An almost linear time algorithm for finding Hamilton cycles in sparse random graphs with minimum degree at least three.

Alan Frieze* and Simi Haber
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh PA15217.

December 20, 2013

Abstract

We describe an algorithm for finding Hamilton cycles in random graphs. Our model is the random graph $G = G_{n,m}^{\delta \geq 3}$. In this model G is drawn uniformly from graphs with vertex set $[n]$, m edges and minimum degree at least three. We focus on the case where $m = cn$ for constant c. If c is sufficiently large then our algorithm runs in $O(n^{1+o(1)})$ time and succeeds w.h.p.

1 Introduction

The threshold for the existence of Hamilton cycles in random graphs has been known very precisely for some time, Komlós and Szemerédi [15], Bollobás [4], Ajtai, Komlós and Szemerédi [1]. Computationally, the Hamilton cycle problem is one of the original NP-complete problems described in the paper of Karp [13]. On the other hand Angluin and Valiant [2] were the first to show that the Hamilton cycle problem could be solved efficiently on random graphs. The algorithm in [2] is randomised and very fast, $O(n \log^2 n)$ time, but requires $Kn \log n$ random edges for sufficiently large $K > 0$. Bollobás, Fenner and Frieze [7] gave a deterministic polynomial time algorithm that works w.h.p. at the exact threshold for Hamiltonicity, it is shown to run in $O(n^{3+o(1)})$ time.

The challenge therefore is to find efficient algorithms for graphs with a linear number of edges. Here we have to make some extra assumptions because a random graph with cn edges is very unlikely to be Hamiltonian. It will have isolated vertices. It is natural therefore to consider models of random graphs with a linear number of edges and minimum degree δ at least two. In fact minimum degree three is required to avoid the event of having three vertices of degree two having a common neighbor. For example, in the case of random r-regular graphs, $r = O(1) \geq 3$, Robinson and Wormald [18], [19] settled the existence question and Frieze, Jerrum, Molloy, Robinson and Wormald [11] gave a polynomial time algorithm for finding a Hamilton cycle. The running time of this algorithm was not given explicitly, but it is certainly $\Omega(n^3)$.

We will work on a model where the assumption is that $\delta \geq 3$ as opposed to all vertices having degree exactly three. It is tempting to think that existence results for the regular case $r = 3$ will help. Unfortunately, this is not true. The proof for the regular case breaks if there is significant variance in the vertex degrees. The

*Research supported in part by NSF Grant CCF2013110

model we consider here, the random graph $G_{n,m}^{\delta\geq 3}$ is uniformly sampled from the set $\mathcal{G}_{n,m}^{\delta\geq 3}$ of graphs with vertex set $[n]$, m edges and minimum degree at least three. A quite natural model for studying Hamilton cycles in sparse random graphs.
Frieze [9] gave an $O(n^{3+o(1)})$ time algorithm for finding large cycles in sparse random graphs and this can be adapted to find Hamilton cycles in $G_{n,cn}^{\delta\geq 3}$ in this time for sufficiently large c. The paper [10] gives an algorithm that reduces this to $n^{1.5+o(1)}$ for $c \geq 10$. The main aim of this paper is to construct an almost linear time algorithm for this model, but only with the assumption of larger c.

Theorem 1.1. *If c is sufficiently large then our algorithm finds a Hamilton cycle in $G_{n,m}^{\delta\geq 3}$, $m = cn$, and runs in $O(n^{1+o(1)})$ time and succeeds w.h.p.*

Remark 1.1. *The $n^{o(1)}$ term here is $(\log n)^{O(\log\log n)}$ which is tantalisingly close to best possible(?) $\log^{O(1)} n$.*

2 Outline of the paper

The paper [8] gave an efficient algorithm for finding the maximum matching in a sparse random graph. Its approach was to (i) use the simple greedy algorithm of Karp and Sipser [14] and then (ii) augment it to a maximum matching using alternating paths. In this paper we replace the Karp-Sipser algorithm with the algorithm 2GREEDY that w.h.p. finds a 2-matching in $G = G_{n,m}^{\delta\geq 3}$ with $O(\log n)$ components and we replace alternating paths with extensions and rotations. (A 2-matching is a spanning subgraph of maximum degree at most two).
In Section 3 we will describe our algorithm. We will describe it in two subsections. We will describe 2GREEDY for finding a good 2-matching M in detail in Section 3.1. In Section 3.2 we will describe an algorithm EXTEND-ROTATE that uses extensions and rotations to convert M into a Hamilton cycle. In Section 4 we discuss some "residual randomness" left over by 2GREEDY. In Section 5 we prove some structural properties of $G_{n,m}^{\delta\geq 3}$. In Section 6 we prove some properties relating the output of 2GREEDY to the execution of EXTEND-ROTATE. In Section 7 we do a final calculation to finish the proof. In Section 8 we point to our difficulties in proving $n\log^{O(1)} n$ and in Section 9 we make some final remarks.

3 Algorithm

As already stated, there are two phases to the algorithm. First we find a good 2-matching M and then we convert it to a Hamilton cycle. We look first at how we find M.

3.1 Algorithm 2GREEDY

We greedily and randomly choose edges to add to M. Edges of M are deleted from the graph. We let $b(v) \in \{0, 1, 2\}$ denote the degree of v in M. Once $b(v) = 2$ its incident edges are no longer considered for selection. The vertex itself is deleted from the graph. Thus the graph from which we select edges will shrink as the algorithm progresses. We will use Γ to denote the current subgraph from which edges are to be selected. When there are vertices v of degree $2 - b(v)$ (or less) in Γ, we take care to choose an edge incident with such a vertex. Our observation being that there is a maximum cardinality 2-matching of Γ that contains such an edge.
If every vertex v of Γ had degree at least $3 - b(v)$ then we choose an edge randomly from edges that are incident with vertices v that have $b(v) = 0$. In this way, we quickly arrive at a stage where every vertex of Γ

has $b(v) = 1$. At this point we use the algorithm of [8] to find a (near) perfect matching M^*, which we add to M to create our final 2-matching.
We describe 2GREEDY in enough detail to make some of its claimed properties meaningful. We let

- μ be the number of edges in Γ,
- $Y_k = \{v \in [n] : d_\Gamma(v) = k \text{ and } b(v) = 0\}$, $k = 0, 1, 2$,
- $Z_k = \{v \in [n] : d_\Gamma(v) = k \text{ and } b(v) = 1\}$, $k = 0, 1$,
- $Y = \{v \in [n] : d_\Gamma(v) \geq 3 \text{ and } b(v) = 0\}$,
- $Z = \{v \in [n] : d_\Gamma(v) \geq 2 \text{ and } b(v) = 1\}$,
- M is the set of edges in the current 2-matching.

Note that $V(\Gamma) = [n] \setminus (Y_0 \cup Z_0)$ and that $b(v) \in \{0, 1\}$ for $v \in V(\Gamma)$.
We will assume that the input to our algorithm is an ordered sequence $\sigma_m = (e_1, e_2, \ldots, e_m)$ where $m = cn$. Here $E_m = \{e_1, e_2, \ldots, e_m\}$ are the edges of $G_{n,m}^{\delta \geq 3}$ and σ_m is a random ordering of E_m. Once these orderings are given, the vertices and edges are processed in a deterministic fashion. Thus for example, if 2GREEDY requires a random edge with some property, then it is required to take the first available edge in the given ordering.
We now give details of the steps of

Algorithm 2GREEDY:
Step 1(a) $Y_1 \neq \emptyset$

Choose $v \in Y_1$. We choose v by finding the first edge in the ordering σ that contains a member of Y_1. Suppose that its neighbour in Γ is w. We delete the edge (v, w) from Γ add (v, w) to M and move v to Z_0.

(i) If w is currently in Y then move it to Z. If it is currently in Y_1 then move it to Z_0. If it is currently in Y_2 then move it to Z_1. Call this re-assigning w.

(ii) If $b(w) = 1$ then we move w to Z_0 and make the requisite changes due to the loss of other edges incident with w. In this case w is no longer a vertex of Γ. Call this *tidying up*.

Step 1(b): $Y_1 = \emptyset$ **and** $Y_2 \neq \emptyset$

Choose $v \in Y_2$. We choose v by finding the first edge in the ordering σ that contains a member of Y_2. Suppose that its neighbours in Γ are w_1, w_2.

We choose one of the neighbors at random, say w_1. We move v to Z_1. We delete the edge (v, w_1) from Γ and place it into M. In addition,

(i) If $b(w_1) = 0$ then put $b(w_1) = 1$. Re-assign w_1 i.e. if $w_i \in Y_k$ then move it to Z_{k-1}.

(ii) If $b(w_1) = 1$ then we delete w_1 from Γ. Tidy up.

Step 1(c): $Y_1 = Y_2 = \emptyset$ **and** $Z_1 \neq \emptyset$

Choose $v \in Z_1$. We choose v by finding the first edge in the ordering σ that contains a member of Z_1. Let u be the other endpoint of the path P of M that contains v. Let w be the unique neighbour of v in Γ. We delete v from Γ and add the edge (v, w) to M. In addition there are two cases.

(1) If $b(w) = 0$ then we re-assign w.

(2) If $b(w) = 1$ then we delete vertex w and tidy up.

Step 2: $Y_1 = Y_2 = Z_1 = \emptyset$ **and** $Y \neq \emptyset$
Choose the first edge $(v, w) \in E(\Gamma)$ in the order σ incident with a vertex $v \in Y$. We delete the edge (v, w) from Γ and add it to M. We move v from Y to Z. There are two cases.

(i) If $b(w) = 0$ then move w from Y to Z.

(ii) If $b(w) = 1$ then we delete vertex w and tidy up.

Step 3: $Y_1 = Y_2 = Z_1 = Y = \emptyset$
At this point Γ will be distributed as $G^{\delta \geq 2}_{\nu,\mu}$ for some ν, μ where $\mu = O(\nu)$. As such, it contains a (near) perfect matching M^* [12] and it can be found in $O(\nu)$ expected time [8]. This step comprises

Step 3a Apply the Karp-Sipser algorithm to Γ. W.h.p. this results in the construction of a matching M_1^* that covers all but $O(\nu^{1/5+o(1)})$ vertices $U = \{u_1, u_2, \ldots, u_\ell\}$.

Step 3b Now find augmenting paths from u_{2i-1} to u_{2i} for $i \leq \ell/2$. This produces the matching M^*.

The output of 2GREEDY is set of edges $M \leftarrow M \cup M^*$.

3.2 Extension-Rotation Algorithm

We now describe an algorithm, EXTEND-ROTATE that w.h.p. converts M into a Hamilton cycle. The main idea is that of a *rotation*. Given a path $P = (u_1, u_2, \ldots, u_k)$ and an edge $e = (u_k, u_i)$ where $i \leq k - 2$ we say that the path $P' = (u_1, \ldots, u_i, u_k, u_{k-1}, \ldots, u_{i+1})$ is obtained from P by a rotation. u_1 is the *fixed* endpoint of this rotation. We say that e is the *inserted* edge.
Given a path P with endpoints a, b we define a *restricted rotation search* $RRS(\nu)$ as follows: We start by doing a sequence of rotations with a as the fixed endpoint. Furthermore

R1 We do these rotations in "breadth first manner", described in detail in Section 6.

R2 We stop this process when we have either (i) created ν endpoints or (ii) we have found a path Q with an endpoint that has a neighbor w outside of Q. The path $Q + w$ will be longer than P. We say that we have found an *extension*.

Let $END(a)$ be the set of endpoints, other than a, produced by this procedure. Assuming that we did not find an extension and having constructed $END(a)$, we take each $x \in END(a)$ in turn and starting with the path P_x that we have found from a to x, we carry out R1,R2 above with x as the fixed endpoint and either find an extension or create a set of ν paths with x as one endpoint and the other endpoints comprising a set $END(x)$ of size ν.
Algorithm EXTEND-ROTATE

Step ER1 Let $K_1, K_2, \ldots, K_r$ be the components of M where $|K_1| = \max\{|K_j| : j \in [r]\}$. If K_1 is a path then we let $P_0 = K_1$, otherwise we let $P_0 = K_1 \setminus \{e\}$ where e is any edge of K_1.

Step ER2 Let P be the component of the current 2-matching M that contains P_0. If P is not a cycle, go directly to ER3. If P is a Hamilton cycle we are done. Otherwise there is a vertex $u \in P$ and a vertex $v \notin P$ such that $f = (u, v)$ is an edge of G, assuming that G is connected, see Lemma 6.3. Let Q be the component containing v. By deleting an edge of P incident to u and (possibly) an edge of Q incident with v and adding f we create a new path of length at least $|P| + 1$ with vertex set equal to $V(P) \cup V(Q)$. Rename this path P.

Step ER3 Carry out $RSS(\nu)$ until either an extension is found or we have constructed ν endpoint sets.

Case a: We find an extension. Suppose that we construct a path Q with endpoints x, y such that y has a neighbour $z \notin Q$.

(i) If z lies in a cycle C then let R be a path obtained from C by deleting one of the edges of C incident with z. Let now $P = x, Q, y, z, R$ and go to Step ER2.

(ii) If $z = u_j$ lies on a path $R = (u_1, u_2, \ldots, u_k)$ where the numbering is chosen so that $j \geq k/2$ then we let $P = x, Q, y, z, u_{j-1}, \ldots, u_1$ and go to Step ER2.

Case b: If there is no extension then we search for an edge $f = (p, q)$ such that $p \in END(a)$ and $q \in END(p)$. If there is no such edge then the algorithm fails. If there is such an edge we let Q be the corresponding path from p to q. We replace P in our 2-matching by the cycle $Q + f$ and go to ER2.

3.3 Execution Time of the algorithm

The expected running time of 2GREEDY is $O(n)$ and w.h.p. it completes in $O(n)$ time with a 2-matching M with at most $K_1 \log n$ components for some constant $K_1 > 0$. This follows from the results of [8] and [10].
To bound the execution time of EXTEND-ROTATE we first observe that it follows from [2] that $RSS(\nu)$ can be carried out in $O(\nu^2 \log n)$ time. We will take

$$\nu = n^{1/2+\varepsilon}$$

where

$$\varepsilon = \frac{K(\log\log n)^2}{\log n} \tag{1}$$

where K is a sufficiently large positive constant.
We now bound the number of executions of $RSS(\nu)$. Each time we execute Step ER3, we either reduce the number of components by one or we halve the size of one of the components not on the current path. So if the component sizes of M are $n_1, n_2, \ldots, n_\kappa$ then the number of executions of Step ER3 can be bounded by

$$\kappa + \sum_{i=1}^{\kappa} \log_2 n_i = O(\log^2 n).$$

So the total execution time is w.h.p. of order

$$n + (n^{1/2+O(\varepsilon)})^2 \log^2 n = O(n^{1+O(\varepsilon)}).$$

This clearly suffices for Theorem 1.1.
We will now turn to discuss the probability that our algorithm succeeds after we have described 2GREEDY. We remind the reader that the analysis assumes that c is sufficiently large.

4 Residual Randomness

Let G be a graph with an ordering of its edges and consider a run of 2GREEDYon that graph. At every point of time each vertex is in one of the sets $Y_0, Z_0, Y_1, Y_2, Z_1, Y$ and Z as defined above.

We let the set of vertices that were removed from the graph while in Z be denoted by R. We call them "regular vertices". These vertices are removed from Γ in the execution of a **Step 1** or **Step 2** of 2GREEDY and they are internal vertices of paths of M at the start of Step 3.
For a vertex v let t_v be the time(=step number) at which 2GREEDY deletes v from Γ. Vertices w that are not deleted before the start of Step 3 are given $t_w = \infty$. A vertex is *early* if $t_v \leq n^{1-\varepsilon}$ and *late* otherwise. An edge e_i is *punctual* if $i \leq (1-\alpha)m$ and *tardy* otherwise, where α is a small positive constant.
When a vertex $v \in R$ gets matching degree two we take the incident non-matching edge e with the lowest index in σ to be its Z-*witness*. The fact that $v \in Z$ just before this happens implies that e exists. We let W denote the set of Z-witnesses. We next define two sets R_0 and Λ_0:
We let

$$R_0 = \{v \in R : \ v \text{ is early and the } Z\text{-witness of } v \text{ is punctual}\}.$$

and

$$\Lambda_0 = \left\{v : v \text{ has punctual degree at least three in } \Gamma(n^{1-\varepsilon}) \ \right\}$$

where $\Gamma(t)$ is the graph Γ after t steps of 2GREEDY. The *punctual degree* of a vertex is the number of punctual edges incident to it.
We may now state and prove the main lemma of this section.

Lemma 4.1. *In what follows R_0, Λ_0 are defined with respect to G and an ordering of its edges. Let $e = \{x, y\}$ be a tardy edge of G where $x \in R_0$ and $y \in \Lambda_0$. Let G' be the graph obtained from G by deleting e. Assume that running* 2GREEDY *on G up until Step 3 gives a 2-matching M and a witness set W and running* 2GREEDY *on G' up until Step 3 gives M', W', R_0' and Λ_0'. Then $M = M', W = W', R_0 = R_0'$ and $\Lambda_0 = \Lambda_0'$.*

Proof We claim that up to time t_x, 2GREEDY will delete the same vertices and edges from $\Gamma(t)$ and $\Gamma(t)'$ and then delete x from both. After this the two graphs will coincide. We do this by induction on t. This is clearly true for $t = 0$ and assume that $\Gamma(t)$ and $\Gamma(t)'$ differ only in e and $t < t_x$. Consider the edge (v, w) chosen by 2GREEDY. We have $x \notin \{v, w\}$ since $t < t_x$. We cannot have $y \in \{v, w\}$ at time t because $y \in \Lambda_0$ and so it will have degree at least three at this point, in both $\Gamma(t)$ and $\Gamma(t)'$.
Note that the induction hypothesis implies that the sets $Y_0, Y_1, \ldots, Z$ are the same in $\Gamma(t), \Gamma(t)'$. Indeed, deleting e can only affect the status of x or y. This cannot affect the status of x because e comes after the Z-witness for x. Deleting e does not affect the status of y because e is tardy and y has punctual degree at least three at time t. Because the sets $Y_0, Y_1, \ldots, Z$ are unchanged, the choice of step is the same in $\Gamma(t)$ and $\Gamma(t)'$.
Because e is tardy, deleting it cannot affect the punctual degree of any vertex and so Λ_0 is unchanged. We have argued that its deletion does not affect whether or not a vertex is early and it cannot affect punctual Z-witnesses and so R_0 remains unchanged. □

Remark 4.1. *Suppose that $e_i = (v, w)$ and that (i) $v \in R_0$, (ii) $w \in \Lambda_0$ and (iii) e_i is tardy. Then replacing e_i by (v', w') such that (i) $v' \in R_0$ and (ii) $w' \in \Lambda_0$ results in the the same output M, W.*
The net effect of this is that if we condition on all edges except for the tardy edges between R_0 and Λ_0 then the unconditioned tardy $R_0 : \Lambda_0$ edges are uniformly random. This is what we mean by there being residual randomness.

5 Degree Sequence of $G_{n,m}^{\delta \geq 3}$

The degrees of the vertices in G are distributed as truncated Poisson random variables $Po(\lambda; \geq 3)$, see for example [3]. More precisely we can generate the degree sequence by taking random variables $Z_1, Z_2, \ldots, Z_n$

where

$$\mathbb{P}(Z_i = k) = \frac{\lambda^k}{k! f_3(\lambda)} \qquad for\ i = 1, 2, \ldots, n\ and\ k \geq 3, \tag{2}$$

where $f_j(\lambda) = e^\lambda - \sum_{k=0}^{j-1} \frac{\lambda^k}{k!}$ for $j \geq 0$. ($f_0(\lambda) = e^\lambda$).
Then we condition on $Z_1 + Z_2 + \cdots Z_n = 2m$. The resulting $Z_1, Z_2, \ldots, Z_n$ can be taken to have the same distribution as the degrees of G. This follows from Lemma 4 of [3]. If we choose λ so that

$$\mathbb{E}(Po(\lambda; \geq 3)) = \frac{2m}{n} \text{ or } \frac{\lambda f_2(\lambda)}{f_3(\lambda)} = \frac{2m}{n}$$

then the conditional probability, $\mathbb{P}(Z_1 + Z_2 + \cdots Z_n = 2m) = \Omega(1/\sqrt{n})$ and so we will have to pay a factor of $O(\sqrt{n})$ for removing the conditioning i.e. to use the simple inequality $\mathbb{P}(A \mid B) \leq \mathbb{P}(A)/\mathbb{P}(B)$. (This factor $O(n^{1/2})$ can be removed but it will not be necessary to do this here).
When c is large we find that λ is close to c. To make this precise we have

Lemma 5.1. *If c is sufficiently large, then $\lambda \geq 2c - 1$.*

Proof We have

$$\begin{aligned} \frac{\lambda}{2c} &= 1 - \frac{\lambda^2}{2f_2(\lambda)} \\ &\geq 1 - \frac{c^2}{2(2c-\lambda)f_1(c)}. \end{aligned}$$

Here we have used the convexity of f_2 and the fact that $f_2' = f_1$.
So if $\lambda < 2c - 1$ then

$$1 - \frac{1}{2c} \geq 1 - \frac{c^2}{2f_1(c)}$$

which is a contradiction for large c. □
The maximum degree Δ in G is less than $\log n$ q.s.[1] and equation (7) of [3] enables us to claim that that if $\nu_k, 2 \leq k \leq \log n$ is the number of vertices of degree k then q.s.

$$\left| \nu_k - \frac{n\lambda^k e^{-\lambda}}{k! f_3(\lambda)} \right| \leq K_1 \left(1 + \sqrt{n\lambda^k e^{-\lambda}/(k! f_3(\lambda))} \right) \log n,\ 2 \leq k \leq \log n. \tag{3}$$

for some constant $K_1 > 0$.
In particular, this implies that if the degrees of the vertices in G are $d_1, d_2, \ldots, d_n$ then q.s.

$$\sum_{i=1}^{n} d_i(d_i - 1) = O(n). \tag{4}$$

Given the degree sequence we make our computations in the configuration model, see Bollobás [5]. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of non-negative integers with $m = cn$. Let $W = [2cn]$ be our set of *points* and let $W_i = [d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_i]$, $i \in [n]$, partition W. The function $\phi : W \to [n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing F (i.e. a partition of W into $m = cn$ pairs) we obtain a (multi-)graph G_F

[1] A sequence of events, $\mathcal{E}_n$ occurs *quite surely* (q.s.) if $\mathbb{P}(\neg\mathcal{E}_n) = o(n^{-C})$ for any $C > 0$.

with vertex set $[n]$ and an edge $(\phi(u),\phi(v))$ for each $\{u,v\}\in F$. Choosing a pairing F uniformly at random from among all possible pairings of the points of W produces a random (multi-)graph G_F.
This model is valuable because of the following easily proven fact: Suppose $G\in\mathcal{G}_{n,\mathbf{d}}$, the set of (simple) graphs with vertex set $[n]$ and degree sequence $\mathbf{d}$. Then

$$\mathbb{P}(G_F=G\mid G_F\text{ is simple})=\frac{1}{|\mathcal{G}_{n,\mathbf{d}}|}.$$

It follows that if G is chosen randomly from $\mathcal{G}_{n,\mathbf{d}}$, then for any graph property $\mathcal{P}$

$$\mathbb{P}(G\in\mathcal{P})\leq\frac{\mathbb{P}(G_F\in\mathcal{P})}{\mathbb{P}(G_F\text{ is simple})}. \tag{5}$$

Furthermore, applying Lemmas 4.4 and 4.5 of McKay [17] we see that if the degree sequence of G satisfies (4) then $\mathbb{P}(G_F\text{ is simple})=\Omega(1)$. In which case the configuration model can substitute for $\mathcal{G}_{n,\mathbf{d}}$ (and hence $G^{\delta\geq3}_{n,m}$) in dealing with events of probability $o(n^{-1/2})$.

Lemma 5.2. *W.h.p.*

(a) $G^{\delta\geq3}_{n,m}$ contains no set $S\subseteq[n], 3\leq s=|S|\leq s_0=\frac{1}{5}\log_c n$ such that S contains at least $s+1$ edges.

(b) Let W_1 denote the set of vertices v that are within distance $\ell_0=2\log\log n$ of a cycle of length at most $2\ell_0$ in $G^{\delta\geq3}_{n,m}$. Then w.h.p. $|W_1|\leq n^{1/2}\log^{7\ell_0}n$.

(c) W.h.p. there does not exist a connected subset of $K_c\log n\leq s\leq n^{3/5}$ vertices that contain $s/10$ vertices of degree at most 30. Here K_c is some sufficiently large constant.

Proof (a) The expected number of sets S containing $|S|+1$ edges can be bounded by

$$O(n^{1/2})\sum_{s=3}^{s_0}\sum_{|S|=s}\sum_{D\geq3s}\sum_{\substack{d_1+\cdots+d_s=D\\ d_1,\ldots,d_s\geq3}}\prod_{i=1}^{s}\frac{\lambda^{d_i}}{f_3(\lambda)d_i!}\binom{D}{s+1}\left(\frac{D}{cn-2s}\right)^{s+1}\leq \tag{6}$$

$$O(n^{1/2})\sum_{s=3}^{s_0}\sum_{|S|=s}\sum_{D\geq3s}\left(\frac{De}{s+1}\right)^{s+1}\left(\frac{D}{cn-2s}\right)^{s+1}\frac{\lambda^D s^D}{D!f_3(\lambda)^s}. \tag{7}$$

Explanation: For (6) we choose a set of size s with vertices of degree $d_1,d_2,\ldots,d_s\geq3$ and $d_1+\cdots+d_s=D$. The term $\prod_{i=1}^s\frac{\lambda^{d_i}}{f_3(\lambda)d_i!}$ (modulo $O(n^{1/2})$) accounts for the probability of these degrees. We then choose $s+1$ configuration points and bound from above the probability that they are all paired with other points associated with s by $\left(\frac{D}{cn-2s}\right)^{s+1}$. We use $\sum_{d_1+\cdots+d_s=D}\prod_{i=1}^s\frac{1}{d_i!}=\frac{s^D}{D!}$ to get (7).
Continuing we observe that $(D/cs)^{2s+2}\leq\left(1+\frac{3}{c}\right)^D$ for $D\geq3s$. This is clearly true for $D\leq cs$ and follows by induction on $D\geq cs$. Therefore,

$$\sum_{D\geq3s}\frac{D^{2s+2}\lambda^Ds^D}{D!}\leq(cs)^{2s+2}\sum_{D\geq3s}\frac{((\lambda+3)s)^D}{D!}\leq(cs)^{2s+2}e^{(\lambda+3)s}.$$

Plugging this into (7) and using $(s+1)(cn-2s)\geq scn$ for $s\leq s_0$ we get a bound of

$$O(n^{1/2})\sum_{s=3}^{s_0}\sum_{|S|=s}\frac{ces}{n}\left(\frac{cese^{\lambda+3}}{nf_3(\lambda)}\right)^s$$

$$
\begin{aligned}
&\le O\left(\frac{cs}{n^{1/2}}\right)\sum_{s=3}^{s_0}\left(\frac{ne}{s}\right)^s\left(\frac{cese^{\lambda+3}}{nf_3(\lambda)}\right)^s\\
&\le O\left(\frac{cs}{n^{1/2}}\right)\sum_{s=3}^{s_0}\left(\frac{ce^{\lambda+5}}{f_3(\lambda)}\right)^s\\
&\le O\left(\frac{cs}{n^{1/2}}\right)\sum_{s=3}^{s_0}(2ce^5)^s \qquad (8)\\
&=o(1).
\end{aligned}
$$

To obtain (8) we use the fact that c large implies that λ is large and then $e^\lambda \le 2f_3(\lambda)$.
(b)

$$
\mathbb{E}(|W_1|) \le O(n^{1/2}) \sum_{k\le \ell_0, \ell \le 2\ell_0} \binom{n}{k+\ell}(k+\ell)!\ell \left(\frac{\Delta^2}{2m-6\ell_0}\right)^{k+\ell} \le O(n^{1/2})\ell_0^2 \log^{6\ell_0} n.
$$

Explanation: We choose the $k+\ell$ vertices in the path plus cycle and then order them in at most $(k+\ell)!$ ways and then choose a place to close the cycle in at most ℓ ways. The factor $\left(\frac{\Delta^2}{2m-6\ell_0}\right)^{k+\ell}$ bounds the probability that the edges exist (in the configuration model).
(We remind the reader that it is possible to remove the $O(n^{1/2})$ factor here. This would be worth doing if we could reduce ε to $O(\log\log n/\log n)$. This should become apparent in the proof of Lemma 6.8, equation (37)).
Part (b) follows from the Markov inequality.
(c) For a fixed s, the probability such a set exists can be bounded by

$$
O(n^{1/2}) \sum_{|S|=s} \binom{s}{s/10} \sum_{D\ge 3s} \sum_{\substack{d_1+\cdots+d_s=D\\ 3\le d_i,\, i\in[s]\\ d_i\le 30,\, i\in[s/10]}} \prod_{i=1}^{s} \frac{\lambda^{d_i}}{d_i! f_3(\lambda)} \binom{D}{s-1}\left(\frac{D}{cn}\right)^{s-1}.
$$

Explanation: We choose a set S and we let the degrees in S be $d_1, d_2, \ldots, d_s$ where D is the total degree. Since the induced subgraph is connected, it must contain a spanning tree. We weaken this to it must contain $s-1$ edges. $\binom{D}{s-1}$ enumerates the lower numbered points of the edges and then D^{s-1} enumerates the other possible endpoints and then $\left(\frac{1}{cn-2s}\right)^{s-1} = \frac{1+o(1)}{(cn)^{s-1}}$ bounds the probability the selected pairs exist.
We bound this by

$$
O(n^{1/2})\binom{n}{s}\binom{s}{s/10}\sum_{D\ge 3s}\binom{D}{s}\left(\frac{D}{cn}\right)^{s-1} f_3(\lambda)^{-s}[x^D]\left(\sum_{i=3}^{30}\frac{\lambda^i x^i}{i!}\right)^{s/10} f_3(\lambda x)^{9s/10} \qquad (9)
$$

$$
\le O(n^{3/2})\left(\frac{e}{s}\right)^s (10e)^{s/10} \sum_{D\ge 3s}\left(\frac{De}{s}\right)^s\left(\frac{D}{c}\right)^{s-1} \frac{1}{f_3(\lambda)^s(1+\xi)^D}\left(\sum_{i=3}^{30}\frac{\lambda^i(1+\xi)^i}{i!}\right)^{s/10} f_3(\lambda(1+\xi))^{9s/10} \qquad (10)
$$

for any positive ξ.

Explanation: The coefficient of x^D in $g(x) = \left(\sum_{i=3}^{30} \frac{\lambda^i x^i}{i!}\right)^{s/10} f_3(\lambda x)^{9s/10}$ is $\displaystyle\sum_{\substack{d_1+\cdots+d_s=D\\ 3\leq d_i,\, i\in[s]\\ d_i\leq 30,\, i\in[s/10]}} \prod_{i=1}^{s} \frac{\lambda^{d_i}}{d_i!}$. This is the explanation for (9). To get (10) we use the usual approximations plus $[x^D]g(x) \leq g(\zeta)/\zeta^D$ for any positive ζ. We use $\zeta = 1+\xi$.

Now if λ is large then $f_3(\lambda) \geq e^\lambda/2$. Also, $f_3(\lambda(1+\xi)) \leq e^{\lambda(1+\xi)}$. Furthermore,

$$\sum_{i=3}^{30} \frac{\lambda^i(1+\xi)^i}{i!} \leq 2\frac{\lambda^{30}e^{30\xi}}{30!} \leq 2\left(\frac{\lambda e^{1+\xi}}{30}\right)^{30}.$$

We will take ξ to be small but fixed. Then the bound becomes

$$O(n^{3/2})(10e)^{s/10}\left(\frac{e^2}{cs^2}\right)^s \frac{2^s e^{9\lambda\xi s/10}}{e^{\lambda s/10}}\left(2\left(\frac{\lambda e^{1+\xi}}{30}\right)^{30}\right)^{s/10} \sum_{D\geq 3s} \frac{D^{2s-1}}{(1+\xi)^D}.$$

We argue next that there exists $C = C(\xi)$ such that

$$\sum_{D\geq 3s} \frac{D^{2s-1}}{(1+\xi)^D} \leq (Cs)^{2s}. \tag{11}$$

Indeed, $D^{2s-1}/(1+\xi)^D$ is log-concave as a function of D and so has a unique maximum for $D \geq 3s$. We then have

$$\sum_{D=3s}^{\infty} \frac{D^{2s-1}}{(1+\xi)^D} \leq 2\int_{x=s}^{\infty} \frac{x^{2s-1}}{(1+\xi)^x}dx = 2s^{2s}\int_{y=1}^{\infty}\left(\frac{y^2}{(1+\xi)^y}\right)^s dy \leq 2s^{2s}\int_{y=1}^{\infty}(Ke^{-\xi y/2})^s dx$$

for $K = K(\xi)$ sufficiently large. This verifies (11). (The factor 2 after the first inequality comes from splitting the range of summation into two places within which the summand is monotone).

Continuing, we get a bound of

$$\begin{aligned}
&\leq O(n^{3/2})(10e)^{s/10}\left(\frac{e^2}{cs^2}\right)^s \frac{2^s e^{9\lambda\xi s/10}}{e^{\lambda s/10}}\left(2\left(\frac{\lambda e^{1+\xi}}{30}\right)^{30}\right)^{s/10} (Cs)^{2s}\\
&= O(n^{3/2})\left(\frac{2C(10e)^{1/10}e^2}{ce^{\lambda(1-9\xi)/10}}\left(\frac{\lambda e^{1+\xi}}{30}\right)^3\right)^s\\
&= o(1)
\end{aligned}$$

if we take $\xi = 1/10$ and c and hence λ sufficiently large.

□

6 Finding a Hamilton cycle

We assume that we have a path P with endpoints a, b and we do rotations with a as the fixed endpoint to try to find an extension. In the next section we show that if no extensions are found, then w.h.p. we create sufficient endpoints other than b on paths of length equal to P. Throughout this description, we will assume that no extension is found i.e. all neighbors of endpoints turn out to be vertices of P. We associate the search with something similar to an alternating tree of matching theory.

6.1 Tree Growth

In this section we describe our search for a longer path than P using EXTEND-ROTATE in terms of growing a tree structure T, where each vertex determines a path. We initialise T to be a sngle vertex b. We expose what happens w.h.p. if we fail to find an extension. Let $A_0 = \{b\}$ and let B_0 be the set of neighbors of b on P, excluding b's path neighbor. We now define the sets $A_i, B_i, i = 1, \ldots,$ and $C_i = \bigcup_{j \leq i}(A_j \cup B_j)$. Here every vertex v in A_i will be the endpoint of a path of the same length as P. It will be obtained from P by exactly i rotations with a as the fixed endpoint. Fix $i \geq 0$ and let $A_i = \{v_1, v_2, \ldots, v_k\}$. We build A_{i+1}, B_{i+1} by examining $v_1, v_2, \ldots, v_k$ in this order. Initially $A_{i+1} = B_{i+1} = \emptyset$ and we will add vertices as we process the vertices of A_i. Fix $v = v_j$. We have a path P_v with endpoints a, v. We consider two cases:
Case 1: $|C_i| \leq i_0 = \frac{1}{20} \log_c n$.
Let $N_v = \{u_1, u_2, \ldots, u_d\}$ be the neighbors of v, excluding its neighbor on P_v. We also exclude from N_v those neighbors already in B_{i+1} (as defined so far). Let w_j be the neighbor of u_j on P_v that lies between u_j and v for $j = 1, 2, \ldots, d$. Let $N'_v = \{w_1, w_2, \ldots, w_d\}$. We exclude from N'_v those vertices already in A_{i+1} (as defined so far). We add N_v to B_{i+1} and N'_v to A_{i+1} and we add edges (v, u_j) and (u_j, w_j) to T. The edge (u_j, w_j) will be called a *lost* edge. Furthermore, we define $P_{w_j} = P_v + (v, u_j) - (u_j, w_j)$ and observe that P_{w_j} has endpoints a, w_j.
Case 2: $|C_i| > i_0$.
Now let $N_v = \{u_1, u_2, \ldots, u_d\}$ be its neighbors as above. We now exclude from N_v those neighbors already in C_{i+1} (as defined so far) as well as those u_j for which $w_j \in C_{i+1}$. We define N'_v and update A_{i+1}, B_{i+1}, T with this restricted N_v.
We define the subgraph $T = T(P, a, k)$ as follows: It has vertex set C_k plus the edges of the form (v, u_j) and (u_j, w_j) used above. T suggests a tree. It is usually a tree, but in rare cases it may be unicyclic. This follows from Lemma 5.2. When this happens, some $v \in A_i$ (Case 1) has a neighbor in $B_j, j \leq i$.
We see from this that w.h.p. T has at most one cycle. By construction, cycles of T are contained in the first i_0 levels. If there are two cycles inside the first i_0 levels then there is a set S (consisting of the two cycles plus a path joining them) with at most $4i_0$ vertices and at least $|S| + 1$ edges.
We argue next that w.h.p. T can be assumed to grow to a certain size and we can control its rate of growth.

Lemma 6.1. *Let β be some small fixed positive constant. If c is sufficiently large, then for all paths P and endpoints a such that extension does not occur, w.h.p.*

(a) There exists k such that $|C_k| \geq L_0 = \frac{1}{15} \log_c n$.

(b) If $L_0 \leq |C_k| \leq n^{.6}$ then $|A_{k+1}| \in [2(1-\beta)c|C_k|, 2(1+\beta)c|C_k|]$, even if only punctual edges are used once $|A_k|$ reaches size at least n^{ε}.

(c) There exists $k_0 = O(\log_c n)$ such that $|A_{k_0}| \in [(2c(1+\beta))^{-1} n^{1/2+9\varepsilon}, 2c(1+\beta)n^{1/2+9\varepsilon}]$.

(d) Let $k_1 = k_0 - \ell_0$ where $\ell_0 = 2 \log \log n$ and let $x \in A_{k_1}$. Let S be the set of descendants of x in A_{k_0} and let $s = |S|$. Let $S_0 = \{y \in S : d(y) \geq 30\}$ and let $s_0 = |S_0|$. Then, where W_1 is as in Lemma 5.2,

  (i) $x \notin W_1$ and $s \geq (2c(1-\beta))^{k_0-k_1}/4$ implies that $s_0 \geq 99s/100$.

  (ii) $s \leq (2c(1+\beta))^{k_0-k_1} \log n$.

Proof (a) Lemma 2.1 of [12] proves the following: Suppose that S is the set of endpoints that can be produced by considering all possible sequences of rotations starting with some fixed path P and keeping one endpoint fixed. Let T be the set of external neighbors of S. Here $S \cap T = \emptyset$. Then $|T| \leq 2|S|$ and $S \cup T$

contains strictly more than $|S \cup T|$ edges. The assumption here is that the graph involved has minimum degree at least three. It follows from Lemma 5.2 that $|S| \geq \frac{1}{15}\log_c n$ w.h.p. As a final check, if $|C_k|$ never reached L_0 in size then it would have explored all possible sets of endpoints i.e. the breadth first search is no restriction.
(b) If the condition in (b) fails then the following structure appears: Let $\delta = 1$ if T is not a tree and 0 otherwise. Let $EVEN(T) = \bigcup_{i=0}^k A_i$ and $ODD(T) = \bigcup_{i=0}^k B_i$ where k is the number of iterations involved in the construction of T. Then with $|EVEN(T)| = l$ and $|N(EVEN(T))| = r$ we have (i) $2(l-1)+\delta$ edges of T connecting $EVEN(T)$ to $ODD(T)$, (ii) $r-l+1$ edges connecting $EVEN(T)$ to $N(EVEN(T)) \setminus ODD(T)$ and (iii) none of the $l(n-r-l)$ edges between $EVEN(T)$ and $V \setminus N(EVEN(T))$ are present.
Assume first that T is actually a tree and that $l \leq n^{\varepsilon}$ so that the edges of T need not be punctual. The next calculation will however be conducted assuming only that $l \leq n^{.6}$. This will enable us to use the calculations for the case $l > n^{\varepsilon}$ too.
Given the vertices of T and $N(EVEN(T))$, the probability of the existence of a T with $L_0 \leq l \leq n^{.6}$ and $r \leq 2(1-\beta)cl$ can be bounded by

$$O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1} \sum_{\substack{d_i \geq 3,\, i \in [r+l-1] \\ \sum_{i=1}^l d_i = r+l-1}} \left(\prod_{i=1}^{l} \frac{\lambda^{d_i} d_i!}{d_i! f_3(\lambda)} \prod_{i=l+1}^{2l-1} \frac{\lambda^{d_i} d_i(d_i-1)}{d_i! f_3(\lambda)} \prod_{i=2l}^{r+l} \frac{\lambda^{d_i} d_i}{d_i! f_3(\lambda)}\right) \tag{12}$$

Explanation: The quantities involving λ give the probability that the relevant degrees are $d_i, i \in [r+l-1]$. The probability that an edge exists between vertices u and v of degrees d_u and d_v, given the existence of other edges in T, is at most $\frac{d_u' d_v'}{2m-2(l+r)+3}$ where $d_u' = d_u$ less the number of edges already assumed to be incident with u. Hence, given the degree sequence, the probability that T exists is at most

$$\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1} \prod_{i=1}^{l} d_i! \prod_{i=l+1}^{2l-1} d_i(d_i-1) \prod_{i=2l}^{r+l} d_i.$$

(We dropped the $+3$ in $2m-2(l+r)+3$).
Here the first product corresponds to $EVEN(T)$, the second product corresponds to $ODD(T)$ and the final product corresponds to neighbours of T (not in T).
We will implicitly use the fact that if c is sufficiently large, then so is λ.
We now simplify the expression (12) obtained for the probability to

$$\begin{aligned}
&O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1} \times \\
&\frac{\lambda^{2r+2l-3}}{f_3(\lambda)^{r+l-1}} \sum_{\sum_{i=1}^l d_i = r+l-1} \left(\sum_{d_i \geq 3,\, i \in [r+l-1]} \prod_{i=l+1}^{2l-1} \frac{\lambda^{d_i-2}}{(d_i-2)!} \prod_{i=2l}^{r+l} \frac{\lambda^{d_i-1}}{(d_i-1)!}\right) \\
&\leq O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1} \times \\
&\frac{\lambda^{2r+2l-3}}{f_3(\lambda)^{r+l-1}} \sum_{\substack{\sum_{i=1}^l d_i = r+l-1 \\ d_i \geq 3, i \in [l]}} \left(\left(\prod_{i=l+1}^{2l-1} \sum_{d_i \geq 3} \frac{\lambda^{d_i-2}}{(d_i-2)!}\right)\left(\prod_{i=2l}^{r+l} \sum_{d_i \geq 3} \frac{\lambda^{d_i-1}}{(d_i-1)!}\right)\right)
\end{aligned}$$

$$\leq O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\frac{\lambda^{2r+2l-3}}{f_3(\lambda)^{r+l-1}}\binom{r}{l}f_1(\lambda)^{l-1}\;\; f_2(\lambda)^{r-l+1} \tag{13}$$

$$\leq O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\frac{\lambda^{2r+2l-3}}{f_3(\lambda)^{r+l-1}}\left(\frac{er}{l}\right)^{l}f_1(\lambda)^{l-1}\;\; f_2(\lambda)^{r-l+1}$$

$$= O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\left(\frac{er}{l}\right)^{l}\frac{(2c\lambda)^r}{\lambda}\left(\frac{2c\lambda f_1(\lambda)}{f_2(\lambda)^2}\right)^{l-1} \tag{14}$$

using $\dfrac{\lambda f_2(\lambda)}{f_3(\lambda)} = 2c$

$$\leq O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}(2c\lambda)^r\left(\frac{4(1-\beta)ec^2\lambda f_1(\lambda)}{f_2(\lambda)^2}\right)^{l-1} \quad \text{using } r/l \leq 2(1-\beta)c \tag{15}$$

$$\leq O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}(2c\lambda)^r\left(\frac{4(1-\beta^2)ec^2\lambda}{f_2(\lambda)}\right)^{l-1} \quad \text{using } \frac{f_1(\lambda)}{f_2(\lambda)} < 1+\beta$$

$$\leq O(\sqrt{n})\left(\frac{1}{2cn}\right)^{l+r-1}e^{3(l+r)^2/2cn}(2c\lambda)^r\left(\frac{4(1-\beta^2)ec^2\lambda}{f_2(\lambda)}\right)^{l-1} \quad \text{using } m = cn$$

$$= O(\sqrt{n})\left(\frac{1}{n}\right)^{l+r-1}e^{o(l)}\;\lambda^r\left(\frac{2(1-\beta^2)ec\lambda}{f_2(\lambda)}\right)^{l-1} \tag{16}$$

since $r = O(l)$.
We now count the number of such configurations. We begin by choosing *EVEN(T)* and the root vertex of the tree in at most $n\binom{n}{l-1}$ ways. We make the following observation about T. The contraction of the lost edges of the tree yields a unique tree on the l even vertices. We note, by Cayley's formula, that the number of trees that could be formed using l vertices is l^{l-2}. Reversing this contraction, we now choose the sequence of l vertices from *ODD(T)*, that connect up vertices in *EVEN(T)* in $(n-l)(n-l-1)...(n-2l+1) = (n-l)_l$ ways. We pick the remaining $r-l$ vertices from the remaining $n-2l$ vertices in $\binom{n-2l}{r-l}$ ways. These $r-l$ vertices can connect to *EVEN(T)* in at most l^{r-l} ways. Hence, the total number of choices for T is at most

$$\binom{n}{l}l^{l-2}(n-l)_l\binom{n-2l}{r-l}l^{r-l} \leq n^{r+l}e^r\left(\frac{l}{r-l}\right)^{r-l}. \tag{17}$$

Combining the bounds for probability and choices of T, we get an upper bound of

$$n^{r+l}e^r\left(\frac{l}{r-l}\right)^{r-l}O(\sqrt{n})\left(\frac{1}{n}\right)^{l+r-1}\;\lambda^r\left(\frac{2ec\lambda}{f_2(\lambda)}\right)^{l-1} \leq O(n^{3/2})\cdot\left(\frac{e\lambda l}{r-l}\right)^{r-l}\left(\frac{2e^2c\lambda^2}{f_2(\lambda)}\right)^{l-1} \tag{18}$$

The expression $\left(\frac{e\lambda l}{x}\right)^x$ is maximized at $x = \lambda l$. Our assumptions imply that $r \leq 2(1-\beta)cl < \lambda l$. Hence, we have the bound

$$O(n^{3/2})\cdot\left(\frac{e\lambda l}{2(1-\beta)cl}\right)^{2(1-\beta)cl}\left(\frac{2e^2c\lambda^2}{f_2(\lambda)}\right)^{l}$$

$$\leq O(n^{3/2})\cdot\left(\left(\frac{e}{1-\beta}\right)^{2(1-\beta)c}\cdot\frac{2e^2c^3}{f_2(\lambda)}\right)^{l} \quad \text{using } \lambda < 2c$$

$$\leq O(n^{3/2})\cdot e^{-\beta^2cl/2} \tag{19}$$

using

$$f_2(\lambda) > \frac{2e^2c^3e^{2(1-\beta)c}}{(1-\beta)^{2(1-\beta)c}}$$

for c sufficiently large. Here we use Lemma 5.1 and $(e/(1-\beta))^{2-2\beta} \le e^{2-\beta^2+O(\beta^3)}$, so that the inequality is true for small positive β.
We sum $O(n^{3/2}) \cdot e^{-\beta^2 cl/2}$ over all r and l with $L_0 \le l \le n^{0.6}$ and $l \le r \le (1-\beta)cl$ and we get the probability to be at most

$$O(n^{7/2})e^{-\beta^2 cL_0/2} = o(1) \tag{20}$$

for c sufficiently large.
We now consider the probability of the existence of a T having $L_0 \le l \le n^{0.6}$ and $r \ge 2(1+\beta)cl$. Note that we can assume $r \le l\Delta \le l \log n$ here.
The bound (14) remains valid. Replacing r by $r+1$ multiplies this by a factor $O(cn^{-1}e^{l/r})$ and so for this bound we can just assume that $r = 2(1+\beta)cl$. This changes the $1-\beta$ in (15) to $1+\beta$ and we replace (16) by

$$O(\sqrt{n}) \left(\frac{1}{n}\right)^{l+r-1} e^{o(l)}\ \lambda^r \left(\frac{2ec(1+\beta)^2\lambda}{f_2(\lambda)}\right)^{l-1}.$$

We re-use (17) and replace (18) by

$$\begin{aligned} & O(n^{3/2}) \cdot \left(\frac{e\lambda l}{r-l}\right)^{r-l} \left(\frac{2e^{2+o(1)}(1+\beta)^2c\lambda^2}{f_2(\lambda)}\right)^{l-1}. \qquad (21) \\ & \le O(n^{3/2}) \cdot \left(\left(\frac{e}{1+\beta-1/2c}\right)^{2(1+\beta)c-1} \cdot \frac{2e^{2+o(1)}(1+\beta)^2c^3}{f_2(\lambda)}\right)^{l} \qquad \text{using } \lambda < c \\ & \le O(n^{3/2}) \cdot e^{-\beta^2 cl/2} \end{aligned}$$

using

$$f_2(\lambda) > \frac{2e^{2+o(1)}(1+\beta)^2c^3e^{(2+2\beta+?)c}}{(2(1+\beta)-1/c)^{2(1+\beta)c-1}}$$

for c sufficiently large. Here we again use Lemma 5.1 and $(e/(1+\beta))^{2+2\beta} \le e^{2-\beta^2+O(\beta^3)}$, so that the inequality is true for small positive β.
We sum $O(n^{3/2}) \cdot e^{-\beta^2 cl/2}$ over all r and l with $L_0 \le l \le n^{0.6}$ and $r \ge 2(1+\beta)cl$ and we get the probability to be at most

$$O(n^{7/2})e^{-\beta^2 cL_0/2} = o(1) \tag{22}$$

for c sufficiently large.
We next consider the case where $l \ge n^{\varepsilon}$ and $r \le 2(1-\beta)c$ and we can use at most n^{ε} tardy edges. We will use (17), which is still a valid upper bound and only modify (12). Let

$$b(d, d', d'', \alpha) = \binom{d}{d', d'', d-d'-d''} \frac{((1-\alpha)m)_{d'}(\alpha m)_{d-d'}}{(m)_d} =$$

$$\binom{d}{d',d'',d-d'-d''}(1-\alpha)^{d'}\alpha^{d-d'}\left(1+O\left(\frac{\log^2 n}{n}\right)\right)$$

for $d \le \Delta \le \log n$.

We modify (12). In the following, t will be the number of tardy edges used. We relax our requirements and allow these edges to occur anywhere in T. In the expression (23) below, d_i is the degree of vertex i and for $i \in [l]$, $d_i^* = d_i' + d_i''$ is its degree in T. For $i \in [l]$ the punctual degree is d_i' and its tardy degree in T is d_i''. The factor $b(d_i, d_i', d_i'', \alpha)$ is justified as follows: Given that i has degree d_i, we count the partition of its incident edges into d_i' punctual edges and d_i'' tardy T-edges and $d_i - d_i^*$ tardy non-T edges. The factor $\frac{((1-\alpha)m)_{d_i'}(\alpha m)_{d_i-d_i'}}{(m)_d}$ is the probability that the relevant edges are punctual or tardy. The final factor of $(1-\alpha)$ in (23) accounts for the $r-l+1$ edges connecting $EVEN(T)$ to $N(EVEN(T)) \setminus ODD(T)$ being punctual.

$$O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\times$$
$$\sum_{d_i\ge 3}\sum_{t=0}^{n^\varepsilon}\sum_{\substack{d_i^*=d_i'+d_i''\le d_i,\, i\in[l]\\ \sum_{i=1}^l d_i'=r+l-1-t\\ \sum_{i=1}^l d_i''=t}}\left(\prod_{i=1}^{l}\frac{\lambda^{d_i}d_i^*!}{d_i!f_3(\lambda)}b(d_i,d_i',d_i'',\alpha)\prod_{i=l+1}^{2l-1}\frac{\lambda^{d_i}d_i(d_i-1)}{d_i!f_3(\lambda)}\prod_{i=2l}^{r+l}\frac{\lambda^{d_i}d_i}{d_i!f_3(\lambda)}(1-\alpha)\right) \tag{23}$$
$$=O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\times\frac{\lambda^{2r+2l-3}(1-\alpha)^{2r}}{f_3(\lambda)^{r+l-1}}\times$$
$$\sum_{d_i\ge 3}\sum_{t=0}^{n^\varepsilon}\sum_{\substack{d_i^*=d_i'+d_i''\le d_i,\, i\in[l]\\ \sum_{i=1}^l d_i'=r+l-1-t\\ \sum_{i=1}^l d_i''=t}}\prod_{i=1}^{l}\frac{(\lambda\alpha)^{d_i-d_i^*}\alpha^{d_i''}d_i^*!}{d_i'!d_i''!(d_i-d_i^*)!}\left(\left(\prod_{i=l+1}^{2l-1}\sum_{d_i\ge 3}\frac{\lambda^{d_i-2}}{(d_i-2)!}\right)\left(\prod_{i=2l}^{r+l}\sum_{d_i\ge 3}\frac{\lambda^{d_i-1}}{(d_i-1)!}\right)\right)$$
$$\le O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\times\frac{\lambda^{2r+2l-3}(1-\alpha)^{2r}}{f_3(\lambda)^{r+l-1}}\times$$
$$\sum_{t=0}^{n^\varepsilon}\sum_{\substack{\sum_{i=1}^l d_i^*=r+l-1\\ \sum_{i=1}^l d_i''=t\\ d_i''\le d_i^*, i\in[l]}}\prod_{i=1}^{l}\binom{d_i^*}{d_i''}\alpha^{d_i''}\left(\sum_{k\ge 0}\frac{(\lambda\alpha)^k}{k!}\right)^l\left(\prod_{i=l+1}^{2l-1}\sum_{d_i\ge 3}\frac{\lambda^{d_i-2}}{(d_i-2)!}\right)\left(\prod_{i=2l}^{r+l}\sum_{d_i\ge 3}\frac{\lambda^{d_i-1}}{(d_i-1)!}\right)$$
$$\le O(\sqrt{n})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\times\frac{\lambda^{2r+2l-3}(1-\alpha)^{2r}}{f_3(\lambda)^{r+l-1}}\times$$
$$\sum_{t=0}^{n^\varepsilon}\sum_{\substack{\sum_{i=1}^l d_i^*=r+l-1\\ \sum_{i=1}^l d_i''=t\\ d_i''\le d_i^*, i\in[l]}}\prod_{i=1}^{l}\left(\frac{\alpha e d_i^*}{d_i''}\right)^{d_i''}e^{\lambda\alpha l}f_1(\lambda)^{l-1}f_2(\lambda)^{r-l} \tag{24}$$
$$\le O(n^{1/2+\varepsilon})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\frac{\lambda^{2r+2l-3}(1-\alpha)^{2r}}{f_3(\lambda)^{r+l-1}}\times$$

$$2^l\binom{r-1}{l-1}\binom{t+l-1}{l-1}e^{\alpha(r+l-1)}e^{\lambda\alpha l}f_1(\lambda)^{l-1}f_2(\lambda)^{r-l} \qquad \text{(See below for an explanation).} \tag{25}$$

$$\leq O(n^{1/2+\varepsilon})\left(\frac{1}{2m-2(l+r)}\right)^{l+r-1}\times\frac{\lambda^{2r+2l-3}(1-\alpha)^{2r}}{f_3(\lambda)^{r+l-1}}\times\binom{r}{l}8^l e^{\alpha(r+l-1)}e^{\lambda\alpha l}f_1(\lambda)^{l-1}f_2(\lambda)^{r-l}. \tag{26}$$

Explanation for (24)$\longrightarrow$(25)**:** We bound $(\alpha e d_i^*/d_i'')^{d_i''}$ by $e^{\alpha d_i^*}$ when $d_i^*\geq 2$ and by one when $d_i^*=1$. If $L=\{i: d_i^*=1\}$ then $\prod_{i\notin L}(\alpha e d_i^*/d_i'')^{d_i''}\leq e^{\alpha(r+l-1)}$. There are at most 2^l choices for L and after this there are at most $\binom{r-1}{l-1}$ choices for the $d_i^*\geq 2$.
Observe now that the expression in (26) is precisely

$$8^l e^{\alpha(r+l-1)}e^{\lambda\alpha l}(1-\alpha)^{2r}\leq 8^l e^{\alpha((\lambda+1)l-r)}$$

times the expression in (13). It follows that the probability bound (19) can be replaced by

$$O(n^{3/2+\varepsilon})\cdot e^{-\beta^2 cl/2}\cdot 8^l e^{\alpha((\lambda+1)l-r)}\leq O(n^{3/2+\varepsilon})\cdot e^{-\beta^2 cl/3}$$

if we take $\alpha=\beta^2/10$.
We sum this over l,r to get the required conclusion.

The case $r\geq 2(1+\beta)cl$ for $l\geq n^{\varepsilon}$, using only punctual edges follows a fortiori from the case where we can use any edge in T, punctual or tardy.
We finally consider the case where T is not a tree. When this happens, it will be because of a unique (Lemma 5.2) edge introduced in Case 1. We can handle this by multiplying our final estimates by $O(i_0^2 n^{-1}\log^2 n)$. The factor $O(i_0^2)$ accounts for choosing a pair of vertices in T in Case 1 and $O(n^{-1}\log^2 n)$ bounds the probability of the existence of this edge, given previous edges.
Part (c) follows from (b).
(d) If we consider the growth of the sub-tree emanating from x then we can argue that it grows as fast as described in (a) and (b). We just have to deal with the edges pointing into the part of T that has already been constructed. We can argue that (23) with $\alpha=o(1)$ and $d_i^*=d_i$ gives a valid upper bound here. This is because the chances of choosing an endpoint in T is $o(1)$ at each point, as opposed to an edge being tardy. If $x\notin W_1$ then the descendants D_i of x at levels k_0+i grow at a rate of at least two (i.e. $|D_{i+1}|\geq 2|D_i|$) for $O(\log\log n)$ steps until $|D_i|\gg\log n$ and after this will grow at a rate of at least $2c(1-\beta)$. In which case the leaves of T_x, the sub-tree of T rooted at x, will constitute a fraction $1-O(1/c)$ of the vertices of T_x. The result now follows from Lemma 5.2(c).
If $x\in W_1$ then $|D_i|$ grows at a rate of at most $2c(1+\beta)$ once it has reached size $\log n$. □

Remark 6.1. *It follows from this lemma that only $O(n^{1/2+O(\varepsilon)})$ tardy edges are needed to build all of the instances of A_{k_0} needed by* EXTEND-ROTATE*. If one looks at Section 4.3.1 of [8] one sees, in conjunction with equation (1) of that paper that the total running time of Step 3b of this paper is $O(n^{.995+o(1)})$ and so we can use this as a bound on the number of punctual edges examined by Step 3b. We can drastically reduce this in the same way we did for building the trees in* EXTEND-ROTATE*, but since we are only claiming our result for c sufficiently large and $\varepsilon\ll .005$, this is not necessary, since there will w.h.p. be $\Omega(n^{1-2\varepsilon})$ tardy $R_0:\Lambda_0$ edges, see Lemma 6.5 below. In other words, almost all of the tardy $R_0:\Lambda_0$ edges are* **not** *used for tree building. They can therefore be called on to close cycles in* **Case b** *of Step ER3 of Algorithm* EXTEND-ROTATE.

The above lemma shows that A_k can be relied on to get large. Unfortunately, we need to do some more analysis because we do not have full independence, having run 2GREEDY. Normally, one would only have to show that $END(a)$ is large for all relevant vertices a and this would be enough to show the existence w.h.p. of an edge joining a to $b \in END(a)$ for some a, b. We will have to restrict our attention to the case where $a \in R_0$ and $b \in \Lambda_0$, see Remark 4.1. So first of all we will show that w.h.p. there are many $a \in R_0$, see Lemma 6.8. For this we show that every path we come across contains many consecutive triples $u, v, w \in R_0$. In which case, an inserted edge (x, v) produces a path with an endpoint in R_0. We also need to show that w.h.p. there are many $b \in \Lambda_0$, see Lemma 6.7. We will also need to show that there are many edges that can be (a, b), see Lemma 6.5.

For the Lemma 6.3 below we need some results from [10]. Let $\mathbf{u} = \mathbf{u}(t)$ denote $(y(t), z(t), \mu(t))$ and let $\hat{\mathbf{u}} = \hat{\mathbf{u}}(t)$ denote $(\hat{y}(t), \hat{z}(t), \hat{\mu}(t))$ where $y(t)$ etc. denotes the value of $y = |Y|, z = |Z|, \mu = |E(\Gamma(t)|$ at time t and $\hat{y}(t)$ etc. denotes the deterministic value for the solution to the associated set of differential equations, summarised in equation (152) of that paper:

$$\frac{d\hat{y}}{dt} = \hat{A} + \hat{B} - \hat{C} - 1; \quad \frac{d\hat{z}}{dt} = 2\hat{C} - 2\hat{A} - 2\hat{B}; \quad \frac{d\hat{\mu}}{dt} = -1 - \hat{D}. \tag{27}$$

where

$$\hat{A} = \frac{\hat{y}\hat{z}\hat{\lambda}^5 f_0(\hat{\lambda})}{8\hat{\mu}^2 f_2(\hat{\lambda}) f_3(\hat{\lambda})}, \quad \hat{B} = \frac{\hat{z}^2\hat{\lambda}^4 f_0(\hat{\lambda})}{4\hat{\mu}^2 f_2(\hat{\lambda})^2}, \quad \hat{C} = \frac{\hat{y}\hat{\lambda} f_2(\hat{\lambda})}{2\hat{\mu} f_3(\hat{\lambda})}, \quad \hat{D} = \frac{\hat{z}\hat{\lambda}^2 f_0(\hat{\lambda})}{2\hat{\mu} f_2(\hat{\lambda})}. \tag{28}$$

and

$$\frac{\hat{y}\hat{\lambda} f_2(\hat{\lambda})}{f_3(\hat{\lambda})} + \frac{\hat{z}\hat{\lambda} f_1(\hat{\lambda})}{f_2(\hat{\lambda})} = 2\hat{\mu}.$$

Lemma 7.1 of [10] proves that $\mathbf{u}(t)$ and $\hat{\mathbf{u}}(t)$ are close w.h.p.:

Lemma 6.2.
$$||\mathbf{u}(t) - \hat{\mathbf{u}}(t)||_1 \leq n^{8/9}, \qquad \textit{for } 1 \leq t \leq \min\left\{T_0, \hat{T}_0\right\} \textit{ w.h.p.}.$$

Here T_0 is a stopping time and $\hat{T}_0$ is a deterministic time such that w.h.p. Step 3 of 2GREEDY begins before $\min\left\{T_0, \hat{T}_0\right\}$.

Note that $\varepsilon \ll 1/9$. Let
$$i_0 = n^{3/4-\varepsilon} \text{ and } \rho = n^{1/4}.$$

Equation (163) of [10] states that w.h.p.

$$|\theta_\xi(\hat{\mathbf{u}}(t)) - \Delta_\xi| = O\left(\rho^{-1}\log^2 n + \frac{||\mathbf{u}(t) - \hat{\mathbf{u}}(t)||_1}{n}\right) \text{ for } \xi = a, b, c, 2. \tag{29}$$

Here
$$\theta_a = 0,\ \theta_b = \hat{A},\ \theta_c = \hat{A} + \hat{B} \text{ and } \theta_2 = 1 - \theta_a - \theta_b - \theta_c$$

and Δ_ξ is the proportion of steps of 2GREEDY in $[t, t + \rho]$ that are Step 1ξ, $\xi = a, b$ or Step 2, if $\xi = 2$. Now if $\hat{z} = o(n)$ $\hat{\mu} = \Omega(n)$ and $\hat{\lambda} = \Omega(1)$ then we have from (28) that $\hat{A}, \hat{B} = O(\hat{z}/n)$ and that

$$\theta_b = O(\hat{z}/n), \theta_c = O(\hat{z}/n), \theta_2 = 1 - o(1).$$

Then from (27) we see that $\hat{z}$ grows at the rate $2 - o(1)$ per time step, so long as $t = o(n)$ and hence $\hat{z} = o(n)$. It is shown in [10] that if $c \geq 10$ then w.h.p. $\hat{\lambda} = \Omega(1)$ up until the (random) time when Step 3 of 2GREEDY begins. See equation (190) of that paper. Furthermore, it follows from Lemma 6.2 that w.h.p.

X1 If $t = \gamma n^{1-\varepsilon}$ for some constant γ then w.h.p. $z(t) \sim 2t$.

X2 If $t = \gamma n^{1-\varepsilon}$ for some constant γ then w.h.p. there will be $O(n^{1-2\varepsilon})$ instances of Step 1 in $[0, t]$.

X3 $\lambda = \Omega(1)$ up until the start of Step 3.

Lemma 6.3. *W.h.p., all the paths in Steps 1 and 2 of* EXTEND-ROTATE *contain at least* $n_0 = \Omega(n^{1-4\varepsilon}/\log n)$ *pairs of consecutive edges* $(u, v), (v, w)$ *such that* $u, v, w \in R_0$.

Proof First consider the steps in the range $[0, i_0\rho/8]$. It follows from **X1** that at the end of this period, there will w.h.p. be at least $i_0\rho/5$ vertices in Z. Now consider the range $[i_0\rho/8, i_0\rho/4]$. We know that w.h.p. $\theta_2 = 1 - o(1)$ throughout this range. Consider the edge (v, w) of Step 2 at some time in $[i_0\rho/8, i_0\rho/4]$. The probability that $w \in Z$ is $\Omega(n^{-\varepsilon})$ and the probability it has a punctual Z-witness is $1 - \alpha - o(1)$. This holds regardless of the previous history, assuming **X1,X2**, and so Z dominates a binomial with mean $\Omega(n^{1-\varepsilon})$ and we can use Chernoff bounds to bound this from below w.h.p.

On the other hand, the probability that $w \in Z$ is $O(n^{-\varepsilon}\Delta) = O(n^{-\varepsilon} \log n)$. This implies that the number of times we create a component of M containing more than two vertices is $O(n^{1-2\varepsilon} \log n)$. Thus w.h.p. almost all components of M at the end of the period $[0, i_0\rho/4]$ consist of isolated edges. Let us assume then that there are at least $A_1 n^{1-\varepsilon}$ such edges where in the following $A_1, A_2, \ldots,$ are positive constants. Let S_1 denote this set of components.

Now consider the steps in the range $[i_0\rho/4, i_0\rho/2]$, which again are almost all Step 2 and consider the edge (v, w) of Step 2. We have $w \in V(S_1)$ with probability at least $A_2 n^{-\varepsilon}$. This is because w.h.p. the total degree of $V(S_1)$ will be $\Omega(n^{1-\varepsilon})$ and the total degree of G is at most $2cn$. The vertex w is early by definition. Also the Z-witness of w will be punctual with probability at least $1 - \alpha - o(1)$. We next observe that with probability at least $\left(1 - \Omega\left(\frac{\Delta}{n}\right)\right)^{n^{1-\varepsilon}/4} = 1 - o(1)$, this component will not be absorbed into a larger component in $[i_0\rho/4, i_0\rho/2]$. Thus, in expectation, at time $i_0\rho/2$ there is a set S_2 of $A_3 n^{1-2\varepsilon}$ components of M consisting of a path of length two with its middle vertex in R_0. A second moment calculation will show concentration around the mean, for $|S_2|$.

We can repeat this argument for the periods $[i_0\rho/2, 3i_0\rho/4]$,$[3i_0\rho/4, i_0\rho]$ to argue that by time $n^{1-\varepsilon}$, M will contain a set S_3 of at least $A_4 n^{1-4\varepsilon}$ components consisting of paths of length four in which the internal vertices are all in R_0.

We can argue that w.h.p. at least half of the components in S_3 will have both end vertices of degree at most $3c$. Indeed the number of edges incident with vertices of degree more than $3c$ is relatively small. The expected number of such edges is asymptotically equal to

$$\sum_{k \geq 3c} \frac{k\lambda^k}{k! f_3(\lambda)} \leq \frac{3c\lambda^{3c}}{(3c)! f_3(\lambda)} \left(1 + \frac{\lambda}{3c+1} + \cdots\right) \leq \frac{6c(3c)^{3c}}{(3c)! f_3(\lambda)} \leq \varepsilon_c = (e/3)^{3c}.$$

The number of such edges is concentrated around its mean. If we assume degrees are independent and less than $\log n$ then we can use Hoeffding's Theorem and then correct by a factor $O(n^{1/2})$ to condition on the total degree. Given this, we see that w.h.p. at least a $2(1 - \varepsilon_c)^2/3$ fraction of the components of S_3 will be created in two executions of Step 2 with the degree v less than $3c$.

Observe now that with probability at least $\left(1 - \frac{6c}{\Omega(n)}\right)^{2cn} = \Omega(1)$ a component $C \in S_3$ will survive as a component of M until the execution of Step 3. The $\Omega(n)$ in the denominator comes from the fact that w.h.p. Step 3 begins with $|Z| = \Omega(n)$. Let S_4 denote this set of components and note that w.h.p. there will be at least $A_5 n^{1-4\varepsilon}$ components in S_4.

Step 3 of 2GREEDY adds a matching M^* that is disjoint from the edges in the contraction of S_4 to a matching. The matching M^* is independent of S_4. This implies that w.h.p. any cycle (or possibly path) of the union of M and M^* of length $\ell \geq n^{8\varepsilon}$, contains at least $A_5 \ell n^{-4\varepsilon}$ members of S_4. Here we are using concentration of the hypergeometric distribution i.e. sampling without replacement.
In EXTEND-ROTATE we start with a path of length $\ell = \Omega(n/\log n)$ and w.h.p. every path is generated by deleting at most $O(\log^2 n)$ edges. This completes the proof of the lemma. □

6.2 Batches

Let $\Gamma(t)$ denote the graph Γ after t steps of 2GREEDY. Suppose that $t_1 < t_2 \leq n^{1-\varepsilon}$ and that 2GREEDY applies Step 2 at times t_1, t_2 and Step 1 at times $t_1 < t < t_2$. We consider the set of edges and vertices removed from time t_1 to time t_2, i.e. the graph $\Gamma(t_1) \setminus \Gamma(t_2)$ and call it a *batch*. Note that batches are connected subgraphs since each edge/vertex removed is incident to some edge that is previously removed.
We also claim that each batch w.h.p. is constructed within $O(\log^2 n)$ steps and contains $O(\log^3 n)$ vertices. This follows from [10] as we now explain. Let $\zeta = y_1 + 2y_2 + z_1$ and let $\mathbf{v}$ be the state vector $(y_0, y_1, \ldots, \mu)$. Equations (67), (68), (69) of [10] show that

$$\mathbb{E}[\zeta' - \zeta \mid |\mathbf{v}|] = -(1-Q) - o(1)$$

where

$$Q = Q(\mathbf{v}) = \frac{yz}{4\mu^2} \frac{\lambda^3}{f_3(\lambda)} \frac{\lambda^2 f_0(\lambda)}{f_2(\lambda)} + \frac{z^2}{4\mu^2} \frac{\lambda^4 f_0(\lambda)}{f_2(\lambda)^2}.$$

Lemma 6.2 of [10] shows that $1 - Q = -\Omega(1)$ if $\lambda = \Omega(1)$, and **X3** is our justification for assuming this. Thus the expected change in ζ is $-\Omega(1)$ when $\zeta > 0$. We carry out Step 2 iff $\zeta = 0$. Now ζ can change by at most $O(\Delta) = O(\log n)$ and has a negative drift whenever it is positive. This implies that it must return to zero within $O(\log^2 n)$ steps. Another $\Delta \leq \log n$ factor will allow at most $\log n$ edges to be removed in one step. By making the hidden constant sufficiently large, we can replace w.h.p. by with probability $1 - O(n^{-10})$.

Lemma 6.4.

(a) W.h.p. there are at most $n^{1-4\varepsilon}$ vertices $v \in G$ that are within distance $\ell_0 = 2\log\log n$ of 6 distinct batches.

(b) W.h.p. no vertex has degree more than 4 in a single batch.

Proof
(a) We bound the probability of vertex v being within distance ℓ_0 of s batches by

$$\rho_s = \binom{n^{1-\varepsilon}}{s} \prod_{i=1}^{s} \mathbb{P}(dist(v, B_i) \leq \ell_0 \mid dist(v, B_j) \leq \ell_0,\, 1 \leq j < i).$$

Explanation: Here $\binom{n^{1-\varepsilon}}{s}$ is the number of choices for the start times of the batches $B_1, B_2, \ldots, B_s$.
We claim that for each i, v,

$$\mathbb{P}(dist(v, B_i) \leq \ell_0 \mid dist(v, B_j) \leq \ell_0,\, 1 \leq j < i) = O\left(\frac{\log^{2+\ell_0} n}{n}\right). \tag{30}$$

This gives

$$\rho_s \leq \exp\left\{-(K-2-o(1))(\log\log n)^2 s\right\} \leq n^{-5\varepsilon},$$

assuming $K \geq 13$ in (1). This implies that the expected number of vertices within distance ℓ_0 of 6 batches is less than $n^{1-5\varepsilon}$. The result now follows from the Markov inequality.

Proof of (30): Suppose that B_i is constructed at time t_i. It is a subgraph of $\Gamma(t_i)$ and depends only on this graph. We argue that

$$\mathbb{P}(\exists w \in B_i : dist(v,w) \leq \ell_0 \mid dist(v,B_j) \leq \ell_0,\ 1 \leq j < i) \leq O(n^{-10}) + O\left(\frac{\log^{2+\ell_0} n}{n}\right). \tag{31}$$

Explanation: The $O(n^{-10})$ term is the probability that the batch B_i is large. The term $O\left(\frac{i\log^{2+\ell_0} n}{n}\right)$ in (31) arises as follows. We can assume that $|N_{\ell_0}(v)| \leq \Delta^{\ell_0} \leq \log^{\ell_0} n$, where $N_{\ell_0}(v)$ is the set of vertices within distance ℓ_0 of v. Suppose as in [3] we expose the graph Γ at the same time that we run 2GREEDY. For us it is convenient to work within the configuration model of Bollobás [5]. Assume that we have exposed $N_{\ell_0}(v)$. At the start of the construction of a batch we choose a random edge (x,y) of the current graph. The probability this edge lies in $N_{\ell_0}(v)$ is $O(\log^{\ell_0} n/n)$. In the middle of the construction of a batch, one endpoint of an edge is known and the the other endpoint is chosen randomly from the set of configuration points associated with $\Gamma(t)$. The probability this new endpoint lies in $N_{\ell_0}(v)$ is also $O(\log^{\ell_0} n/n)$ and there are only $O(\log^2 n)$ steps in the creation of a batch.

(b) The probability that vertex v appears $k+3$ times in a fixed batch can be bounded above by $\binom{O(\log^2 n)}{k}\left(O\left(\frac{\Delta}{n}\right)\right)^k = O\left(\frac{\log^{k+3} n}{n^k}\right)$. Indeed, if v has degree at least 3 at any time, then the probability its degree in the current batch increases in any step is $O\left(\frac{\Delta}{n}\right)$. □

We now argue that there will be a sufficient number of tardy $R_0 : \Lambda_0$ edges.

Lemma 6.5. *W.h.p. there will be $\Omega(n^{1-2\varepsilon})$ tardy $R_0 : \Lambda_0$ edges.*

Proof We first consider the set F_1 of tardy edges $e = (u,v)$ such that (i) u appears at least twice in the first $n^{1-\varepsilon}/10$ edges and in at least 30 other punctual edges and (ii) vertex v has punctual degree at least 30 and does not appear in the first $n^{1-\varepsilon/2}$ edges in σ. To bound the number of choices Z_1 for u we can consider the number of times the vertex w in a step of 2GREEDY is a repeat. This has expectation $\Omega(n^{1-2\varepsilon})$ since in the interval $[n^{1-\varepsilon}/20, n^{1-\varepsilon}/10]$ the chances of a repeat are $\Omega(n^{1-\varepsilon}/n)$. Furthermore, there is always an $\Omega(1)$ chance that the vertex w chosen is in 40 edges $e_i, i \in [n^{1-\varepsilon}, (1-\alpha)n]$. It is however conceivable that u is adjacent to more than 10 vertices whose degree in Γ is reduced in the first $n^{1-\varepsilon}$ steps. In which case u's punctual degree could be less than 30. However, the expected number of such u is $O(n \times (n^{-\varepsilon}\log^2 n)^{10})$. (Each loss requires that the chosen w is within distance two of u). Thus the Markov inequality implies that w.h.p. the number of such u is $o(n^{1-2\varepsilon})$. A change of choice for w only changes Z_1 by one or less. Thus an application of a martingale tail inequality shows that $Z_1 = \Omega(n^{1-2\varepsilon})$ q.s. Here the probability space is a random permutation of W in the configuration model that we get after conditioning on the degree sequence and permuting the edges. We can use a result of McDiarmid [16] or Lemma 11 of Frieze and Pittel [12] to prove concentration, as interchanging a pair in the permutation only affects Z_1 by at most one. Given these choices, the expected size of $|F_1|$ is $\Omega(mn^{-2\varepsilon})$ and a similar martingale argument shows that q.s. we have $|F_1| = \Omega(mn^{-2\varepsilon})$.

Suppose that u satisfies (i). Lost edges are all part of batches. Thus it loses at most 24 edges (Lemma 6.4(a),(b)) before the second edge incident with u is chosen and then u will be in R_0. This is because u will be in Z just before this point and will then be placed in R. and it will have at least six choices for a

punctual Z-witness. We use the fact that almost all of the first $n^{1-\varepsilon}$ steps are Step 2 to see that the edges incident with u occurring in the first $n^{1-\varepsilon}/10$ steps will indeed be selected before time $n^{1-\varepsilon}$.
If v satisfies (ii) and loses at most 24 punctual edges because of Step 1 in the first $n^{1-\varepsilon}$ steps then v will be in Λ_0. This is because it will have punctual degree at least six in $\Gamma_{n^{1-\varepsilon}}$. □
We now consider the probability that A_{k_0} contains many vertices that lie in $\Lambda_1 = \Lambda_2 \cup \Lambda_3$ where

$$\Lambda_2 = \left\{v : v \text{ appears in the first } n^{1-\varepsilon/2} \text{ edges in } \sigma\right\}$$
$$\Lambda_3 = \left\{v \notin \Lambda_2 : v \text{ loses 24 edges because of Step 1 in the first } n^{1-\varepsilon} \text{ steps}\right\}.$$

In the proof of Lemma 6.5 we used the fact that

$$\text{if the degree of } v \text{ is at least 30 and } v \notin \Lambda_1 \text{ then } v \in \Lambda_0. \tag{32}$$

Lemma 6.6. *W.h.p., every extension-rotation tree T has $|A_{k_0} \cap \Lambda_1| \leq |A_{k_0}|/30$.*

Proof We first estimate $|A_{k_0} \cap \Lambda_2|$. We go back to (21) and estimate, for fixed r, l,

$$\mathbb{P}(\exists T : |A_{k_0} \cap \Lambda_2| \geq l/100) \leq O(n^{3/2}) \cdot \left(\frac{e\lambda l}{r-l}\right)^{r-l} \left(\frac{4e^{2+o(1)}(1+\beta)^2 c\lambda^2}{f_2(\lambda)}\right)^{l-1} \binom{l}{l/100} \left(\frac{n^{1-\varepsilon/2}\Delta}{m}\right)^{l/100}. \tag{33}$$

Explanation: We have taken the RHS of (21) and multiplied by a bound on the probability that there are at least $l/100$ members of $EVEN(T)$ appearing in the first $n^{1-\varepsilon/2}$ edges of σ. Note the the permutation σ is independent of T. We use (21) and not (18) because we can only assume that $r \leq 2(1+\beta)cl$.
Thus,

$$\mathbb{P}(\exists T : |A_{k_0} \cap \Lambda_2| \geq l/100) \leq$$
$$O(n^{3/2}) \cdot \left(\left(\frac{e\lambda l}{r-l}\right)^{(r-l)/l} \cdot \frac{4e^{2+o(1)}(1+\beta)^2 c\lambda^2}{f_2(\lambda)} \cdot (100e)^{1/100} \cdot \left(\frac{n^{1-\varepsilon/2}\log n}{cn}\right)^{1/100}\right)^l \leq n^{-\varepsilon l/300}. \tag{34}$$

Now we are interested here in the case where $l = n^{1/2+o(1)}$ and so the RHS of (34) is easily strong enough so that we can apply the union bound over r, l.
We next estimate $|A_{k_0} \cap \Lambda_3|$. We replace (33) by

$$\mathbb{P}(\exists T : |A_{k_0} \cap \Lambda_3| \geq l/100) \leq$$
$$O(n^{3/2}) \cdot \left(\frac{e\lambda l}{r-l}\right)^{r-l} \left(\frac{4e^{2+o(1)}(1+\beta)^2 c\lambda^2}{f_2(\lambda)}\right)^{l-1} \binom{l}{l/100} \sum_{k=23l/100}^{24l/100} \binom{n^{1-\varepsilon}}{k} \left(\frac{l\log n}{100m}\right)^k. \tag{35}$$

Explanation; We have taken the RHS of (21) and multiplied by a bound on the probability that there is a set of leaves S of size $l/100$ such that at least $24l/100$ times during the first $n^{1-\varepsilon}$ steps the vertex w, the neighbor of the selected v, is in S. This is computed assuming that we have exposed the edges of T. The count $k \in [23l/100, 24l/100]$ excludes the edges of T that are assumed to be exposed. The sum in (35) is dominated by $k = 23l/100$. Equation (34) is therefore replaced by

$$\mathbb{P}(\exists T : |A_{k_0} \cap \Lambda_3| \geq l/100) \leq$$

$$O(n^{3/2}) \cdot \left(\left(\frac{e\lambda l}{r-l} \right)^{(r-l)/l} \cdot \frac{4e^{2+o(1)}(1+\beta)^2 c\lambda^2}{f_2(\lambda)} \cdot (100e)^{1/100} \cdot \left(\frac{100en^{1-\varepsilon}}{23l} \right)^{23/100} \cdot \left(\frac{l \log n}{100cn} \right)^{23/100} \right)^l \leq n^{-\varepsilon l/20}.$$

□

The next lemma puts a lower bound on $|\Lambda_0 \cap A_{k_0}|$ (see Lemma 6.1).

Lemma 6.7. *W.h.p.* $|\Lambda_0 \cap A_{k_0}| \geq |A_{k_0}|/2$.

Proof Let $k_1 = k_0 - \ell_0$ where $\ell_0 = 2 \log \log n$. and consider $A_{k_1} = \{a_1, a_2, \ldots, a_\rho\}$. Note that

$$\rho \geq \frac{n^{1/2+9\varepsilon}}{(2c(1+\beta))^{1+\ell_0}}.$$

This is because $|A_{k-1}| \geq |A_k|/(2c(1+\beta)$ for $k_1 < k \leq k_0$ and $|A_{k_0}| \geq n^{1/2+9\varepsilon}/(2c(1+\beta))$, see Lemma 6.1. Let s_i be the number of descendents of a_i in A_{k_0} and let s_i' be the number of early descendents of a_i in $A_{k_0} \cap \Lambda_1$.

Let s_i'' be the number of descendents of a_i in A_{k_0} that have degree at most 30. We observe from Lemma 6.1(b) that

$$|A_{k_0}| = \sum_{i=1}^{\rho} s_i \geq \rho(2c(1-\beta))^{k_0-k_1}. \tag{36}$$

Next let $I = \left\{ i \in [\rho] : a_i \notin W_1 \text{ and } s_i \geq (2c(1-\beta))^{k_0-k_1}/4 \right\}$ (where W_1 is from Lemmas 5.2, 6.1) and observe that

$$\sum_{i \notin I} s_i \leq \rho(2c(1-\beta))^{k_0-k_1}/4 + n^{1/2} \log^{4\ell_0+1} n (2c(1+\beta))^{k_0-k_1} \leq |A_{k_0}|/3. \tag{37}$$

It follows from Lemma 6.1(d) that

$$s_i'' \leq s_i/100 \qquad \text{for } i \in I.$$

It follows from Lemma 6.6 that w.h.p.

$$\sum_{i=1}^{\rho} s_i' \leq |A_{k_0}|/30.$$

Now, after using (32), we see that

$$|\Lambda_0 \cap A_{k_0}| \geq \sum_{i \in I} (s_i - s_i'') - \sum_{i=1}^{\rho} s_i' \geq \left(\frac{99}{100} \cdot \frac{2}{3} - \frac{1}{30} \right) |A_{k_0}|.$$

□

We now consider going one iteration further and building A_{k_0+1}.

Lemma 6.8. *W.h.p.* A_{k_0+1} *contains at least* $\Omega(n^{1/2+2\varepsilon})$ *vertices of* R_0*. Furthermore, we can find these* R_0 *vertices by examining* $n^{1-3\varepsilon} \log n$ *tardy* $R_0 : \Lambda_0$ *edges.*

Proof Assume from Lemmas 6.1 and 6.7 that A_{k_0} contains at least $n_1 = \frac{n^{1/2+9\varepsilon}}{4c(1+\beta)}$ vertices in Λ_0. Assume also from Lemma 6.3 that all of the paths corresponding to A_{k_0} have $n_0 = \Omega(n^{1-4\varepsilon}/\log n)$ consecutive triples

$u, v, w \in R_0$. If the middle vertex v is the neighbour of an endpoint, then it yields a new endpoint of A_{k_0+1} in R_0. Then the expected number of rotations leading to an endpoint in R_0 is at least

$$C_1 \times n^{1-3\varepsilon} \log n \times \frac{n^{1/2+9\varepsilon}}{4c(1+\beta)} \times \frac{n^{1-4\varepsilon}}{\log n} \times \frac{1}{|R_0| \times n} = \Omega(n^{1/2+2\varepsilon})$$

for some constant $C_1 > 0$, assuming that $|R_0| = O(n^{1-\varepsilon} \log n)$.
We can claim a q.s. lower bound because almost all of the tardy $R_0 : \Lambda_0$ edges are unconditioned, see remark 6.1. □

7 Finishing the proof

We have argued that we only need to do $\ell_1 = O(\log^2 n)$ extensions w.h.p. The tardy $R_0 : \Lambda_0$ edges are our scarce resource of residual randomness. Remark 6.1 explains that we only need to use an $o(1)$ proportion in building trees up to the k_0th level. We will only use the result of Lemma 6.8 for growing the first extension-rotation tree of each of the $O(\log^2 n)$ path extensions. Lemma 6.8 tells us that we only need to use an $o(1)$ fraction of the available $R_0 : \Lambda_0$ edges for producing many paths that have an R_0 endpoint.
Consider a round of EXTEND-ROTATE where we are trying to extend path P. We start with a path and then we construct a BFS "tree". After the first tree construction of each round, we construct A_{k_0} and create one more level A_{k_0+1}. From Lemma 6.8, we should obtain $\Omega(n^{1/2+2\varepsilon})$ paths with early endpoints. Now we grow trees from each of these paths and try to close them using the set $E_L = \{f_1, f_2, \dots, f_M\}$ of unused tardy $R_0 : \Lambda_0$ edges. We can examine these edges in σ order. The probability that the next edge f_i fails to close a path to a cycle is $p = \Omega(n^{1/2+2\varepsilon} \times n^{1/2+9\varepsilon} \times n^{-2})$. So the probability we fail is at most $\mathbb{P}(Bin(M, p) < \ell_1)$. Now $Mp = \Omega(n^{9\varepsilon}) \gg \ell_1$ and so the Chernoff bounds imply that we succeed w.h.p.

Remark 7.1. *As a final thought, although we have proved that we can find a Hamilton cycle quickly, being very selective in our choice of edges for certain purposes, the breadth first nature of our searches imply that we can proceed in a more natural manner and use all edges available to us. In the worst-case we would have to use the designated ones.*

8 Why not $\varepsilon = O\left(\frac{\log\log n}{\log n}\right)$?

In the proof of Lemma 6.1 we need to choose $\ell_0 = 2 \log\log n$ so that $2^{\ell_0} \gg L_0$ of that lemma. But then in (30) we want $n^\varepsilon \gg \log^{\ell_0} n$. With some work we could replace the bound $\log^{\ell_0} n$ by $O(c)^{\ell_0}$ which would allow us to take $\varepsilon = \frac{K \log\log n}{\log n}$. The catch here is that in this case we would need K to grow with c. This is not satisfactory and so we content ourselves for now with (1).

9 Final Remarks

We have shown that a Hamilton cycle can w.h.p. be found in $O(n^{1+o(1)})$ time. It should be possible to replace $n^{o(1)}$ by $\log^{O(1)} n$ and we have explained the technical difficulty in Section 8. We think that $O(n \log^2 n)$ should be possible. It should also be possible to apply the ideas here to speed up the known algorithms for random regular graphs, or graphs with a fixed degree sequence.

We have seen that our approach is to use extensions and rotations and show that a carefully selected set E_L of (near) random edges can be used to close cycles when necessary. The reader might feel that one could possibly simplify the paper by splitting the edge set into two random sets. The first could be used to do the extensions and rotations and the second could be used to close cycles. This was the approach taken in [10]. Unfortunately, if we remove more than $o(n^{1/2})$ random edges, then we seem to substantially change the distribution of the remaining edges and then we cannot apply the results of [10] to argue directly that 2GREEDY works as claimed. If we only remove $o(n^{1/2})$ edges then we need to build endpoint sets of size $n^{3/4+o(1)}$, explaining the $n^{3/2+o(1)}$ running time of the algorithm in [10]. The approach in this paper has been to "re-use" edges, barely looked at by 2GREEDY, giving us $n^{1-o(1)}$ random edges to close cycles.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi. *The first occurrence of Hamilton cycles in random graphs.*, Annals of Discrete Mathematics 27 (1985), 173–178.

[2] D. Angluin and L.G. Valiant, *Fast probabilistic algorithms for Hamilton circuits and matchings*, Journal of Computer and System Sciences 18 (1979), 155–193.

[3] J. Aronson, A.M. Frieze and B.G. Pittel, *Maximum matchings in sparse random graphs: Karp-Sipser re-visited*, Random Structures and Algorithms 12 (1998), 111–178.

[4] B. Bollobás, *The evolution of sparse graphs*, In: Graph Theory and Combinatorics, Proc. Cambridge Combin. Conf. in honour of Paul Erdős (B. Bollobás, ed.), Academic Press, 1984, pp. 35–57.

[5] B. Bollobás, *A probabilistic proof of an asymptotic formula for the number of labelled regular graphs*, European Journal on Combinatorics 1 (1980), 311–316.

[6] T.I.Fenner and A.M.Frieze, *On the existence of Hamiltonian cycles in a class of random graphs*, Discrete Mathematics 45 (1983), 301–305.

[7] B.Bollobás, T.I.Fenner and A.M.Frieze, *An algorithm for finding Hamilton paths and cycles in random graphs*, Combinatorica 7 (1987), 327–341.

[8] P. Chebolu, A.M. Frieze and P. Melsted, *Finding a Maximum Matching in a Sparse Random Graph in $O(n)$ Expected Time* JACM 57, (2010), 161–172.

[9] A.M. Frieze, *Finding Hamilton cycles in sparse random graphs*, Journal of Combinatorial Theory B 44 (1988), 230–250.

[10] A.M. Frieze, *On a Greedy 2-Matching Algorithm and Hamilton Cycles in Random Graphs with Minimum Degree at Least Three*, to appear.

[11] A.M. Frieze, M.R.Jerrum, M.Molloy, R.Robinson and N.C.Wormald, *Generating and counting Hamilton cycles in random regular graphs*, Journal of Algorithms 21 (1996) 176-198.

[12] A.M. Frieze and B. Pittel, *On a sparse random graph with minimum degree three: Likely Posa's sets are large*, to appear.

[13] R.M. Karp, *Reducibility among combinatorial problems*, Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), Plenum, New York, 1972, pp. 85–103.

[14] R.M. Karp and M. Sipser, *Maximum matchings in sparse random graphs*, Proceedings of the 22nd Annual IEEE Symposium on Foundations of Computing (1981) 364-375.

[15] J. Komlós and E. Szemerédi, *Limit distributions for the existence of Hamilton circuits in a random graph*, Discrete Mathematics 43 (1983), 55–63.

[16] C. McDiarmid, *Concentration for independent permutations*, Combinatorics, Probability and Computing 1 (2002) 163-178.

[17] B. McKay, *Asymptotics for 0-1 matrices with prescribed line sums*, in Enumeration and Design, (Academic Press, 1984) 225-238.

[18] R.W. Robinson and N.C. Wormald, *Almost all cubic graphs are Hamiltonian*, Random Structures and Algorithms 3 (1992) 117-125.

[19] R.W. Robinson and N.C. Wormald, *Almost all regular graphs are Hamiltonian*, Random Structures and Algorithms 5 (1994) 363-374.