shall show that there exists r_0 such that for $r \geqslant r_0$ almost all G in $RG(n, r)$ have property LC:

> a graph G has *property LC* if a longest cycle of G has the same number of vertices as a longest path of G.

However, any connected graph having property LC is Hamiltonian: if the longest cycle C had $k < n$ vertices, then at least one vertex of C would be adjacent to a vertex not in C and then a path of length $k+1$ could be formed. ∎

In order to show that for $r \geqslant r_0$ almost all G in $RG(n, r)$ have property LC, we use the following standard construction. Suppose that $P = (v_1, v_2, ..., v_k)$ is a longest path in a simple graph $G = (V, E)$. Then, if $\{v_k, v_t\}$ is in E, $t \neq k-1$, $P' = (v_1, ..., v_t, v_k, v_{k-1}, ..., v_{t+1})$ is also a longest path of G. If, in addition, $\{v_{t+1}, v_s\} \in E$, $s \neq t$ or $t+2$, we can construct yet another longest path P'' using a similar "flip." We now let G be a *multigraph* and define the above construction in a similar manner.

Keeping v_1 fixed, let $\mathrm{EP}(v_1)$ be the set of the other endpoints of all the longest paths formed by doing all possible sequences of flips. The proof of the following lemma may be found in Pósa [8].

LEMMA 3.2. *If w is in $V - \mathrm{EP}(v_1)$ then w is adjacent to some vertex of $\mathrm{EP}(v_1)$ in G if and only if w is in P and is adjacent to some vertex of $\mathrm{EP}(v_1)$ on P.*

The following inequality is a straightforward consequence of Lemma 3.2:

$$|\delta_G(\mathrm{EP}(v_1))| \leqslant 2\,|\mathrm{EP}(v_1)| - 1. \tag{3.1}$$

Now let Λ_G be the set of endpoints of *all* longest paths in G and, for each $v \in \Lambda_G$, let $\Lambda_G(v)$ be the set of other endpoints of all longest paths having v as one endpoint. Then (3.1) yields the following corollary.

COROLLARY 3.3. *If $|\delta_G(R)| \geqslant 2\,|R|$ for all $R \subseteq V$ with $|R| \leqslant m$, then $|\Lambda_G(v)| > m$ for all $v \in \Lambda_G$.*

It is not difficult to show that, for $r \geqslant 7$ (and possibly with more work, for $r \geqslant 3$), Corollary 3.3 implies that, for all $v \in \Lambda_G$, $|\Lambda_G(v)| > \alpha(r)n$ for almost all G in $RG(n, r)$ for some $\alpha(r) > 0$ (see Lemma 3.4). Thus, if we could ignore conditioning problems, it seems likely that there would be a high enough probability that G would contain an edge joining the endpoints of some longest path, yielding property LC. We overcome the conditioning problems by using a colouring argument analogous to that we used in [4], but at the cost of increasing our estimate of r_0.

We now prove some properties of the blue subgraphs $H(F, \sigma)$.

LEMMA 3.4. *For any real positive $\alpha < \frac{1}{3}$ and integers $r \geqslant s \geqslant 7$, let $B(n, r, \alpha, s)$ be the set of all configurations F in $\Omega(n, r)$ for which there exists $R \subseteq V(n)$ such that* (i) $1 \leqslant |R| \leqslant \alpha n$, *and* (ii) $|\delta_H(R)| \leqslant 2\,|R|$, *where*

$H = H(F, \sigma_1)$. *If* $\pi_1(n) = \text{Prob}(F \in B(n, r, \alpha, s)$ *then* $\lim_{n\to\infty} \pi_1(n) = 0$, *provided*

$$3\alpha + (s\alpha/r)^{1/2} < 1, \tag{3.2}$$

$$\varepsilon = s - \sup_{0<x\leqslant\alpha} \left(\frac{x \log x + 2x \log 2x + (1-3x)\log(1-3x)}{x \log(3x + (sx/r)^{1/2})} \right) > 0. \tag{3.3}$$

(Note that for $\alpha < \frac{1}{3}$

(i) (3.2) is satisfied if s/r is sufficiently small,

(ii) the supremum in (3.3) is finite but tends to infinity as $3\alpha + (s\alpha/r)^{1/2}$ tends to 1.)

Proof. Let F be a random configuration in $\Omega(n, r)$. Now let $R = \{1, 2,..., k\}$, $T = \{k+1, k+2,..., 3k\}$, and $U = \{3k+1, 3k+2,..., n\}$; we first bound the probability that $\delta_H(R) \subseteq T$. In terms of F, this is equivalent to

$$\text{no blue point of } A_R \text{ is paired with a point of } A_U \tag{3.4}$$

and

$$\text{no green point of } A_R \text{ is paired with a blue point of } A_U. \tag{3.5}$$

As the method we are using at present is not capable of getting an estimate of r_0 close to 3 (which we conjecture to be its least value), we ignore (3.5) in order to simplify the calculations.

If (3.4) holds then for some $t \leqslant sk/2$ there will be $2t$ blue points of A_R paired together and $sk - 2t$ blue points of A_R each paired either with a green point of A_R or with a point of A_T. For a fixed set of $2t$ blue points of A_R, the probability of this happening is bounded above by

$$\frac{2t-1}{rn-1} \cdot \frac{2t-3}{rn-3} \cdots \frac{1}{rn-2t+1} \cdot \left(\frac{3k}{n}\right)^{sk-2t} \leqslant (2t/rn)^t (3k/n)^{sk-2t}$$
$$\leqslant (3k/n)^{sk} (sn/9rk)^t.$$

Thus the probability that (3.4) holds is bounded above by

$$\left(\frac{3k}{n}\right)^{sk} \sum_{t=0}^{\lfloor sk/2 \rfloor} \binom{sk}{2t} \left(\frac{sn}{9rk}\right)^t \leqslant (3k/n)^{sk} (1 + (sn/9rk)^{1/2})^{sk}.$$

It then follows that

$$\pi_1(n) \leqslant \sum_{k=1}^{\lfloor \alpha n \rfloor} \frac{n!}{k!\, 2k!(n-3k)!} \left(\frac{3k}{n}\right)^{sk} \left(1 + \left(\frac{sn}{9rk}\right)^{1/2}\right)^{sk}$$
$$< c \sum_{k=1}^{\lfloor \alpha n \rfloor} \left(\frac{n}{2k^2(n-3k)}\right)^{1/2}$$
$$\times \left(\frac{3k}{n} + \left(\frac{sk}{rn}\right)^{1/2}\right)^{sk} \Big/ \left(\frac{k}{n}\right)^{k} \left(\frac{2k}{n}\right)^{2k} \left(1 - \frac{3k}{n}\right)^{n-3k}$$

for some $c > 0$, using Stirling's inequalities. Thus, using (3.2) and (3.3), we see that

$$\pi_1(n) \leqslant \frac{c}{(2-6\alpha)^{1/2}} \sum_{k=1}^{\lfloor \alpha n \rfloor} \frac{1}{k} \left(\frac{3k}{n} + \left(\frac{sk}{rn}\right)^{1/2}\right)^{\varepsilon k}$$
$$= O(n^{-\varepsilon/2}). \quad \blacksquare$$

LEMMA 3.5. *For any $\beta > 0$, $0 < \gamma < 1$, and integer $s > 0$, let $C(n, r, \beta, \gamma, s)$ be the set of all configurations F in $\Omega(n, r)$ for which there exists σ in $K_1(n, r, s)$ such that $|\{v \in V(n) |\ \deg_H(v) \geqslant \gamma r\}| \geqslant \beta n$, where $H = H(F, \sigma)$. There exists $r_1 = r_1(\beta, \gamma, s)$ such that if $r \geqslant r_1$ and $\pi_2(n) = \text{Prob}(F \in C(n, r, \beta, \gamma, s))$ then $\lim_{n\to\infty} \pi_2(n) = 0$.*

Proof. Let F be a random configuration in $\Omega(n, r)$ and let σ be any fixed colouring in $K_1(n, r, s)$. For any $S \subseteq V(n)$, let E_S denote the event that $\deg_H(v) \geqslant \gamma r$ for all $v \in S$.

Suppose now that $|S| = \lceil \beta n \rceil$. If E_S occurs, there must be at least $t = (\gamma r - 2s)\lceil \beta n \rceil$ edges of F pairing a green point of A_S (i.e., a point of $Y = A_S - X_S$) with a blue point of $A_{V(n)-S}$ (i.e., a point of $Z = X_{V(n)-S}$).

We now consider the points of Y sequentially. Whatever the history of previous pairings, the probability that $p \in Y$ is paired with some point $q \in Z$ is at most

$$\theta = \frac{|Z|}{rn - 2|Y|} = \frac{(1-\beta)s}{r - 2\beta(r-s)} + O\left(\frac{1}{n}\right).$$

Thus, if $B(m, \theta)$ denotes a binomial random variable,

$$\text{Prob}(E_S) \leqslant \text{Prob}(B(|Y|, \theta) \geqslant t).$$

(Edges $\{p, q\} \subseteq Y$ have not been ignored; if we conditioned on the set T of such edges, $|Y|$ would be replaced by $|Y| - 2|T|$ in the above inequality.)

On using the Chernoff bound [3], this yields

$$\text{Prob}(E_S) \leqslant e^{-arn}, \qquad \text{where} \quad ar = \frac{((\gamma-\theta)\,r-(2-\theta)\,s)^2\,\beta}{3\theta(r-s)},$$

provided that $(\gamma-\theta)r > (2-\theta)s$. We note that for s fixed a increases unboundedly with r. It now follows that

$$\pi_2(n) \leqslant (r-s+1)^n \binom{n}{\lceil \beta n \rceil} e^{-arn}, \tag{3.6}$$

so $\pi_2(n) \to 0$ for r sufficiently large. ∎

An error in our original proof of Lemma 3.5 was brought to our attention by McDiarmid and a referee. The proof above is due to McDiarmid.

We note that, at the expense of increasing the minimal value of $r_1(\beta, \gamma, s)$, Lemma 3.5 and its proof can be considerably simplified: if $r > (1+\beta)s/(\beta\gamma)$ then $t > |Z|$, so E_S cannot occur and thus $C(n, r, \beta, \gamma, s)$ is empty. This would also yield some simplification in the proof of Lemma 3.6 below, but our upper bound on r_0 (of 796) would then increase considerably.

Using the previous two lemmas, we can now prove the existence of r_0 such that for $r \geqslant r_0$ almost all G in $RG(n, r)$ have property LC.

LEMMA 3.6. *Let $D(n,r) = \{F \in \Omega(n,r) \mid G(F)$ does not have property LC$\}$. There exists r_0 such that if $r \geqslant r_0$ and $\pi_3(n) = \text{Prob}(F \in D(n,r))$ then $\lim_{n\to\infty} \pi_3(n) = 0$.*

Proof. First choose any $\alpha < \frac{1}{3}$ and any positive $\beta < \alpha$ and $\gamma < 1$. Next choose r and s such that (3.2) and (3.3) are satisfied with s replaced by $s-1$, and $r \geqslant r_1(\beta, \gamma, s)$. Let $\Omega = \Omega(n, r)$ and $K = K(n, r, s)$.

For $(F, \sigma) \in \Omega \times K$, let $a(F, \sigma) = 1$ if the following three conditions hold:

no edge of $G(F)$ joins the two endpoints of any longest path of H, (3.7a)

$$|\delta_H(R)| > 2\,|R| \text{ for all } R \subseteq V(n) \text{ with } |R| \leqslant \alpha n, \tag{3.7b}$$

$$|\{v \in V(n) | \deg_H(v) \geqslant \gamma r\}| < \beta n, \tag{3.7c}$$

and let $a(F, \sigma) = 0$ otherwise.

Let $B = B(n, r, \alpha, s-1)$, $C = C(n, r, \beta, \gamma, s)$, and $D = D(n, r)$. We now show that if $\hat{\Omega} = \Omega - (B \cup C)$ and $F \in \hat{\Omega} \cap D$ then $a(F, \sigma) = 1$ for at least one $\sigma \in K_1 = K_1(n, r, s)$. Indeed, since $F \notin C$, (3.7c) is satisfied for all $\sigma \in K_1$. Moreover, since $F \notin B$, (3.7b) is also satisfied for all $\sigma \in K_1$. Now let $P = (v_1, v_2, \ldots, v_k)$ be a longest path in $G(F)$. Consider points p_i and q_i, $1 \leqslant i \leqslant k-1$, with $p_i \in A_{v_i}$, $q_i \in A_{v_{i+1}}$, and $\{p_i, q_i\} \in F$. Such points must exist since P is a path in $G(F)$. At least one $\sigma \in K_1$ colours $p_1, p_2, \ldots, p_{k-1}$ blue, so for such a colouring (3.7a) holds as F is in D.

It therefore follows that

$$|D| \leqslant \sum_{F \in \hat{\Omega}} \sum_{\sigma \in K_1} a(F, \sigma) + |B \cup C|.$$

In order to bound this double sum we introduce an equivalence relation $\cong$ on $\hat{\Omega} \times K_1$ by letting $(F, \sigma) \cong (F', \sigma')$ if and only if

$$\sigma = \sigma', \tag{3.8a}$$

$$F_\sigma = F'_\sigma, \qquad \text{i.e., } H(F, \sigma) = H(F', \sigma). \tag{3.8b}$$

Let $Q = (\hat{\Omega} \times K_1)/\cong$ be the set of equivalence classes. We note that

$$[(F, \sigma)] \in Q \qquad \text{implies (3.7b) and (3.7c) hold.}$$

For $\Delta \in Q$, let $n_\Delta = |\{(F, \sigma) \in \Delta \mid a(F, \sigma) = 1\}|$ and let σ_Δ and F_Δ denote the values of σ and F_σ, respectively, for any $(F, \sigma) \in \Delta$.

We shall later show that if $\theta = ((1-\lambda)\, e^{\lambda})^{(r-s)/2} < 1$, where $\lambda = (1-\gamma)(\alpha - \beta) r/(r-s)$, then

$$n_\Delta \leqslant c_0 \theta^n \, |\Delta| \qquad \text{for some constant } c_0 > 0. \tag{3.9}$$

Hence

$$\begin{aligned} \sum_{F \in \hat{\Omega}} \sum_{\sigma \in K_1} a(F, \sigma) &= \sum_{\Delta \in Q} n_\Delta \\ &\leqslant c_0 \theta^n \sum_{\Delta \in Q} |\Delta| \\ &= c_0 \theta^n \, |\hat{\Omega}| \, |K_1|. \end{aligned}$$

Therefore,

$$\pi_3(n) \leqslant (c_0 \theta^n \, |\hat{\Omega}| \, |K_1| + |B \cup C|)/|\Omega| \leqslant c_0(\theta(r-s+1))^n + \pi_1(n) + \pi_2(n).$$

Thus if we choose r sufficiently large that $\theta(r-s+1) < 1$ we see that $\pi_3(n) \to 0$.

It now only remains to prove (3.9). We suppose first that $\Delta \in Q$ is fixed. Let $Z_\Delta = \bigcup F_\Delta$, i.e., the set of points incident with blue edges, let $Z_i = Z_\Delta \cap A_i$ and let $Y_i = A_i - Z_i$ for $i = 1, 2, \ldots, n$. Now let $Y_\Delta = \bigcup_{i=1}^n Y_i$ and let Ω_Δ be the set of partitions of Y_Δ into 2-element subsets. We then see that

$$(F, \sigma) \in \Delta \qquad \text{if and only if } \sigma = \sigma_\Delta \text{ and } F = F_\Delta \cup I \text{ for some } I \in \Omega_\Delta \tag{3.10}$$

and thus $|\Omega_\Delta| = |\Delta|$. ($I$ is the set of green edges of F.)

Next let $L = \{i \mid |Z_i| \geqslant \gamma r\}$. Since $|Z_i| = \deg_H(i)$ and (3.7c) holds, we have $|L| < \beta n$.

Now let $H = H(F, \sigma)$; H is independent of the choice of (F, σ) in Δ. Since (3.7b) holds, Corollary 3.3 implies that $|A_H(v)| > \alpha n$ for all $v \in A_H$. It then follows that

$$|A_H(v) - L| > (\alpha - \beta)n \qquad \text{for all } v \in A_H, \tag{3.11a}$$

$$|A_H - L| > (\alpha - \beta)n. \tag{3.11b}$$

If we now let $U = \bigcup_{i \in A_H} Y_i$ and $U(v) = \bigcup_{i \in A_H(v)} Y_i$, it follows from (3.11) that, for all $v \in A_H$,

$$|U| \geqslant |U(v)| > (1 - \gamma) r (\alpha - \beta) n.$$

Now let $(F, \sigma) \in \Delta$ be such that $a(F, \sigma) = 1$ and let I be as in (3.10). Then, by (3.7a),

$$\begin{array}{c}\text{there is no } e \in I \text{ such that, if } \psi_r(e) = \{v_1, v_2\}, \\ v_1 \in A_H \qquad \text{and} \qquad v_2 \in A_H(v_1).\end{array} \tag{3.12}$$

Then, if we let $M = (1 - \gamma) r (\alpha - \beta) n$ and $N = (r - s) n \geqslant |Y_\Delta|$, the proportion of I in Ω_Δ that satisfy (3.12) is bounded above by

$$w = \prod_{t=0}^{\lfloor M/2 \rfloor} (N - M - 1)/(N - 2t - 1). \tag{3.13}$$

To show this, we choose I at random so that each such I in Ω_Δ is equally likely to be chosen. We then examine the points in $U = \{p_1, p_2, p_3, ...\}$ in turn until either we find some p_i such that the point q paired with p_i in I is in $U(\lceil p_i/r \rceil)$, or we exhaust I. If q is not in $U(\lceil p_i/r \rceil)$ then we eliminate both p_i and q from further consideration. We observe that I will be exhausted if and only if I satisfies (3.12).

Let us suppose that t pairs have been examined and the process has not terminated. Let p_j be the uneliminated point of U having the smallest index. Then there are at least $M - 2t$ points of $U(\lceil p_j/r \rceil)$ that have not been eliminated. Thus the probability that the process ends with p_j given that it has not previously terminated is at least $(M - 2t)/(N - 2t - 1)$, from which (3.13) follows.

Now, from (3.13),

$$\begin{aligned} w &\leqslant \frac{(N - M)^{\lfloor M/2 \rfloor + 1} (\lfloor N/2 \rfloor - \lfloor M/2 \rfloor - 1)!}{2^{\lfloor M/2 \rfloor + 1} \lfloor N/2 \rfloor !} \\ &\leqslant c_0 \left(1 - \frac{M}{N}\right)^{N/2} e^{M/2} \end{aligned}$$

for some $c_0 > 0$, on using Stirling's inequalities; (3.9) then follows since $n_\Delta/|\Delta| \leqslant w$. ∎

From Lemma 3.6, on using Remark 2.1, we obtain

COROLLARY 3.7. *There exists a least integer r_0 such that if $r \geqslant r_0$ then almost all G in $RG(n, r)$ have property* LC.

This completes the proof of Theorem 3.1.

By straightforward computation, one can show that $r_0 \leqslant 796$ by taking $\alpha = 0.26$, $\beta = 0.019$, $\gamma = 0.3$, and $s = 31$ in Lemma 3.6. Additional computation might reduce this bound, but not significantly.

In spite of the size of this upper bound, it is not unreasonable to conjecture that $r_0 = 3$. It is interesting to note that property LC holds for all 2-regular graphs and since it also holds for almost all r-regular graphs, for r large, a gap might seem unnatural.

Remark. As this manuscript was being typed we were interested to learn from Bollobás [9] that he has just obtained the same result. However, his upper bound on r_0 is approximately 10^7.

REFERENCES

1. B. BOLLOBÁS, Random graphs, *in* "Combinatorics" (H. N. V. Temperley, Ed.), London Math. Soc. Lecture Notes Ser. 52, pp. 80–102, Cambridge Univ. Press, Cambridge, 1981.
2. P. ERDÖS AND A. RÉNYI, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* 5 (1960), 17–61.
3. P. ERDÖS AND J. SPENCER, "Probabilistic Methods in Combinatorics," Academic Press, New York/London, 1974.
4. T. I. FENNER AND A. M. FRIEZE, On the existence of Hamiltonian cycles in a class of random graphs, *Discrete Math.* 45 (1983), 1–5.
5. J. KOMLÓS AND E. SZEMERÉDI, Limit distribution for the existence of Hamiltonian cycles in a random graph, *Discrete Math.* 43 (1983), 55–63.
6. A. D. KORSHUNOV, Solution of a problem of Erdös and Rényi on Hamiltonian cycles in non-oriented graphs, *Soviet Math. Dokl.* 17 (1976), 760–764.
7. C. MCDIARMID, General percolation and random graphs, *Adv. in Appl. Probab.* 13 (1981), 40–60.
8. L. PÓSA, Hamiltonian circuits in random graphs, *Discrete Math.* 14 (1976), 359–364.
9. B. BOLLOBÁS, Almost all regular graphs are Hamiltonian, *European J. Combin.*, to appear.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium