A note on the vacant set of random walks on the hypercube and other regular graphs of high degree

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Abstract

We consider a random walk on a d-regular graph G where $d \to \infty$ and G satisfies certain conditions. Our prime example is the d-dimensional hypercube, which has $n = 2^d$ vertices. We explore the likely component structure of the vacant set, i.e. the set of unvisited vertices. Let $\Lambda(t)$ be the subgraph induced by the vacant set of the walk at step t. We show that if certain conditions are satisfied then the graph $\Lambda(t)$ undergoes a phase transition at around $t^* = n \log_e d$. Our results are that if $t \leq (1-\varepsilon)t^*$ then w.h.p. as the number vertices $n \to \infty$, the size $L_1(t)$ of the largest component satisfies $L_1 \gg \log n$ whereas if $t \geq (1+\varepsilon)t^*$ then $L_1(t) = o(\log n)$.

1 Introduction

The problem we consider can be described as follows. We have a finite graph $G = (V, E)$, and a simple random walk $\mathcal{W} = \mathcal{W}_u$ on G, starting at $u \in V$. In this walk, if $\mathcal{W}(t)$ denotes the position of the walk after t steps, then $\mathcal{W}(0) = u$ and if $\mathcal{W}(t) = v$ then $\mathcal{W}(t+1)$ is equally likely to be any neighbour of v. We consider the likely component structure of the subgraph $\Lambda(t)$ induced by the unvisited vertices of G at step t of the walk.

Initially all vertices V of G are unvisited or *vacant*. We regard unvisited vertices as colored *red*. When $\mathcal{W}_u$ visits a vertex, the vertex is re-colored *blue*. Let $\mathcal{W}_u(t)$ denote the position of $\mathcal{W}_u$ at step t. Let $\mathcal{B}_u(t) = \{\mathcal{W}_u(0), \mathcal{W}_u(1), \ldots, \mathcal{W}_u(t)\}$ be the set of blue vertices at the end of step t, and $\mathcal{R}_u(t) = V \setminus \mathcal{B}_u(t)$. Let $\Lambda_u(t) = G[\mathcal{R}_u(t)]$ be the subgraph of G induced by $\mathcal{R}_u(t)$. Initially $\Lambda_u(0)$ is connected, unless u is a cut-vertex. As the walk continues, $\Lambda_u(t)$ will shrink to the empty graph once every vertex has been visited. We wish to determine, as far as possible, the likely evolution of the component structure as t increases.

For several graph models, it has been shown that the component structure of $\Lambda(t) = \Lambda_u(t)$ undergoes a phase transition of some sort. In this paper we add results for some new classes of graphs. What we expect to happen is that there is a time t^*, such that if $t \geq (1+\varepsilon)t^*$ then w.h.p. all components of $\Lambda(t)$ are "small" and if $t \leq (1-\varepsilon)t^*$ then w.h.p. $\Lambda(t)$ contains some "large" components. Here ε is some arbitrarily small positive constant and the meanings of small, large will be made clear.

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1.1 Previous work

We begin with the paper by Černý, Teixeira and Windisch [3]. They consider a sequence of n-vertex graphs G_n with the following properties:

A1 G_n is d-regular, $3 \leq d = O(1)$.

A2 For any $v \in V(G_n)$, there is at most one cycle within distance $\alpha \log_{d-1} n$ of v for some $\alpha \in (0,1)$.

A3 The second eigenvalue λ_2 of the random walk transition matrix satisfies $\lambda_2 \leq 1 - \beta$ for some constant $\beta \in (0,1)$.

Let

$$t^* = \frac{d(d-1)\log(d-1)}{(d-2)^2} n. \tag{1}$$

In which case, it is shown in [3] that for $t \leq (1-\varepsilon)t^*$ there is w.h.p. a unique giant component in $\Lambda(t)$ of size $\Omega(n)$ and other components are all of size $o(n)$. Furthermore, if $t \geq (1+\varepsilon)t^*$ then all components of $\Lambda(t)$ are of size $O(\log n)$.

The most natural class of graphs which satisfy A1,A2,A3 w.h.p. are random d-regular graphs, $3 \leq d = O(1)$. For this class of graphs Cooper and Frieze [9] tightened the above results in the following ways. (i) they established the asymptotic size of the giant component for $t \leq (1-\varepsilon)t^*$, and proved all other components have size $O(\log n)$; (ii) they proved almost all small components are trees, and gave a detailed census of the number of trees of sizes $O(\log n)$. Subsequent to this work, Černy and Teixeira [4] built on the methodology of [9] and analysed the component structure at time t^* itself. More recently, for random d-regular graphs, $3 \leq d = O(1)$, Cooper and Frieze [10] determined the phase transition for a related structure, the *vacant net*, which by analogy with vacant set, they define as the subgraph induced by the unvisited edges of the graph G. Initially all edges are unvisited. The random walk *visits an edge* by making a transition using the edge.

In the paper [9], Cooper and Frieze also considered the class of Erdős-Reńyi random graphs $G_{n,p}$ with edge probabilities p above the connectivity threshold $p = \log n/n$. For $G_{n,p}$ where $p = c\log n/n$, $(c-1)\log n \to \infty$, they established that $\Lambda(t)$ undergoes a phase transition around $t^* = n \log\log n$. For these graphs, at $t_{-\varepsilon} = (1-\varepsilon)t^*$ the size L_1 of the largest component cannot be $\Omega(n)$ since the vacant set has size $|\mathcal{R}(t_\varepsilon)| = o(n)$ w.h.p. On the other hand it was shown that $L_1 = \Omega(|\mathcal{R}(t_\varepsilon)|)$ w.h.p. More recently, Wassmer [16] found the phase transition in $\Lambda(t)$ when the underlying graph is the giant component of $G_{n,p}$, $p = c/n$, $c > 1$.

There has also been a considerable amount of research on the d-dimensional grid $\mathbb{Z}^d$ and the d-dimensional torus $(\mathbb{Z}/n\mathbb{Z})^d$. Here the results are less precise. Benjamini and Sznitman [2] and Windisch [17] investigated the structure of the vacant set of a random walk on a d-dimensional torus. The main focus of this work is to apply the method of random interlacements. For toroidal grids of dimension $d \geq 5$, it is shown that there is a value $t^+(d)$, linear in n, above which the vacant set is sub-critical, and a value of $t^-(d)$ below which the graph is super-critical. It is believed that there is a phase transition for $d \geq 3$. A recent monograph by Černy and Teixeira [5] summarizes the random interlacement methodology. The monograph also gives details for the vacant set of random r-regular graphs.

1.2 New results

In this note we consider certain types of d-regular graphs with n vertices, where $d \to \infty$ with n. Our main example of interest is the hypercube Q_d which has $n = 2^d$ vertices. The vertex set of the hypercube is sequences $\{0,1\}^d$ where two sequences are defined as adjacent iff they differ in exactly one coordinate. In order to be slightly more general, we identify those properties of the hypercube that underpin our results.

Given certain properties (listed below), we can show that w.h.p. the graph $\Lambda(t)$ exhibits a change in component structure at around the time $t^* = n \log d$ which is asymptotically equal to the expression in (1). We show that if $t \leq t_{-\varepsilon} = (1-\varepsilon)t^*$ then w.h.p. there are components in $\Lambda(t)$ of size much larger than $\log n$, whereas if $t \geq t_\varepsilon = (1+\varepsilon)t^*$ then all components of $\Lambda(t)$ are of size $o(\log n)$.

We use the notation $\mathbf{Pr}(\mathcal{W}_x(t) = y)$ and $P_x^t(y)$ for the probability that a ergodic random walk starting from vertex x is at vertex y at step t. If t is sufficiently large, so that the walk is very close to stationarity and the starting point x is arbitrary, we may also use the simplified notation $\mathbf{Pr}(\mathcal{W}(t) = y)$. Let $\pi_v = d(v)/2m$ to denote the stationary probability of vertex v, where $m = |E|$ is the number of edges of the graph G and $d(v)$ is the degree of v. For regular graphs, $\pi_v = 1/n$. The rate of convergence of the walk is given by

$$|P_x^t(y) - \pi_y| \leq (\pi_y/\pi_x)^{1/2}\lambda^t, \tag{2}$$

where $\lambda = \max(\lambda_2, |\lambda_n|)$ is the second largest eigenvalue of the transition matrix in absolute value. See for example, Lovasz [15] Theorem 5.1.

The hypercube Q_d is bipartite, and a simple random walk does not have a stationary distribution on bipartite graphs. To overcome this, we replace the simple random walk by a *lazy* walk, in which at each step there is a 1/2 probability of staying put. Let $N_G(v)$ denote the neighbours of v in G, and $d_G(v) = |N_G(v)|$. The lazy walk $\mathcal{W}$ has transition probabilities given by

$$P(v,w) = \begin{cases} \frac{1}{2} & w = v \\ \frac{1}{2d_G(v)} & w \in N_G(v) \\ 0 & \text{Otherwise} \end{cases} .$$

We can obtain the underlying simple random walk, which we refer to as the *speedy* walk, by ignoring the steps when the particle does not move. For large t, asymptotically half of the steps in the lazy walk will not result in a change of vertex. Therefore w.h.p. properties of the speedy walk after approximately t steps can be obtained from properties of the lazy walk after approximately $2t$ steps. Unless explicitly stated otherwise, all future discussions and proofs refer to the lazy walk which we will denote by $\mathcal{W}$.

The effect of making the walk lazy is to shift the eigenvalues of the simple random walk upwards so that, for the lazy walk $\lambda = \lambda_2$. For a lazy walk we define a mixing time T, such that for all vertices x, y

$$T = \min_{t \geq 1} \left\{ t : \left| P_x^t(y) - \frac{1}{n} \right| \leq \frac{1}{n^3} \right\}. \tag{3}$$

For the lazy walk, the spectral gap is $1-\lambda$, so using this in (2), property **P1** (defined below) implies that we can take $T = O(d^{\rho_1} \log n)$ in (3). Note that we will always assume there is a lower bound on T given by

$$T \geq K \log n, \tag{4}$$

for some large $K > 0$.

The graph properties we assume for our analysis

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For $S \subset V$, define $N_G(S) = \{w \in V \setminus S : \exists v \in S \ s.t. \ \{v, w\} \in E\}$.

We assume that the graph $G = (V, E)$ is d-regular, connected, and has the properties **P1**–**P4** listed below. The bounds in properties **P2**–**P4** are parameterised by the ε used to define $t_{\pm\varepsilon}$ for the vacant set. We will point out later where we use these bounds, so that the reader can see their relevance.

P1 The spectral gap for the lazy walk is $\Omega(1/d^{\rho_1})$ for some constant $0 < \rho_1 \leq 3$. This implies that we can take $T = O(d^{\rho_1} \log n)$ in (3), (see [14], Chapter 12).

P2 $(\log \log n)^{2/\varepsilon} \ll d = O\left(\frac{n}{\log n}\right)^{1/5}$.

P3 For $u, v \in V$, the graph distance $dist_G(u, v)$ is the length of the shortest path from u to v in G. Let $\nu(u, v)$ be the number of neighbours w of v for which $dist_G(w, u) \leq dist_G(u, v)$. Then for all u, v such that $dist_G(u, v) \leq d^{\varepsilon}$, there exists an $\rho_2 = O(1)$, such that $\nu(u, v) \leq \rho_2 \ dist_G(u, v)$.

P4 For $S \subseteq V$, let $e(S)$ denote the number of edges induced by S. If $|S| = o(\log n)$, then $e(S) = o(d|S|)$.

Properties **P1**–**P4** are various measures of expansion. Our results for the structure of the vacant set $\Lambda(t)$ based on these properties are as follows.

Theorem 1 *Let $\varepsilon = \varepsilon(n)$ be a function such that $\varepsilon \gg 1/\log d$. Let $t^* = n \log d$ and let $t_{\pm\varepsilon} = (1 \pm \varepsilon)t^*$. Let $L_1(t)$ denote the size of the largest component in $\Lambda(t)$. At step t of the speedy walk, the following results for $L_1(t)$ hold.*

(a) *If G satisfies* **P1**, **P2**, **P3**, **P4**, *and $t \leq t_{-\varepsilon}$ then w.h.p. $L_1(t) \geq e^{\Omega(d^{\varepsilon/2})}$.*
Note that $d^{\varepsilon/2}$ can be replaced by $d^{\gamma\varepsilon}$ for any constant $0 < \gamma < 1$.

(b) *If G satisfies* **P1**, **P2**, **P3**, *and $t \geq t_{+\varepsilon}$ then w.h.p. $L_1(t) = o(\log n)$.*

We next give examples of graphs which satisfy Theorem 1(a),(b). Random regular graphs with degree d satisfying **P2** can be shown to satisfy properties **P1**, **P3**, **P4** w.h.p. The hypercube Q_d satisfies **P1**–**P4**. This can be shown as follows. Property **P1** is satisfied with $\rho_1 = 1$, as the spectral gap for the lazy walk is $2/d$ (see [14] page 162). As $d = \log_2 n$, **P2** is clearly satisfied. For **P3**, without loss of generality, let $v = (0, 0, \ldots, 0)$ and let $u = (1, 1, \ldots, 1, 0, \ldots, 0)$ (k 1's) be vertices of Q_d. There are exactly $\nu(u, v) = k$ neighbours w of v which satisfy $dist_G(u, w) \leq dist_G(u, v)$, so we can take $\rho_2 = 1$. For **P4** we can use the edge isoperimetric inequality of Hart [12] which states that the number of edges between S and $V - S$ is at least $s(d - \log_2 s)$, where $|S| = s$. This implies that S induces at most $(s/2) \log_2 s$ edges. If $s = o(d)$ then $e(S) \leq (s/2) \log_2 s = o(ds)$.

2 The main tools for our proofs

Given a graph G and random walk $\mathcal{W}$, let T be the mixing time given in (3). For a vertex v, let $R_v = R_v(G)$ denote the expected number of visits to v by the walk $\mathcal{W}_v$ within T steps. Thus

$$R_v = \sum_{k=0}^{T} P_v^k(v). \tag{5}$$

Note that, as $P_v^0(v) = 1$, $R_v \geq 1$.

Our main tool is a lemma (Lemma 1) that we have found very useful in analysing the cover time of various classes of random graphs. A more general form of Lemma 1 which originally appeared in [6], and simplified in [7] required a certain technical condition to be satisfied. It was shown in [8] that provided $R_v = O(1)$ for all $v \in V$, this condition is always true. For graphs which satisfy **P2** and **P3**, it follows that $R_v = 2 + O(1/d) = O(1)$ as required. We will prove this in Lemma 6. The probabilities given in Lemma 1 and Corollary 2 are with respect to a random walk on a fixed graph G.

Lemma 1 (First visit lemma) *Let $v \in V$ be such that $R_v = O(1)$, $T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$. Let*

$$f_t(u, v) = \mathbf{Pr}(t = \min\{\tau > T : \mathcal{W}_u(\tau) = v\})$$

be the probability that the first visit to v after time T occurs at step t.

There exists

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}, \tag{6}$$

and constant $K > 0$ such that for any $u \in V$, and all $t \geq T$,

$$f_t(u, v) = (1 + O(T\pi_v))\frac{p_v}{(1 + p_v)^{t+1}} + O(T\pi_v e^{-t/KT}). \tag{7}$$

□

Corollary 2 *For $t \geq T$ let $\mathcal{A}_v(t)$ be the event that $\mathcal{W}_u$ does not visit v at steps $T, T+1, \ldots, t$. Then, under the assumptions of Lemma 1,*

$$\mathbf{Pr}(\mathcal{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} + O(T^2\pi_v e^{-t/KT}). \tag{8}$$

The result (8) follows by adding up (7) for $s > t$. □

Remark 3 Let $K > 0$ as in (7) and let L be given by

$$L = 2KT \log n. \tag{9}$$

Provided $p_v = o(1/T)$ and $t \geq L$ then, as $p_v = O(\pi_v)$, the bounds (7) and (8) can be written as

$$f_t(u, v) = (1 + O(T\pi_v))\; p_v \; e^{-tp_v(1+O(p_v))}$$

and

$$\mathbf{Pr}(\mathcal{A}_v(t)) = (1 + O(T\pi_v))\; e^{-tp_v(1+O(p_v))}$$

respectively. For the graphs we consider $\pi_v = 1/n$. From **P1**, $T = O(d^{\rho_1} \log n)$ and from **P2**, $d = O(n/\log n)^{1/4}$. Thus for $\rho_1 \leq 3$, $p_v = o(1/T)$ as required.

Contraction lemma

Let $H = (V(H), E(H))$ be given. Let S be a subset of vertices of H. In order to estimate the probability of a first visit to a set S of vertices, we proceed as follows. Contract S to a single vertex $\gamma(S)$. This forms a multi-graph $\Gamma = \Gamma(H, S) = (V', E')$ in which the set S is replaced by $\gamma = \gamma(S)$. The edges of H, including loops and multiple edges formed by contraction, are retained. For $(v, w) \in E(H)$ the equivalent edge in E' is given as follows. If $v, w \notin S$ then $(v, w) \in E'$, whereas if $v \in S, w \notin S$ then $(\gamma, w) \in E'$. For the case $v, w \in S$ replace $(v, w) \in E$ by $(\gamma, \gamma) \in E'$. It follows that $|E'| = |E(H)|$, so that $\pi_\gamma = \pi_S = \sum_{v \in S} \pi_v$.

Note that if T is a mixing time for $\mathcal{W}$ in H, then T is a mixing time for the walk in Γ. It is proved in [1, Ch. 3], Corollary 27, that if a subset S of vertices is contracted to a single vertex, then the second eigenvalue of the transition matrix cannot increase. Thus $\lambda_2(H) \geq \lambda_2(\Gamma)$. We used the second eigenvalue $\lambda_2(H) = \lambda$ of the lazy walk in (2) to obtain the mixing time bound T in (3). Thus T is also a mixing time bound for (3) in Γ. For $\mathcal{W}_u^H, u \in S$, the equivalent walk in Γ is $\mathcal{W}_\gamma^\Gamma$.

If we apply Lemma 1 to γ in Γ, the probability of a first visit to S in H can be found (up to an additive error of $O(|S|/n^3)$ from the probability of a first visit to γ in Γ. This is proved next.

Lemma 4 *[7] Let $H = (V(H), E(H))$, let $S \subseteq V(H)$, let $\gamma(S)$ be vertex obtained by the contraction of S. Let $V' = V - S + \gamma$, and let $\Gamma(H, S) = (V', E')$. Let $\mathcal{W}_u^H$ be a random walk in H starting at $u \notin S$, and let $\mathcal{W}_u^\Gamma$ be a random walk in Γ. Let T be a mixing time satisfying (3) in both H and Γ.*

For graphs $G = H, \Gamma$, let $\mathcal{A}_w^G(t)$ be the event that in graph G, no visit was made to w at any step $T \leq s \leq t$. Then

$$\mathbf{Pr}(\cap_{v \in S} \mathcal{A}_v^H(t)) = \mathbf{Pr}(\mathcal{A}_\gamma^\Gamma(t)) + O(|S|/n^3).$$

For graphs $G = H, \Gamma$, let $\mathcal{E}_w^G(t)$ be the event that in graph G, the first visit to w after time T occurs at step t, (i.e. $t = \min\{\tau > T : \mathcal{W}^G(\tau) = w\}$). Then

$$\mathbf{Pr}(\cup_{v \in S} \mathcal{E}_v^H(t)) = \mathbf{Pr}(\mathcal{E}_\gamma^\Gamma(t)) + O(|S|/n^3).$$

Proof

Note that $|E(H)| = |E'| = m$, say. Let $W_x(j)$ (resp. $X_x(j)$) be the position of walk $\mathcal{W}_x = \mathcal{W}_x^H$ (resp. $\mathcal{X}_x = \mathcal{W}_x^\Gamma$) at step j. For graphs $G = H, \Gamma$, let $P_u^s(x; G)$ be the s step transition probability for the corresponding walk to go from u to x in G.

$$\begin{aligned}
\mathbf{Pr}(\mathcal{A}_\gamma^\Gamma(t)) &= \sum_{x \neq \gamma} P_u^T(x; \Gamma)\, \mathbf{Pr}(X_x(s-T) \neq \gamma,\ T \leq s \leq t; \Gamma) && (10)\\
&= \sum_{x \neq \gamma} \left(\frac{d(x)}{2m} + O\left(1/n^3\right) \right) \mathbf{Pr}(X_x(s-T) \neq \gamma,\ T \leq s \leq t; \Gamma) && (11)\\
&= \sum_{x \notin S} \left(P_u^T(x; H) + O\left(1/n^3\right) \right) \mathbf{Pr}(W_x(s-T) \notin S,\ T \leq s \leq t; H) && (12)\\
&= \sum_{x \notin S} \left[\mathbf{Pr}(W_u(T) = x)\, \mathbf{Pr}(W_x(s-T) \notin S,\ T \leq s \leq t; H) + O(1/n^3) \right] \\
&= \mathbf{Pr}(W_u(t) \notin S,\ T \leq s \leq t; H) + O(|S|/n^3) \\
&= \mathbf{Pr}(\cap_{v \in S} \mathcal{A}_v^H(t)) + O(|S|/n^3). && (13)
\end{aligned}$$

In (10), if $\mathcal{A}_\gamma^\Gamma(t)$ occurs then $X_u(T) \neq \gamma$. Given $X_u(T) = x$, by the Markov property $X_u(s)$ is equivalent to the walk $X_x(s-T)$. After T steps, the walk X_u on Γ is close to stationarity. We use (3) to approximate $P_u^T(x;\Gamma)$ by $\pi_x = d(x)/2m = 1/n$ in (11). The second factor in equation (12) follows because there is a natural measure preserving map ϕ between walks in H that start at $x \notin S$ and avoid S, and walks in Γ that start at $x \neq \gamma$ and avoid γ.

The proof argument for $\mathcal{E}_\gamma^\Gamma(t)$ is identical to that for $\mathcal{A}_\gamma^\Gamma(t)$. □

We use Lemma 4 throughout the rest of this paper. Indeed most of the proofs rely on contracting some set of vertices S to a vertex $\gamma(S)$. In this case, although a different graph Γ, and different walk $\mathcal{X}$ are used to estimate the probability, provided

$$\frac{|S|}{n^3} = o(\mathbf{Pr}(\mathcal{A}_\gamma^\Gamma(t))),$$

the probability estimate we obtain for the walk $\mathcal{W}$ in the base graph H is correct. It follows from (2) and (3) that by increasing the mixing time T by a constant factor we can, if necessary, reduce the error term $|S|/n^3$ to $|S|/n^c$ for any $c > 0$.

Visits to sets of vertices

Given the walk made a first visit to set of vertices S, we need the probability this first visit was to a given $v \in S$.

Lemma 5 *Let $S = \{v_1, ..., v_k\}$ be a set of vertices of a regular graph G, such that the assumptions of Lemma 1 hold in G for all $v \in S$, and also for $\gamma(S)$ in $\Gamma(G)$. For $t \geq T$, let $\mathcal{E}_v = \mathcal{E}_v(t)$ be the event that the first visit to v after time T occurs at step t, (i.e. $t = \min\{\tau > T : \mathcal{W}(\tau) = v\}$), and let $\mathcal{E}_S = \cup_{v\in S}\mathcal{E}_v$. Suppose $t \geq 2(T+L)$ where $L = 2KT\log n$, where $K > 0$ is some suitably large constant. Let p_w be as defined by (6), (7) in Lemma 1 for the walk on G. Then for $v \in S$*

$$\mathbf{Pr}(\mathcal{E}_v \mid \mathcal{E}_S) \leq \frac{p_v}{p_{\gamma(S)}}(1 + O(L\pi_S). \tag{14}$$

Proof It is enough to prove the lemma for $S = \{u, v\}$, i.e. for two vertices, as vertex u can always be a contraction of a set. Specifically, if $|S| > 2$ let $u = \gamma(S \setminus \{v\})$.

Write t as $t = T + s + T + L$, where $s \geq L$. Let $\mathcal{A}_u$ be the event that $\mathcal{W}(t) = u$, but that $\mathcal{W}(\sigma) \notin \{u, v\}$ for $\sigma \in [T, s+T-1]$, and that $\mathcal{W}(\sigma) \neq u$ for $\sigma \in [s+2T, t-1]$. Contract S to $\gamma = \gamma(S)$ and apply Corollary 2 and Lemma 4 to γ in $[T, T+s-1]$. The probability of no visit to S is $(1+O(T\pi_S))/(1+p_\gamma)^s$. Next, apply Lemma 1 to u in $[s+2T, t] = [t-L, t]$. The probability of a first visit to u at L is $(1+O(T\pi_u))p_u/(1+p_u)^L$. Thus

$$\mathbf{Pr}(\mathcal{A}_u) \leq (1 + O(T\pi_S))p_u/((1+p_\gamma)^s\ (1+p_u)^L). \tag{15}$$

Let $\mathcal{B}_u$ be the event that $\mathcal{W}(t) = u$ but $\mathcal{W}(\sigma) \notin \{u, v\}$ for $\sigma \in [T, t-1]$. Then $\mathcal{B}_u \subseteq \mathcal{A}_u$ and so $\mathbf{Pr}(\mathcal{B}_u) \leq \mathbf{Pr}(\mathcal{A}_u)$. By contracting S we have that the probability of a first visit to γ (and hence S) at step t is

$$\mathbf{Pr}(\mathcal{B}_u \cup \mathcal{B}_v) = (1 + O(T\pi_S))p_\gamma/(1+p_\gamma)^t.$$

As $\mathcal{E}_S = \mathcal{B}_u \cup \mathcal{B}_v$, the upper bound follows from

$$\mathbf{Pr}(\mathcal{E}_v \mid \mathcal{E}_S) = \frac{\mathbf{Pr}(\mathcal{B}_v)}{\mathbf{Pr}(\mathcal{B}_u \cup \mathcal{B}_v)} \leq \frac{\mathbf{Pr}(\mathcal{A}_v)}{\mathbf{Pr}(\mathcal{B}_u \cup \mathcal{B}_v)} = \frac{p_v}{p_\gamma}(1 + O(L\pi_S)).$$

□

3 Proof of Theorem 1(a)

To apply the lemmas of the previous section we will need to estimate R_v as given by (5).

Lemma 6 *If* **P1**, **P2**, **P3** *hold, then in the lazy walk, for any* $v \in V$

(i)
$$R_v = 2 + \frac{2}{d} + O\left(\frac{1}{d^2}\right).$$

(ii) Suppose $\mathcal{W}(0)$ *is at distance at least 2 from* v *(resp. at least 3 from* v*). The probability* $\mathcal{W}$ *visits* $N(v)$ *within* $L = O(T \log n)$ *steps is* $P(2, L) = O(1/d)$ *(resp.* $P(3, L) = O(1/d^2)$*).*

(iii) Let $C \subseteq N(v)$*. For a walk starting from* $u \in C$*, let* R_C *denote the expected number of returns to* C *during* T*. Then* $R_C = 2 + O(1/d)$*.*

Proof *Proof of (i).* We write

$$R_v = 1 + \sum_{k=1}^{T} \frac{1}{2^k} + \sum_{k=0}^{T-1} \frac{1}{2^k} \sum_{w \in N_G(v)} \frac{1}{2d} R(w, T-k-1),$$

where for $w \in N_G(v)$, $R(w, \tau)$ is the expected number of visits to v in τ steps by $\mathcal{W}_w$.

For a lower bound, let $R_v(t)$ be the expected number of returns to v in t steps and let $R_v = R_v(T)$ as usual. Then

$$R(w, \tau) \geq \sum_{j=0}^{\tau-1} \frac{1}{2^j} \frac{1}{2d} R_v(T-\tau) = \frac{R_v(T-\tau)}{d}\left(1 - \frac{1}{2^\tau}\right).$$

This is the probability that for $\tau - 1$ steps the walk loops at vertex w, and then moves to v, giving $R_v(T-\tau)$ expected returns to v. In $t \geq T/2$ steps $P_v^t(v) = (1/n)(1 + o(1))$ (see (2), (3)). Thus if $\tau \leq T/2$, $R_v(T-\tau) = R_v - O(T/n)$, and

$$R_v \geq 2 - \frac{1}{2^{T+1}} + (R_v - O(T/n)) \frac{1}{2d} \sum_{k=0}^{T/2} \frac{1}{2^k}\left(1 - \frac{1}{2^{T-k-1}}\right).$$

Assuming $T \geq K \log n$ (see (4)) it follows that $T2^{-T} = O(d^{-2})$. Thus

$$R_v \geq 2 + \frac{2}{d} + O(1/d^2) - O(T/2^T) - O(T/nd) = 2 + \frac{2}{d} + O(1/d^2)$$

We next prove we can bound $R(w, T)$ from above by

$$R(w, T) \leq R_v\left(\frac{1}{d} + O\left(\frac{1}{d^2}\right)\right). \tag{16}$$

Let $N_G^i(v)$ be the set of vertices at distance i from v in G, let $N_G(v) = N_G^1(v)$, and let $R_i^* = \max_{w \in N_G^i(v)} R(w, T)$. By definition $R(w, T) \leq R_1^*$ for all $w \in N_G(v)$ and

$$R_1^* \leq \sum_{j \geq 0}\left(\frac{1}{2} + \frac{\rho_2}{2d}\right)^j \frac{1}{2d} R_v + \sum_{j \geq 0}\left(\frac{1}{2} + \frac{2\rho_2}{2d}\right)^j \frac{1}{2d} R_1^* + R_3^*. \tag{17}$$

The first summation term counts the case that for some number of steps the walk loops at a vertex of $N_G(v)$, or moves around in $N_G(v)$, which by **P3** has probability at most $\rho_2/2d$. At some point, the walk either moves to v, giving a R_v expected returns, or moves to $N_G^2(v)$. In the latter case, the second term counts moves back to $N_G(v)$, and the third term moves to $N_G^3(v)$, giving the R_3^* upper bound.

We next show that $R_3^* = O(1/d^2)$. To do this we couple the walk on G starting from v, and up to graph distance ρ_3, with a biassed random walk on the line $\{0, 1, \dots, \rho_3\}$, with reflecting barriers at $0, \rho_3$. Once the walk on G has reached graph distance ρ_3, it either moves back towards v immediately or at some future step $t < T$, in which case we continue the coupling from distance $\rho_3 - 1$; or it stays at distance at least ρ_3 until step T in which case there are no further returns to v during T steps. To provide an upper bound R_3^*, we make a worst case analysis where we assume that, on reaching distance ρ_3 the walk immediately moves back towards v and this is repeated T times.

Let $\mathcal{X}$ be random walk on $\{0, 1, \dots, \rho_3\}$, with reflecting barriers at $0, \rho_3$, and transition probabilities for $\mathcal{X}(i)$ for $0 < i < \rho_3$ given by

$$\mathcal{X}(i+1) = \begin{cases} \mathcal{X}(i) - 1 & \text{Probability } q = \frac{\rho_2 \rho_3}{d} \\ \mathcal{X}(i) & \text{Probability } r = \frac{1}{2} \\ \mathcal{X}(i) + 1 & \text{Probability } p = \frac{1}{2} - \frac{\rho_2 \rho_3}{d} \end{cases}.$$

Starting $\mathcal{W} = \mathcal{W}_z$ from $z \in N_G^3(v)$ is equivalent to starting $\mathcal{X} = \mathcal{X}_3$ from $j = 3$. We couple $\mathcal{W}_z$ and $\mathcal{X}_3$ so that $\mathcal{X}_3$ is always as close to 0 as $\mathcal{W}_z$ is to v. Let $u = \mathcal{W}_z(t)$. If $dist(v, u) \geq \rho_3$ then we hold $\mathcal{X}_3$ at ρ_3 until $\mathcal{W}_z$ moves back to graph distance $\rho_3 - 1$. Provided $\rho_3 \leq d^\varepsilon$ and $dist(v, u) \leq \rho_3$, then referring to **P3**, $\nu(v, u) \leq \rho_2 \rho_3$. Thus the probability that $\mathcal{W}_z(t)$ moves towards v is at most the probability that $\mathcal{X}$ moves towards 0.

For a random walk on $0, 1, \dots, \ell$ starting from $j = 0, 1, 2, \dots, \ell$ and with probabilities p, q, r of moving right or left, or looping respectively, it follows from XIV(2.4) of Feller [11] that the probability π_j of the walk visiting 0 before visiting ℓ is

$$\pi_j = \frac{\xi^j - \xi^\ell}{1 - \xi^\ell} \leq 2\xi^j \tag{18}$$

where $\xi = q/p$. Thus for $\mathcal{X}$ as given above, $\xi = \rho_2 \ell / (d - 2\rho_2 \ell)$, where $\ell = \rho_3$.

To finish the proof of (i), we choose $\ell = \rho_3 = \lceil d^\delta \rceil$, for some $\varepsilon/2 < \delta < \varepsilon$. The probability π_3 that $\mathcal{X}$ reaches 0 before ρ_3 is $O(1/d^{3-3\delta}) = O(1/d^2)$. Once the walk $\mathcal{X}$ has reached $\ell = \rho_3$, we restart it at $\rho_3 - 1$. As explained above, to make a worst case assumption, we repeat this process T times. The probability $\mathcal{X}$ reaches to the origin before a return to ρ_3 is given by $\pi_{\rho_3 - 1} = O(\xi^{\rho_3 - 1})$. From **P1**, $T = O(d^{\rho_1} \log n)$, and we find

$$R_3^* \leq T \pi_{\rho_3 - 1} + \pi_3 = O(\log n \; d^{\rho_1 + 1 - \rho_3(1-\delta)}) + O(1/d^2) = O(1/d^2).$$

For the last inequality, we used $\delta > \varepsilon/2$ and **P2** to give

$$d^\delta \geq (\log \log n)^{2\delta/\varepsilon} > \log \log n.$$

Proof of (ii). Let $C = \{v\} \cup N(v)$. The property **P3** holds in G for any vertex at distance $\ell \leq d^\varepsilon$ from v. Because moving closer to C implies moving closer to v, a vertex within distance ℓ of v

has at most $\rho_2\ell$ neighbours closer to C. Thus the probability of a transition from $N_G^2(v)$ to C is at most $2\rho_2/d$. If the walk starts at distance 2 from v, it either loops or moves within $N_G^2(v)$, or, conditional on making a transition away from $N_G^2(v)$, with probability $O(2\rho_2/d)$ it moves to C, and with probability $1-O(1/d)$ moves to $N_G^3(v)$.

To complete the proof we use the same coupling argument as the proof of (i). Assume the walk starts at a distance 3 from v. We define a graph Γ_C obtained from G by contracting the vertices in C to a single vertex γ_C. As explained before Lemma 4, we can still use the same mixing time T. If we replace v by γ_C, we can still use the coupling with the random walk $\mathcal{X}$ on $\{0,1,...,\rho_3\}$. As moving closer to γ_C means moving closer to v, choosing $\rho_3 = \lfloor d^\varepsilon \rfloor - 1$, it follows from **P3** as outlined above that the transition probabilities are correct. By the argument of part (i), the walk next moves to γ_C with probability at most $\pi_2 = O(1/d^2)$ and to a distance ρ_3 from γ_C with probability $1-O(1/d^2)$. After this we use the argument of (i) as before. In conclusion, for a set $C \subseteq N(v)$ and a walk which moves away from C to a distance 2 from v, (resp. distance 3 from v) the probability of a return to $\{v\} \cup N(v)$ within L steps is $O(1/d)$ (resp. $O(1/d^2)$).

Proof of (iii). Let $C \subseteq \{v\} \cup N(v)$. Contract C to γ_C as above. We claim that $R_{\gamma_C} = 2 + O\left(\frac{1}{d}\right)$. The 2 comes from the loop at each vertex and a factor of $O(\rho_2/d)$ comes from possible loops at γ_C arising from G-edges inside C. If the walk moves to $N_G^2(v)$, then by (ii) the probability of a return to C is $O(1/d)$. □

Analysis for $t \leq t_{-\varepsilon}$

Recall that $t_{-\varepsilon} = (1-\varepsilon)n \log d$. Let U denote the set of vertices unvisited by the lazy walk in the time interval $[1, 2t_{-\varepsilon}]$ and let U_0 denote the set of vertices unvisited by the lazy walk in the time interval $[T, 2t_{-\varepsilon}]$. Note that $|U_0 \setminus U| \leq T$. Given Lemma 7 below holds, using **P1**, **P2** it follows that $T = o(|U_0|)$ and thus $|U| = (1-o(1))|U_0|$.

Lemma 7 *w.h.p.*

$$|U_0| \sim \frac{n}{d^{1-\varepsilon}}.$$

Proof Fix a vertex v. Corollary 2 and Remark 3 tell us that

$$\mathbf{Pr}(v \in U_0) = \left(1 + O\left(\frac{T}{n}\right)\right) \exp\left\{-\frac{2t_{-\varepsilon}}{nR_v} + O\left(\frac{t_{-\varepsilon}}{n^2}\right)\right\} + O(e^{-\Omega(t_{-\varepsilon}/T)}). \tag{19}$$

By Lemma 6, $R_v = 2 + \frac{2}{d} + O\left(\frac{1}{d^2}\right)$. This gives $\mathbf{Pr}(v \in U_0) \sim d^{1-\varepsilon}$ and thus

$$\mathbf{E}\,|U_0| \sim \frac{n}{d^{1-\varepsilon}}.$$

Now consider a pair of vertices v, w at distance 5 or more in G. Let Γ_{vw} be obtained from G by contracting v, w to a single vertex γ_{vw}. Referring to Lemma 4 we have

$$\mathbf{Pr}(v, w \in U_0) = \mathbf{Pr}(\mathcal{A}^{\Gamma}_{\gamma_{vw}}(2t_{-\varepsilon})) + O(1/n^3). \tag{20}$$

Working in Γ_{vw}, it follows more or less verbatim by using the arguments of Lemma 6(i) that $R_{\gamma_{vw}} = 2 + \frac{2}{d} + O\left(\frac{1}{d^2}\right)$. As v, w are sufficiently far apart, only minor modifications are needed for the analysis of $\mathcal{X}$. Thus

$$\frac{2}{R_{\gamma_{vw}}} = \left(1 + O\left(\frac{1}{d^2}\right)\right)\left(\frac{1}{R_v} + \frac{1}{R_w}\right). \tag{21}$$

Similarly to (19), from Corollary 2 and Remark 3, with $\pi_{\gamma_{vw}} = 2/n$, and $p_{\gamma_{vw}} = (1+O(T/n))\, 2/(nR_{\gamma_{vw}})$ we find that

$$\mathbf{Pr}(\mathcal{A}^{\Gamma}_{\gamma_{vw}}(2t_{-\varepsilon})) = \left(1+O\left(\frac{T}{n}\right)\right)\exp\left\{-\frac{2}{R_{\gamma_{vw}}}\frac{2t_{-\varepsilon}}{n} + O\left(\frac{t_{-\varepsilon}}{n^2}\right)\right\} + O(e^{-\Omega(t_{-\varepsilon}/T)}). \tag{22}$$

Using $t_{-\varepsilon} = (1-\varepsilon)n\log d$ in (22) it follows from (20) and (21) that

$$\mathbf{Pr}(v,w \in U_0) = \left(1+O\left(\frac{\log d}{d^2}\right)\right)\mathbf{Pr}(v\in U_0)\,\mathbf{Pr}(w\in U_0) + O(1/n^3). \tag{23}$$

We prove concentration using Chebychev's inequality. This states that for a random variable X with finite mean μ and variance σ^2, then for $k>0$,

$$\mathbf{Pr}(|X-\mu|\geq k) \leq \sigma^2/k^2.$$

Let X_{vw} be the indicator for $v,w\in U_0$. Let S be the set of pairs of vertices at distance at least 5, and let S' be the set of distinct pairs at distance at most 4. Then

$$\begin{aligned}\mathbf{E}\,|U_0|^2 &= \mathbf{E}\,|U_0| + \sum_{(v,w)\in S}\mathbf{E}\,X_{vw} + \sum_{(v,w)\in S'}\mathbf{E}\,X_{vw}\\ &\leq \mathbf{E}\,|U_0| + \left(1+O\left(\frac{\log d}{d^2}\right)\right)\mathbf{E}\,|U_0|^2 + O(d^4\,\mathbf{E}\,|U_0|).\end{aligned}$$

The second term on the second line follows from (23). The third term uses the observation that there are $O(d^4)$ vertices at distance at most 4 from a given $v\in U_0$. Thus

$$\mathbf{Var}(|U_0|) = O(d^4\,\mathbf{E}\,|U_0|) + O\left(\frac{\log d}{d^2}\,\mathbf{E}\,|U_0|^2\right).$$

From **P2** we have that $d^4 = o(\mathbf{E}\,|U_0|)$. Thus for some ω tending to infinity

$$\mathbf{Pr}\left(||U_0| - \mathbf{E}\,|U_0|| \leq \frac{\mathbf{E}\,|U_0|}{\sqrt{\omega}}\right) \leq O\left(\frac{\omega\log d}{d^2}\right) + O\left(\frac{\omega d^4}{\mathbf{E}\,|U_0|}\right) = o(1).$$

□

Lemma 8 *A vertex is* bad *if it has fewer than $d^{\varepsilon}/2$ neighbours in U. Let B denote the set of bad vertices. Then w.h.p. $|B| \leq ne^{-d^{\varepsilon}/10}$.*

Proof Fix a vertex v and denote $N_G(v)$ by $W = \{w_1, w_2, \ldots, w_d\}$. Let $X = |W\cap U|$. In the proof of Lemma 7 we showed that for a given vertex x, $\mathbf{Pr}(x\in U) = \tilde{p} \sim d^{-(1-\varepsilon)}$. Thus $\mathbf{E}\,X \sim d^{\varepsilon}$ and if X was distributed as $Bin(d,\tilde{p})$ then it would follow from Hoeffding's inequality that

$$\mathbf{Pr}\left(X \leq \frac{1}{2}d^{\varepsilon}\right) \leq e^{-\Omega(d^{\varepsilon})}. \tag{24}$$

The bound (24) is our target. We establish it is true, in spite of X not having a binomial distribution. For $S\subseteq W$, let $\mathcal{A}_S = \{W\cap U = W\setminus S\}$, i.e. exactly the vertices S of W are visited by the walk. So,

$$\mathbf{Pr}\left(X\leq\frac{1}{2}d^{\varepsilon}\right) = \sum_{D=d-d^{\varepsilon}/2}^{d}\ \sum_{\substack{S\subseteq W\\ |S|=D}}\mathbf{Pr}(\mathcal{A}_S). \tag{25}$$

If $\mathcal{A}_S$ occurs then there is a sequence of times $\mathbf{t} = (t_0 = 1 \leq t_1 < t_2 \cdots < t_D \leq t_{D+1} = 2t_{-\varepsilon})$ and a bijection $f : S \to [D]$ such that for $x \in S$ there is a first visit to w_x at time $t_{f(x)}$. Let $\mathcal{B}(S, \mathbf{t})$ denote this event. For a sequence $\mathbf{t}$, let $\Phi(\mathbf{t}) = \{i : |t_{i+1} - t_i| \leq L\}$, where $L = 2KT \log n$ is given by (9). Let $\mathcal{T}_h = \{\mathbf{t} : |\Phi(\mathbf{t})| = h\}\,|$. For $h \geq 0$, let

$$S_h = \sum_{\mathbf{t} \in \mathcal{T}_h} \mathbf{Pr}(\mathcal{B}(S, \mathbf{t})).$$

Then,

$$\mathbf{Pr}(\mathcal{A}_S) \leq \sum_{h=0}^{D} S_h. \tag{26}$$

The main content of the proof of this lemma will be to establish that

$$\mathbf{Pr}(\mathcal{A}_S) = O(1) \left(e^{-2pt_{-\varepsilon}}\right)^{(d-D)} \left(1 - e^{-2pt_{-\varepsilon}}\right)^{D}. \tag{27}$$

Given (25) and (27) we see that

$$\mathbf{Pr}\left(X \leq \frac{1}{2} d^{\varepsilon}\right) = O(1) \sum_{D \geq d - d^{\varepsilon}/2} \binom{d}{D} \left(e^{-2pt_{-\varepsilon}}\right)^{(d-D)} \left(1 - e^{-2pt_{-\varepsilon}}\right)^{D}.$$

The expected value of $Bin(d, e^{-2pt_{-\varepsilon}})$ is $d^{\varepsilon}(1 + o(1))$, so from the Hoeffding inequality,

$$\mathbf{Pr}\left(X \leq \frac{1}{2} d^{\varepsilon}\right) = O\left(e^{-d^{\varepsilon}/8}\right).$$

Thus the expected number of bad vertices is

$$\mathbf{E}\,|B| = O\left(n\, e^{-d^{\varepsilon}/8}\right),$$

and the lemma follows from the Markov inequality.

Proof of (27)**.** We begin with S_0. Our upper bound for S_0 will contain some terms that should properly be assigned to some $S_h, h > 0$, but this is allowable as we proving an upper bound. We repeat this warning below. Let

$$p = \frac{1}{\left(2 + O\left(\frac{1}{d}\right)\right) n}, \tag{28}$$

then we have

$$\begin{aligned} S_0 \leq D! \sum_{t_1 < t_2 \cdots < t_D} &\left(\prod_{i=1}^{D} \frac{(1 + O(T/n))p}{(1 + (d - i + 1)p)^{t_i - t_{i-1}}} + o(e^{-\Omega(\frac{t_i - t_{i-1}}{T})}) \right) \\ &\times \left(\frac{1 + O(Td/n)}{(1 + (d - D)p)^{2t_{-\varepsilon} - t_D}} + o(e^{-\Omega(\frac{t_{-\varepsilon} - t_D}{T})}) \right). \end{aligned} \tag{29}$$

Proof of (29)**.** Assume for the moment that $S = \{w_1, \ldots, w_D\}$ and that $f(w_i) = i$ for $i = 1, 2, \ldots, D$. Let $A_i = \{w_i, w_{i+1}, \ldots, w_D\}$ for $i = 1, 2, \ldots, D$. We assign times $t_1, t_2, \ldots, t_D$ to S in $D!$ ways. Now consider a term

$$\Psi_i = \frac{(1 + O(T/n))p}{(1 + (d - i + 1)p)^{t_i - t_{i-1}}} + o(e^{-\Omega((t_i - t_{i-1})/T)}). \tag{30}$$

We claim this is an upper bound for the probability that there were no visits to $w_i, \dots, w_d$ during $[t_{i-1}+T, t_i-1]$ followed by a first visit to w_i at t_i. If so, it is also an upper bound for the probability there is no visit to $w_i, \dots, w_d$ during $[t_{i-1}+1, t_i-1]$ followed by a visit to w_i at t_i. This bound hold regardless of the first t_{i-1} steps of the walk. In fact this bound allows for visits to $w_i, w_{i+1}, \dots, w_d$ during the time interval $[t_{i-1}+1, t_{i-1}+T-1]$. This is allowable as Ψ_i is an upper bound. Thus some terms properly attributed to $S_h, h > 0$ are overcounted.

To prove (30), define a graph Γ_{A_i} obtained from G by contracting the vertices in A_i to a single vertex γ_{A_i}. The mixing time T does not increase, as explained above Lemma 4. By Lemma 1, the probability a first visit to γ_{A_i} in $[t_{i-1}+T, t_i]$ occurs at t_t can be written as $(d-i+1)\Psi_i$. Given a first visit has been made to A_i, we need to upper bound the probability of the event $\mathcal{E}_v$ that this first visit was made to a given $v \in A_i$. Lemma 5 gives the answer as

$$\mathbf{Pr}(\mathcal{E}_v \mid \mathcal{E}_{A_i}) \leq \frac{p_v}{p_{\gamma_{A_i}}}(1+O(Ld/n)) = (1+O(1/d))\frac{1}{d-i+1}.$$

To obtain the right hand side above, we used $\pi_{\gamma(A_i)} = (1+O(Td/n))(d-i+1)/n$ and $R_v, R_\gamma = 2+O(1/d)$,which follows directly from Lemma 6(i),(iii). This establishes (30).

The final term in (29), given by $\frac{1+O(Td/n)}{(1+(d-D)p)^{2t_{-\varepsilon}-t_D}} + o(e^{-\Omega((t_{-\varepsilon}-t_D)/T)})$ bounds the probability that the vertices in $\{w_{D+1}, \dots, w_d\}$ are not visited in the interval $[t_D, 2t_{-\varepsilon}]$. This follows directly from Corollary 2.

End of proof of (29)**.**

The next step is to evaluate (29). Considering (30), the term $\frac{p}{(1+(d-i+1)p)^{t_i-t_{i-1}}} = \Omega((1/n)e^{(t_i-t_{i-1})/n})$, whereas the term $o(e^{-\Omega((t_i-t_{i-1})/T}) = o(e^{(t_i-t_{i-1})/T})$. As $t_i - t_{i-1} \geq L = KT\log n$ the latter term can be absorbed into the $O(d^{-1})$ in the definition of p. Furthermore,

$$\frac{1}{1+(d-i+1)p} = \exp\left\{-(d-i+1)p + O\left(\frac{d^2}{n^2}\right)\right\}.$$

Noting that

$$\sum_{i=1}^{D+1}(d-i+1)(t_i-t_{i-1}) = (d-D)t_{D+1} + (t_1 + \cdots + t_D),$$

we can write

$$\begin{aligned}
S_0 &\leq 2D!p^D \sum_{t_1<t_2\cdots<t_D} \exp\left\{-p\sum_{i=1}^{D+1}(d-i+1)(t_i-t_{i-1})\right\} \\
&= 2D!p^D e^{-2(d-D)pt_{-\varepsilon}} \sum_{t_1<t_2\cdots<t_D} \exp\left\{-p\sum_{i=1}^{D} t_i\right\} \\
&\leq 2e^{-2(d-D)pt_{-\varepsilon}}\left(p\sum_{t=1}^{2t_{-\varepsilon}} e^{-pt}\right)^D \\
&\leq 3e^{-2(d-D)pt_{-\varepsilon}}\left(p\int_{t=0}^{2t_{-\varepsilon}} e^{-pt}dt\right)^D \\
&= 3e^{-2(d-D)pt_{-\varepsilon}}(1-e^{-2pt_{-\varepsilon}})^D.
\end{aligned} \tag{31}$$

We next show that $S_1, S_2 \dots, S_D$ are not much larger in total than S_0.

We say a visit to vertex u is T-*distinct*, if it occurs at least T steps after a previous T-distinct visit, or from the start of the walk. Thus if $\mathcal{W}(t) = u$, and this visit is T-distinct, the next T-distinct visit to u will be at the first step $s \geq t + T$ such that $\mathcal{W}(s) = u$. Once a T-distinct visit has taken place, several *secondary visits* to the vertex u may occur within the next $T - 1$ steps, and thus before the next T-distinct visit. We will consider such secondary visits separately in our proof.

We consider the case $t_i - t_{i-1} \leq L$ in two parts, namely $t_i - t_{i-1} < T$, and $T \leq t_i - t_{i-1} \leq L$. The first case is for *secondary visits*, and the second case *close (together) visits*. These require a separate analysis.

Given $\mathbf{t} = (t_1, \dots, t_D)$ for arbitrary $D \leq d$, let $Z \geq D - k$ be an upper bound on the total number of secondary visits to $W = N(v)$ occurring as a result of $k \leq D$ first visits to W which are T-distinct. Let $N_2(v)$ denote the set of vertices at distance 2 from v. Then

$$Z(\mathbf{t}) = N_1 + \cdots + N_k$$

where N_i are the number of secondary visits to $W = N(v)$ (i.e. returns to W via $\{v\} \cup N_2(v)$) which occur during $[t_i, t_i + T]$, $i = 1, ..., k$.

The values of N_i are independent and geometrically distributed with failure probability $O(1/d)$. From $W = N(v)$ the particle moves to $\{v\} \cup N(v)$ with probability $O(1/d)$, (this follows from **P3**). Otherwise the particle moves to distance 2 away from v with probability $1 - O(1/d)$, and we can use the value of $P(2, T) = O(1/d)$ from Lemma 6(ii). For any $D \leq d$, the probability $\widehat{P}(\ell)$ of at least ℓ secondary visits is

$$\widehat{P}(\ell) = \binom{D + \ell - 1}{\ell} \left(\frac{O(1)}{d}\right)^\ell \leq \left(\frac{O(1)D}{\ell d}\right)^\ell \leq \left(\frac{O(1)}{\ell}\right)^\ell = e^{-\Theta(\varepsilon d^\varepsilon \log d)},$$

on choosing $\ell = d^\varepsilon/100$. Provided $\varepsilon \gg 1/\log d$, the probability of at least $d^\varepsilon/100$ secondary visits to W is $o(e^{-d^\varepsilon})$.

We next consider close together visits. For convenience, replace D by $D' = D - Z$ i.e. remove any entries in $\mathbf{t}$ corresponding to secondary visits. Let h count those T-distinct first visits which are close together i.e. $T \leq t_i - t_{i-1} \leq L$. After $t \geq T$ steps, the distribution of the walk is close to stationary, so the probability that the walk is within distance 2 of vertex v is $O(d^2/n)$. If the walk is at least distance 3 from v, by Lemma 6(ii) the probability of a visit to $W = N(v)$ in L steps is at most $P(3, L) = O(1/d^2)$. It follows that, independently of any previous ones, each close visit has probability $O(d^2/n) + O(1/d^2) = O(1/d^2)$, assuming $d = o(n^{1/4})$ (see **P2**).

To bound S_h we note that the remaining $k = D - h$ first visits are 'well spaced' i.e. $L \leq t_i - t_{i-1}$. There are $\binom{D-1}{h}$ ways to assign the h 'close together' events to the $k = D - h$ 'well spaced' ones. To do so, we choose an allocation $n_1, n_2, \dots, n_k \geq 0$ such that $n_1 + n_2 + \cdots + n_k = h$.

Note that $S_0 = S_0(D)$ so changing D to $D - h$, for $h \geq 1$, from (31) we have

$$S_h(D) \leq S_0(D - h)\binom{D-1}{h}\left(\frac{O(1)}{d^2}\right)^h \leq S_0(D)\left(\frac{e^{2pt_{-\varepsilon}}}{1 - e^{-2pt_{-\varepsilon}}}\right)^h \left(\frac{O(D)}{hd^2}\right)^h \leq S_0(D)\left(\frac{O(d^{-\varepsilon})}{h}\right)^h. \tag{32}$$

The value of p is from (28), and $t_{-\varepsilon} = (1 - \varepsilon)n \log d$. Inequality (32), along with (31) completes the proof of (27), and the lemma follows. $\square$

We can now easily show that w.h.p. at time $2t_{-\varepsilon}$, there is a component of size much larger than $\log n$.

Lemma 9 *W.h.p. the graph induced by unvisited vertices contains a component of size at least $e^{\Omega(d^{\varepsilon/2})}$.*

Proof Let $n_0 = \frac{n}{5(ed^{1-\varepsilon/2})^{d^{\varepsilon/2}} d^{1-\varepsilon}}$. We begin by greedily choosing $v_1, v_2, \ldots, v_{n_0} \in U$ such that v_i, v_j are at distance greater than $d^{\varepsilon/2}$. This is easily done, because there are $1 + \binom{d}{1} + \binom{d}{2} + \cdots + \binom{d}{d^{\varepsilon/2}} < 2\binom{d}{d^{\varepsilon/2}} \leq 2(ed^{1-\varepsilon/2})^{d^{\varepsilon/2}}$ vertices within distance $d^{\varepsilon/2}$ of any given vertex. Having chosen $v_1, v_2, \ldots, v_k$, $k \leq n_0$, there will w.h.p. be at least $\frac{n}{2d^{1-\varepsilon}} - 2k(ed^{1-\varepsilon/2})^{d^{\varepsilon/2}} > 0$ choices for v_{k+1}. For each i let V_i denote the set of vertices within distance $d^{\varepsilon/2}$ of v_i. The V_i are disjoint and so from Lemma 8 there are w.h.p. at least $n_0 - ne^{-d^{\varepsilon}/10} > 0$ indices i such that $V_i \cap B = \emptyset$.

Choose i such that $V_i \cap B = \emptyset$. From v_i we can do breadth first search, but only including vertices in U. If L_r denotes the rth level of this search where $L_0 = \{v_i\}$ then we see that $|L_{r+1}| \geq \frac{d^{\varepsilon}|L_r|}{2\rho_2(r+1)}$. Thus V_i contains a component of size at least

$$\sum_{i=0}^{d^{\varepsilon/2}/2} \binom{d^{\varepsilon}/2}{i} \frac{1}{(2\rho_2)^i} = e^{\Omega(d^{\varepsilon/2})}.$$

□

4 Proof of Theorem 1(b)

Let

$$s = \frac{2\log n}{\varepsilon \log d} = o(\log n).$$

We will show that w.h.p. there is no component of size s or more at time $t \geq 2t_{+\varepsilon}$ in $\Gamma(t)$, with respect to the lazy walk.

Lemma 10 *For $v \in V$ there are at most $(ed)^{s-1}$ sets S such that (i) $v \in S$, (ii) $|S| = s$ and (iii) $G[S]$ is connected.*

Proof The number of such sets is bounded by the number of distinct s-vertex trees which are rooted at v. This in turn is bounded by the number of distinct d-ary rooted trees with s vertices. This is equal to $\binom{ds}{s}/((d-1)s+1)$, see Knuth [13]. □

We fix a set S of size s that induces a connected subgraph of G. To estimate the probability that S is unvisited at time $t \geq 2t_{+\varepsilon}$ we contract S to a vertex γ_S as in the proofs of Lemmas 7 and 8. We need to estimate the probability that γ_S is unvisited by a lazy random walk on the associated graph Γ_S during the time interval $[T, 2t_{+\varepsilon}]$. For this we need to prove

Lemma 11 $R_{\gamma_S} = 2 + o(1)$.

Proof Let $e(S)$ denote the number of edges contained in S. It follows from **P4** that $e(S) = o(ds)$. This means that γ_S has degree ds, of which $o(ds)$ comes from loops associated with internal edges of S. It then follows that when the walk on Γ_S is at γ_S then it leaves γ_S with probability $\frac{1}{2} - o(1)$. It is then straightforward to use the argument of Lemma 6 to finish the proof of the lemma. $\Box$

Using Lemma 10 and Lemma 11 we see that if $p_\gamma = \frac{(1+o(1))s}{2}$ then

$$\begin{aligned}
\mathbf{Pr}(\text{there exists a component of size } s) &\leq n(ed)^{s-1} \left(\frac{1 + O(Ts/n))}{(1+p_\gamma)^{2t_{+\varepsilon}}} + O(T^2 s e^{-\Omega(t_{+\varepsilon}/T)}) \right) \\
&\leq 2n(ed \cdot e^{-(1-o(1))(1+\varepsilon)\log d})^s \\
&\leq 2nd^{-2\varepsilon s/3} = o(1).
\end{aligned}$$

$\Box$

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