GIANT DESCENDANT TREES AND MATCHING SETS IN THE PREFERENTIAL ATTACHMENT GRAPH.

HUSEYIN ACAN, ALAN FRIEZE, AND BORIS PITTEL

Abstract. We study the δ-version of the preferential attachment graph with m attachments for each of incoming vertices. We show that almost surely the scaled size of a breadth-first (descendant) tree rooted at a fixed vertex converges, for $m = 1$, to a limit whose distribution is a mixture of two beta distributions, and that for $m > 1$ the limit is 1. We also analyze the likely performance of a greedy matching algorithm for all $m \geq 1$ and establish an almost sure lower bound for the size of the matching set.

1. Introduction

It is widely accepted that graphs/networks are an inherent feature of life today. The classical models $G_{n,m}$ and $G_{n,p}$ of Erdős and Rényi [17] and Gilbert [22], respectively, lacked some salient features of observed networks. In particular, they failed to have a degree distribution that decays polynomially. Barabási and Albert [3] suggested the Preferential Attachment Model (PAM) as a more realistic model of a "real world" network. There was a certain lack of rigour in [3] and later Bollobás, Riordan, Spencer and Tusnády [6] gave a rigorous definition.

Many properties of this model have been studied. Bollobás and Riordan [7] studied the diameter and proved that with high probability (whp) PAM with n vertices and $m > 1$ attachments for every incoming vertex has diameter $\approx \log n / \log\log n$. Earlier result by Pittel [30] implied that for $m = 1$ whp the diameter of PAM is of exact order $\log n$. Bollobás and Riordan [9, 10] studied the effect on component size from deleting random edges from PAM and showed that it is quite robust whp. The degree distribution was studied in Mori [27, 28], Flaxman, Frieze and Fenner [18], Berger, Borgs, Chayes and Saberi [4]. Peköz, Röllin and Ross [29] established convergence, with rate, of the joint distribution of the degrees of finitely many vertices. Acan and Hitczenko [2] found an alternative proof, without rate, via a memory game. Pittel [32] used the Bollobás-Riordan pairing model to

2010 *Mathematics Subject Classification.* 05C05, 05C07, 05C30, 05C80, 60C05.

Key words and phrases. random, preferential attachment graphs, asymptotics.

Research of the second author supported in part by NSF Grant DMS 1661063.

approximate, with explicit error estimate, the degree sequence of the first $n^{m/(m+2)}$ vertices, $m \geq 1$, and proved that, for $m > 1$, PAM is connected with probability $\approx 1 - O((\log n)^{-(m-1)/3})$. Random walks on PAM have been considered in the work of Cooper and Frieze [14, 15]. In the first paper there are results on the proportion of vertices seen by a random walk on an evolving PAM and the second paper determines the asymptotic cover time of a fully evolved PAM. Frieze and Pegden [21] used random walk in a "local algorithm" to find vertex 1, improving results of Borgs, Brautbar, Chayes, Khanna and Lucier [12]. The mixing time of such a walk was analyzed in Mihail, Papadimitriou and Saberi [23] who showed rapid mixing. Interpolating between Erdős-Rényi and preferential attachment, Pittel [31] considered birth of a giant component in a graph process G_M on a fixed vertex set, when G_{M+1} is obtained by inserting a new edge between vertices i and j with probability proportional to $[\deg(i)+\delta]\cdot[\deg(j)+\delta]$, with $\delta > 0$ being fixed.

The previous paragraph gives a small sample of results on PAM that can be related to its role as a model of a real world network. It is safe to say that PAM has now been accepted into the pantheon of random graph models that can be studied purely from a combinatorial aspect. For example, Cooper, Klasing and Zito [16] studied the size of the smallest dominating set and Frieze, Pérez-Giménez, Prałat and Reiniger [19] studied the existence of perfect matchings and Hamilton cycles.

2. Our Results

We study the number of descendants of a given vertex and also analyse the performance of an on-line greedy algorithm for finding a large matching. We carry out this analysis in the context of a generalization of the model from [6]. The precise model is taken from Hofstad [26, Ch. 8], and is described next.

Preferential Attachment, δ-extension:
Vertex 1 has m loops, so its degree is $2m$ initially. Recursively, vertex $t+1$ has m edges, and it uses them one at a time either to connect to a vertex $x \in [t]$ or to loop back on itself.

More precisely, at step $i \in [m]$:
Step $t+1$:

- **(a)** vertex $t+1$ attaches itself to $x \in [t]$, thus increasing degrees of both x and $t+1$ by 1, with probability $C \cdot (d_{t,i-1}(x) + \delta)$, where $d_{t,i-1}(x)$ is the degree of vertex x just before the arrival of vertex $t+1$ *plus* the number of times vertex $t+1$ connected to vertex x in the preceding $i-1$ steps;
- **(b)** vertex $t+1$ loops back on itself with probability $C \cdot (d_{t,i-1}(t+1) + 1 + i\delta/m)$, thus increasing the degree of $t+1$ by 2, where $d_{t,i-1}(t+1)$ is the

degree of $t+1$ after $i-1$ attachments. (The summand 1 in the formula for the probability is the contribution to the degree of $t+1$ coming from the i-th edge whose other endpoint has not been chosen yet.)

(c) C is determined from the condition "the $t+1$ probabilities add up to 1".

In other words, denoting by w the random receiving end of the i-th edge of $t+1$, we have

$$\mathbb{P}(w = x) = \begin{cases} C \times (d_{t,i-1}(x) + \delta) & \text{if } x \in [t] \\ C \times (d_{t,i-1}(t+1) + 1 + i\delta/m) & \text{if } x = t+1, \end{cases} \tag{2.1}$$

where

$$C = \left((t + i/m)\delta + 1 + \sum_{x=1}^{t+1} d_{t,i-1}(x) \right)^{-1} = \frac{1}{(t + i/m)\delta + 2(mt + i) - 1}.$$

Remark 2.1. Note that the process is well defined for $\delta \geq -m$ since for such δ, all the probabilities defined in (2.1) are nonnegative and add up to 1. However, we will see that $G_{m,-m}(t)$ is the star centered at vertex 1, and the key problems we want to solve have trivial solutions in that extreme case.

We will use the notation $\{G_{m,\delta}(t)\}$ for the resulting graph process. In particular, for $m = 1$ we have: using "$|\circ$" to indicate conditioning on pre-history,

$$\mathbb{P}\big(t + 1 \text{ selects } x|\circ\big) = \begin{cases} \dfrac{1+\delta}{(2+\delta)t + (1+\delta)}, & x = t+1, \\ \dfrac{d_t(x)+\delta}{(2+\delta)t + (1+\delta)}, & x \in [t]. \end{cases} \tag{2.2}$$

The total degree of $G_{m,\delta}(t)$ is $2mt$.

2.1. **Number of Descendants.** Fix a positive integer r and let $X(t)$ denote the number of descendants of r at time t. Here r is a descendant of r and x is a descendant of $r = O(1)$ if and only if x chooses to attach itsef to at least one descendant of r in Step x. In other words, if we think of the graph as a directed graph with edges oriented towards the smaller vertices, vertex x is a descendant of r if and only if there is a directed path from x to r. We prove two theorems:

Theorem 2.2. *Suppose that $m = 1$ and $\delta > -1$ and $p(t) = X(t)/t$. Then almost surely (i.e. with probability 1), $\lim p(t)$ exists, and its distribution is the mixture of two beta-distributions, with parameters $a = 1$, $b = r - \frac{1}{2+\delta}$ and $a = \frac{1+\delta}{2+\delta}$, $b = r$, weighted by $\frac{1+\delta}{(2+\delta)r-1}$ and $\frac{(2+\delta)(r-1)}{(2+\delta)r-1}$ respectively. Consequently a.s. $\liminf_{t\to\infty} p(t) > 0$.*

Note. **(i)** The proof is based on a new family of martingales $M_\ell(t) := \frac{\left(X(t)+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(t+\beta)^{(\ell)}}$, $(z)^{(\ell)}$ standing for the rising factorial. This family definitely resembles the martingales Mori [27, 28] used for the vertex degrees. Our proof of the martingale property is, unsurprisingly, quite different. **(ii)** Whp $G_{1,\delta}$ is a forest of $\Theta(\log t)$ trees rooted at vertices with loops. For the preferential attachment tree (no loops), Janson [25] recently proved that the scaled sizes of the *principal* subtrees, those rooted at the root's children and ordered chronologically, converge a.s. to the *GEM* distributed random variables. His techniques differ significantly.

When $m > 1$ we have the somewhat surprising result that, for $r = O(1)$, almost surely all but a vanishingly small fraction of vertices are descendants of r, (cf. [25]).

Theorem 2.3. *Let $m > 1$ and $\delta > -m$ and let $p_X(t) = X(t)/t$, $p_Y(t) = Y(t)/(2mt)$, where $Y(t)$ is the total degree of the descendants of r at time t. Then almost surely $\lim p_X(t) = \lim p_Y(t) = 1$.*

2.2. **Greedy Matching Algorithm.** We analyze a greedy matching algorithm; a.s. it delivers a surprisingly large matching set even for relatively small m. This algorithm generates the increasing sequence $\{M(t)\}$ of partial matchings on the sets $[t]$, with $M(1) = \emptyset$. Suppose that $X(t)$ is the set of unmatched vertices in $[t]$ at time t. If $t+1$ chooses a vertex $u \in X(t)$ to attach itself to then $M(t+1) = M(t) \cup \{\{u, t+1\}\}$, otherwise $M(t+1) = M(t)$. (If $t+1$ chooses multiple vertices from $X(t)$, then we pick one of those as u arbitrarily.) Let

$$h(z) = h_{m,\delta}(z) := 2\left[1 - \left(\frac{m+\delta}{2m+\delta}\right) z\right]^m - z - 1$$

and let $\rho = \rho_{m,\delta}$ be the unique root $\rho = \rho_{m,\delta}$ in the interval $[0,1]$ of $h(z) = 0$: $\rho_{m,\delta} \in (0,1)$ if $\delta > -m$. Denoting $x(t) = X(t)/t$, we have

Theorem 2.4. *If $\delta > -m$, then, for any $\alpha < 1/3$, almost surely,*

$$\lim_{t\to\infty} t^\alpha \max\{0, x(t) - \rho_{m,\delta}\} = 0.$$

In consequence, the Greedy Matching Algorithm a.s. finds a sequence of nested matchings $\{M(t)\}$, with $M(t)$ of size at least $(1-o(1))(1-\rho_{m,\delta})t/2$.

Remark 2.5. Observe that $\rho_{m,-m} = 1$, which makes it plausible that the maximum matching size is miniscule compared to t. In fact, by Remark 2.1, $G_{m,-m}(t)$ is the star centered at vertex 1 and hence the maximum matching size is 1.

Remark 2.6. Consider the case $\delta = 0$. Let $r_m := 1 - \rho_{m,0}$; some values of r_m are:

$$\begin{aligned} &r_1 = 0.5000, &&r_2 = 0.6458, &&r_5 = 0.8044,\\ &r_{10} = 0.8863, &&r_{20} = 0.9377, &&r_{70} = 0.9803. \end{aligned} \tag{2.3}$$

With a bit of calculus, we obtain that $r_m = 1 - 2m^{-1}\log 2 + O(m^{-2})$.

Remark 2.7. Close to the PAM is the Uniform Attachment Model: vertex $t+1$ selects uniformly at random (repetitions allowed) m vertices from the set $[t]$. (See Acan and Pittel [1] for connectivity and bootstrap percolation results.) An argument, broadly analogous to the one for Theorem 2.4, gives the following theorem.

Theorem 2.8. *Let r_m denote a unique positive root of $2(1-z^m) - z = 0$: $r_m = 1 - m^{-1}\log 2 + O(m^{-2})$. Then, for any $\alpha < 1/3$, almost surely*

$$\lim_{t\to\infty} t^\alpha \big|1 - r_m - x(t)\big| = 0$$

for the uniform attachment model.

[Note that $x(t)$ is the fraction of unmatched vertices.]
Some values of r_m in this case are:

$$\begin{aligned} &r_1 = 0.6667, &&r_2 = 0.7808, &&r_5 = 0.8891,\\ &r_{10} = 0.9386, &&r_{20} = 0.9674, &&r_{35} = 0.9809. \end{aligned}$$

3. Proof of Theorem 2.2

For $t \geq r > 1$, let $X(t) = X_{m,\delta}(t) = X_{m,\delta}(t,r)$, $Y(t) = Y_{m,\delta}(t) = Y_{m,\delta}(t,r)$ denote the size and the total degree of the vertices in the vertex set of the sub-tree $T(t) = T_{m,\delta}(t,r)$ rooted at r; so $X(r) = X_{m,\delta}(r,r) = 1$ and $Y(r) = Y_{m,\delta}(r,r) \in [m, 2m]$, m ($2m$ resp.) attained when $t+1$ forms no loops (forms m loops resp.) at itself. Introduce $p(t) = p_Y(t) = \frac{Y(t)}{2mt}$ and $p_X(t) = \frac{X(t)}{t}$. This notation will be used in the proof of Theorem 2.3 as well, but of course $m = 1$ in the proof of Theorem 2.2.

Here

$$Y(t) = \begin{cases} 2X(t), & \text{if } r \text{ looped on itself,}\\ 2X(t) - 1, & \text{if } r \text{ selected a vertex in } [r-1]. \end{cases}$$

(In particular, $p_X(t) = p(t) + O(t^{-1})$.) So, by (2.2),

$$\mathbb{P}(X(t+1) = X(t)+1|\circ) = \frac{Y(t) + \delta X(t)}{(2+\delta)t + (1+\delta)}$$
$$= \begin{cases} \dfrac{(2+\delta)X(t)}{(2+\delta)t+(1+\delta)}, & \text{if } r \text{ looped on itself,} \\ \dfrac{(2+\delta)X(t)-1}{(2+\delta)t+(1+\delta)}, & \text{if } r \text{ selected a vertex in } [r-1]. \end{cases}$$

Thus we are led to consider the process $X(t)$ such that

$$\mathbb{P}(X(t+1) = X(t)+1|\circ) = \frac{(2+\delta)X(t)+\gamma}{(2+\delta)t+(1+\delta)}, \tag{3.1}$$
$$\mathbb{P}(X(t+1) = X(t)|\circ) = 1 - \mathbb{P}(X(t+1) = X(t)+1|\circ);$$

$\gamma = 0$ if r looped on itself, $\gamma = -1$ if r selected a vertex in $[r-1]$.

Note. Suppose $\delta = -1$, (see Remark 2.1). Then, by (2.2), vertex r selects a vertex in $[r-1]$, so that $\gamma = -1$. Since $X(r) = 1$, it follows from (3.1) that $X(t) \equiv 1$ for $t \geq r$. This means that $G_{1,-1}(t)$ is the star with vertex 1 being the star's center, cf. [26, Exercise 8.5].

Lemma 3.1. *Let $(z)^{(\ell)} = \prod_{j=0}^{\ell-1}(z+j)$, $\beta = \frac{1+\delta}{2+\delta}$. Then, conditioned on the attachment record during the time interval $[r,t]$, i.e. starting with attachment decision by vertex r, we have*

$$\mathbb{E}\left[\left(X(t+1) + \frac{\gamma}{2+\delta}\right)^{(\ell)} \Bigg| \circ\right] = \frac{t+\beta+\ell}{t+\beta}\left(X(t) + \frac{\gamma}{2+\delta}\right)^{(\ell)}.$$

Consequently $M(t) := \frac{\left(X(t)+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(t+\beta)^{(\ell)}}$ is a martingale.

For $\delta = 0$ this claim was proved in Pittel [32].

Proof. First of all,

$$\frac{(2+\delta)X(t)+\gamma}{(2+\delta)t+(1+\delta)} = \frac{X(t)+\alpha}{t+\beta}, \qquad \alpha = \frac{\gamma}{2+\delta},\ \beta = \frac{1+\delta}{2+\delta}.$$

Introduce $Z(t) = X(t) + \alpha$. By (2.2), we have: for $k \geq 1$, and $t \geq r$,

$$\begin{aligned}
\mathbb{E}[Z^k(t+1)|\circ] &= (Z(t)+1)^k \frac{Z(t)}{t+\beta} + Z^k(t)\left(1 - \frac{Z(t)}{t+\beta}\right) \\
&= \frac{Z(t)}{t+\beta}\sum_{j=0}^{k}\binom{k}{j}Z^j(t) + Z^k(t)\left(1-\frac{Z(t)}{t+\beta}\right) \\
&= Z^k(t) + \frac{Z(t)}{t+\beta}\sum_{j=0}^{k-1}\binom{k}{j}Z^j(t) \\
&= Z^k(t)\frac{t+\beta+k}{t+\beta} + \frac{1}{t+\beta}\sum_{j=1}^{k-1}\binom{k}{j-1}Z^j(t).
\end{aligned} \tag{3.2}$$

Next recall that

$$z^{(\ell)} = \sum_{k=1}^{\ell} z^k s(\ell,k), \tag{3.3}$$

where $s(\ell,k)$ is the signless, first-kind, Stirling number, i.e. the number of permutations of the set $[\ell]$ with k cycles. In particular,

$$\sum_{\ell\geq 1}\eta^\ell\frac{s(\ell,k)}{\ell!} = \frac{1}{k!}\log^k\frac{1}{1-\eta}, \quad |\eta|<1, \tag{3.4}$$

Comtet [13, Section 5.5]. Using (3.2) and (3.3), we have

$$\begin{aligned}
\mathbb{E}\big[Z^{(\ell)}(t+1)|\circ\big] &= \sum_{k=1}^{\ell}s(\ell,k)\mathbb{E}\big[Z^k(t+1)|\circ\big] \\
&= (t+\beta)^{-1}\sum_{k=1}^{\ell}s(\ell,k)\cdot\left((t+\beta+k)Z^k(t) + \sum_{j=0}^{k-1}\binom{k}{j-1}Z^j(t)\right) \\
&=: (t+\beta)^{-1}\sum_{i=1}^{\ell}\sigma(\ell,i)Z^i(t),
\end{aligned}$$

$$\sigma(\ell,i) = \begin{cases} (t+\beta+\ell)s(\ell,\ell), & \text{if } i=\ell, \\ (t+\beta)s(\ell,i) + \sum_{k=i}^{\ell}s(\ell,k)\binom{k}{i-1}, & \text{if } i<\ell. \end{cases}$$

We need to show that $\sigma(\ell,i) = (t+\beta+\ell)s(\ell,i)$ for $k<\ell$, which is equivalent to

$$\ell s(\ell,i) = \sum_{k=i}^{\ell}s(\ell,k)\binom{k}{i-1}.$$

To prove the latter identity, it suffices to show that, for a fixed i, the exponential generating functions of the two sides coincide. By (3.4),

$$\begin{aligned}
&\sum_{\ell\ge 1}\frac{\eta^\ell}{\ell!}\sum_{k=i}^{\ell} s(\ell,k)\binom{k}{i-1} = \sum_{k\ge i}\binom{k}{i-1}\sum_{\ell\ge k}\frac{\eta^\ell}{\ell!}s(\ell,k)\\
&=\sum_{k\ge i}\binom{k}{i-1}\frac{1}{k!}\log^k\frac{1}{1-\eta} = \frac{1}{(i-1)!}\left(\log^{-1}\frac{1}{1-\eta}\right)\sum_{s\ge 1}\frac{1}{s!}\log^s\frac{1}{1-\eta}\\
&=\frac{1}{(i-1)!}\left(\log^{i-1}\frac{1}{1-\eta}\right)\left(\frac{1}{1-\eta}-1\right) = \frac{1}{(i-1)!}\left(\log^{i-1}\frac{1}{1-\eta}\right)\frac{\eta}{1-\eta}.
\end{aligned}$$

And, using (3.4) again,

$$\begin{aligned}
&\sum_{\ell\ge 1}\frac{\eta^\ell}{\ell!}\,\ell s(\ell,i) = \eta\sum_{\ell\ge 1}\frac{\ell\eta^{\ell-1}}{\ell!}\,s(\ell,i)\\
&=\eta\frac{d}{d\eta}\left(\frac{1}{i!}\log^i\frac{1}{1-\eta}\right) = \frac{1}{(i-1)!}\left(\log^{i-1}\frac{1}{1-\eta}\right)\frac{\eta}{1-\eta}.
\end{aligned}$$

$\square$

To identify the $\lim p(t)$, recall that the classic beta probability distribution has density

$$f(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1},\quad x\in(0,1),$$

parametrized by two parameters $a>0$, $b>0$, and moments

$$\int_0^1 x^\ell f(x;a,b)\,dx = \prod_{j=0}^{\ell-1}\frac{a+j}{a+b+j}. \tag{3.5}$$

We can now complete the proof of Theorem 2.2. By Lemma 3.1, we have $\mathbb{E}[M(t)|\gamma] = M(r)$, i.e.

$$\mathbb{E}\left[\frac{\left(X(t)+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(t+\beta)^{(\ell)}}\,\middle|\,\gamma\right] = \frac{\left(1+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(r+\beta)^{(\ell)}}.$$

For every $\ell\ge 1$, by martingale convergence theorem, conditioned on γ, there exists an integrably finite $\mathcal{M}_{\gamma,\ell}$ such that a.s.

$$\lim_{t\to\infty}\frac{\left(X(t)+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(t+\beta)^{(\ell)}} = \mathcal{M}_{\gamma,\ell},\quad \ell\ge 0,$$

and

$$\mathbb{E}[\mathcal{M}_{\gamma,\ell}] = \frac{\left(1+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(r+\beta)^{(\ell)}}.$$

So, using the notation $p_X(t) = X(t)/t$, we have: a.s.

$$\lim_{t\to\infty} (p_X(t))^\ell = \mathcal{M}_{\gamma,\ell} = (\mathcal{M}_{\gamma,1})^\ell, \tag{3.6}$$

and

$$\mathbb{E}\big[(\mathcal{M}_{\gamma,1})^\ell\big] = \frac{\left(1+\frac{\gamma}{2+\delta}\right)^{(\ell)}}{(r+\beta)^{(\ell)}} = \prod_{j=0}^{\ell-1} \frac{1+\frac{\gamma}{2+\delta}+j}{r+\beta+j}.$$

This means that $\mathcal{M}_{\gamma,1}$ is beta-distributed with parameters $1+\frac{\gamma}{2+\delta}$ and $r+\beta-1-\frac{\gamma}{2+\delta}$. By the definition of γ and (2.2), we have

$$\mathbb{P}(\gamma = 0) = \frac{1+\delta}{(2+\delta)(r-1)+(1+\delta)} = \frac{1+\delta}{2r-1+\delta r}.$$

We conclude that $\lim_{t\to\infty} p(t)$ has the distribution which is the mixture of the two beta distributions, with parameters $a=1$, $b=r-\frac{1}{2+\delta}$, and $a=\frac{1+\delta}{2+\delta}$, $b=r$, weighted by $\frac{1+\delta}{(2+\delta)r-1}$ and $\frac{(2+\delta)(r-1)}{(2+\delta)r-1}$ respectively.

This completes the proof of Theorem 2.2.

3.1. **Proof of Theorem 2.3.** We need to derive tractable formulas/bounds for the conditional distribution of $Y(t+1)-Y(t)$. First, let us evaluate the conditional probability that selecting the second endpoints of the m edges incident to vertex $t+1$ no loops will be formed. Suppose there has been no loop in the first $i-1$ steps, $i\in[m]$; call this event $\mathcal{E}_{i-1}$. On event $\mathcal{E}_{i-1}$, as the i-th edge incident to $t+1$ is about to attach its second end to a vertex in $[t]\cup\{t+1\}$, the total degree of all these vertices is $2mt+i-1$ ($1\le i\le m$). So, by the definition of the transition probabilities (items **(a), (b), (c)**) we have

$$\mathbb{P}(\mathcal{E}_i|\circ) = \frac{2mt+i-1+t\delta}{2mt+2(i-1)+t\delta+1+\frac{i\delta}{m}},$$

"$\circ$" indicating conditioning on the full record of $i-1$ preceding attachments such that the event $\mathcal{E}_{i-1}$ holds. Crucially this conditional probability depends on i only. Therefore the probability of a given full *loops-free* record of the m attachments is equal to the corresponding probability for the "no loops in m attachments process", multiplied by

$$\Pi_m(t) := \prod_{i=1}^m \frac{2mt+i-1+t\delta}{2mt+2(i-1)+t\delta+1+\frac{i\delta}{m}} = 1-O(t^{-1}). \tag{3.7}$$

Lemma 3.2. *If no loops are allowed in the transition from t to $t+1$, then*

$$\mathbb{P}(Y(t+1)=Y(t)+m+a|\circ)$$
$$=\binom{m}{a}\frac{(Y(t)+\delta X(t))^{(a)}\big(2mt-Y(t)+\delta(t-X(t))^{(m-a)}}{\big((2m+\delta)t\big)^{(m)}},\quad (a\in[m]),$$
$$\mathbb{P}(Y(t+1)=Y(t)\,|\,\circ)=\frac{\big(2mt-Y(t)+\delta(t-X(t))^{(m)}}{\big((2m+\delta)t\big)^{(m)}}.$$

Proof. Vertex $t+1$ selects, in m steps, a sequence $\{v_1,\dots,v_m\}$ of m vertices from $[t]$, with t choices for every selection. The total vertex degree of $[t]$ (of $V(T(t))$ respectively) right before step i is $2mt+i-1$ $\big(Y(t)+\mu_i$ respectively, $\mu_i:=|\{j<i: v_j\in V(H(t))\}|\big)$. Conditioned on this prehistory,

$$\mathbb{P}(v_i\in V(T(t)))=\frac{Y(t)+\delta X(t)+\mu_i}{2mt+\delta t+i-1},$$
$$\mathbb{P}(v_i\in[t]\setminus V(T(t)))=\frac{2mt-Y(t)+\delta(t-X(t))+i-1-\mu_i}{2mt+\delta t+i-1}.$$

Therefore a sequence $\mathbf{v}=\{v_1,\dots,v_m\}$ will be the outcome of the m-step selection with probability

$$\mathbb{P}(\mathbf{v})=\left(\prod_{i\in[m]}\Big((2m+\delta)t+i-1\Big)\right)^{-1}$$
$$\times\prod_{i:v_i\in V(T(t))}\big(Y(t)+\delta X(t)+\mu_i\big)\cdot\prod_{i:v_i\in[t]\setminus V(T(t))}\big(2mt-Y(t)+\delta(t-X(t))+i-1-\mu_i\big).$$

Furthermore, for $a\in[m]$, on the event $\{Y(t+1)=Y(t)+m+a\}$ for each admissible $\mathbf{v}$ we have

$$\{\mu_i\}=\{0,1,\dots,a-1\},\quad \{i-1-\mu_i\}=\{0,1,\dots,m-a-1\},$$

so that

$$\mathbb{P}(\mathbf{v})=\frac{(Y(t)+\delta X(t))^{(a)}\big(2mt-Y(t)+\delta(t-X(t))^{(m-a)}}{\big((2m+\delta)t\big)^{(m)}}$$

Since the total number of admissible sequences is $\binom{m}{a}$, we obtain the first formula in Lemma 3.2. The second formula is the case of $\mathbb{P}(\mathbf{v})$ with $a=0$. $\square$

It is clear from the proof that $\{\mathbb{P}_m(a)\}_{0\le a\le m}$,

$$\mathbb{P}_m(a):=\binom{m}{a}\frac{(Y(t)+\delta X(t))^{(a)}\big(2mt-Y(t)+\delta(t-X(t))^{(m-a)}}{\big((2m+\delta)t\big)^{(m)}},$$

is a probability distribution of a random variable D, a "rising-factorial" counterpart of the binomial $\mathcal{D}=\text{Bin}(m,p=Y(t)/2mt)$. Define the falling factorial $(x)_\ell = x(x-1)\cdots(x-\ell+1)$. It is well known that $\mathbb{E}[(\mathcal{D})_\mu] = (m)_\mu p^\mu$, $(\mu \le m)$. For D we have

$$\mathbb{E}[(D)_\mu] = \sum_a (a)_\mu \mathbb{P}_m(a) = \frac{(m)_\mu \,(Y(t)+\delta X(t))^{(\mu)}}{\big((2m+\delta)t\big)^{(\mu)}} \cdot \sum_{a\ge\mu} \binom{m-\mu}{a-\mu}$$
$$\times \frac{(Y(t)+\delta X(t)+\mu)^{(a-\mu)}\big((2m+\delta)t+\mu-(Y(t)+\delta X(t)+\mu)\big)^{((m-\mu)-(a-\mu))}}{(2mt+\mu)^{(m-\mu)}}$$
$$= \frac{(m)_\mu \,(Y(t)+\delta X(t))^{(\mu)}}{\big((2m+\delta)t\big)^{(\mu)}}, \quad (3.8)$$

since the sum over $a \ge \mu$ is $\sum_{\nu\ge 0} \mathbb{P}_{m-\mu}(\nu) = 1$.

From Lemma 3.2 and (3.8) we have: if no loops during the transition from t to $t+1$ are allowed, then

$$\mathbb{E}[Y(t+1)-Y(t)|\circ] = \sum_{a=1}^{m} (a+m)\mathbb{P}_m(a)$$
$$= \frac{m\big(Y(t)+\delta X(t)\big)}{(2m+\delta)t} + m\left(1 - \frac{\big((2m+\delta)t - Y(t) - \delta X(t)\big)^{(m)}}{\big((2m+\delta)t\big)^{(m)}}\right), \quad (3.9)$$

and

$$\mathbb{E}[X(t+1)-X(t)|\circ] = 1 - \frac{\big((2m+\delta)t - Y(t) - \delta X(t)\big)^{(m)}}{\big((2m+\delta)t\big)^{(m)}} \quad (3.10)$$

What if the ban on loops at the vertex $t+1$ is lifted? From the discussion right before Lemma 3.2, we see that both $\mathbb{E}[\big(Y(t+1)-Y(t)\big)\mathbb{I}(\text{no loops})|\circ]$ and $\mathbb{E}[\big(X(t+1)-X(t)\big)\mathbb{I}(\text{no loops})|\circ]$ are equal to the respective RHS's in (3.9) and (3.10) *times* $\Pi_m(t) = 1 - O(t^{-1})$. Consequently, adding the terms $O(t^{-1})$ to the RHS of (3.9) and to the RHS of (3.10) we obtain the sharp asymptotic formulas for $\mathbb{E}[Y(t+1)-Y(t)|\circ]$ and $\mathbb{E}[X(t+1)-X(t)|\circ]$ in the case of the loops-allowed model.

Let
$$p(t) = \frac{2m}{2m+\delta}\, p_Y(t) + \frac{\delta}{2m+\delta}\, p_X(t),$$
where $p_Y(t) = \frac{Y(t)}{2mt}$ and $p_X(t) = \frac{X(t)}{t}$ as defined in the beginning of the section. Theorem 2.3 asserts

Theorem 3.3. *Let $m > 1$ and $\delta > -m$. Then almost surely* $\lim p_Y(t) = \lim p_X(t) = 1$.

Proof. First of all, we note that $mX(t) \le Y(t) \le 2mX(t)$. The lower bound is obvious. The upper bound follows from induction on t: Suppose $Y(t) \le 2mX(t)$. If $X(t+1) = X(t)$, then $Y(t+1) = Y(t) \le 2mX(t) = 2mX(t+1)$.

If $X(t+1) = X(t)+1$, then $Y(t+1) \le Y(t)+2m \le 2mX(t)+2m = 2mX(t+1)$. Therefore, by the definition of $p(t)$, we have

$$\frac{p_X(t)}{2} \le p_Y(t) \le p_X(t) \Longrightarrow \frac{m+\delta}{2m+\delta}\, p_X(t) \le p(t) \le p_X(t); \tag{3.11}$$

in particular, $p(t) \in [0,1]$ since $\delta \ge -m$. We will also need

$$\frac{\big((2m+\delta)t - Y(t) - \delta X(t)\big)^{(m)}}{\big((2m+\delta)t\big)^{(m)}} = (1-p(t))^m + O(t^{-1}).$$

So, using (3.9), we compute

$$\begin{aligned}
\mathbb{E}[p_Y(t+1)|\circ] &= \mathbb{E}\left[\frac{Y(t+1)}{2mt}\cdot\frac{t}{t+1}\Big|\circ\right]\\
&= \frac{t}{t+1}\left(p_Y(t) + \frac{1}{2t}\big[1+p(t)-(1-p(t))^m\big] + O(t^{-2})\right) = p_Y(t) + q_Y(t),
\end{aligned}$$

$$q_Y(t) := \frac{1}{2(t+1)}\big[1+p(t)-2p_Y(t)-(1-p(t))^m\big] + O(t^{-2}). \tag{3.12}$$

Likewise

$$\begin{aligned}
\mathbb{E}[p_X(t+1)|\circ] &= p_X(t) + q_X(t),\\
q_X(t) &= \frac{1}{t+1}\big[1-p_X(t)-(1-p(t))^m\big] + O(t^{-2}).
\end{aligned} \tag{3.13}$$

Multiplying the equation (3.12) by $\frac{2m}{2m+\delta}$, the equation (3.13) by $\frac{\delta}{2m+\delta}$, and adding them, we obtain

$$\begin{aligned}
\mathbb{E}[p(t+1)|\circ] &= p(t) + q(t),\\
q(t) &= \frac{m+\delta}{(2m+\delta)(t+1)}\big[1-p(t)-(1-p(t))^m\big] + O(t^{-2}).
\end{aligned} \tag{3.14}$$

Since $1-z-(1-z)^m \ge 0$ on $[0,1]$, the equation (3.14) implies that $\sum_t \mathbb{E}[|q(t)|] < \infty$. So, a.s. there exists $Q := \lim_{\tau\to\infty}\sum_{1\le\tau\le t} q(\tau)$, with $\mathbb{E}[|Q|] \le \sum_t \mathbb{E}[|q(t)|] < \infty$, i.e. a.s. $|Q| < \infty$. Introducing $Q(t+1) = \sum_{\tau\le t} q(\tau)$, we see from (3.14) that $\{p(t+1)-Q(t+1)\}_{t\ge1}$ is a martingale with $\sup_t |p(t+1)-Q(t+1)| \le 1 + \sum_{\tau\ge1}|q(\tau)|$. By the martingale convergence theorem we obtain that there exists an integrable $\lim_{t\to\infty}(p(t)-Q(t+1))$, implying that a.s. there exists a random $p(\infty) = \lim_{t\to\infty} p(t)$. The (3.14) also implies that

$$1 \ge \mathbb{E}[p(\infty)] = \frac{m+\delta}{2m+\delta}\sum_{t\ge1}\frac{1}{t+1}\mathbb{E}\big[1-p(t)-(1-p(t))^m\big] + O(1).$$

Since $m+\delta > 0$ and

$$\lim_{t\to\infty}\mathbb{E}\big[1-p(t)-(1-p(t))^m\big] = \mathbb{E}\big[1-p(\infty)-(1-p(\infty))^m\big],$$

and the series $\sum_{t\geq 1} t^{-1}$ diverges, we obtain that $\mathbb{P}(p(\infty)\in\{0,1\})=1$.

Recall that $p(t)\geq \frac{m+\delta}{2m+\delta}\,p_X(t)$. If we show that a.s. $\liminf_{t\to\infty} p_X(t)>0$, it will follow that a.s. $p(\infty)>0$, whence a.s. $p(\infty)=1$, implying (by $p(t)\leq p_X(t)$) that a.s. $p_X(\infty)$ exists, and is 1, and consequently (by the formula for $p(t)$) a.s. $p_Y(\infty)$ exists, and is 1.

So let's prove that a.s. $\liminf_{t\to\infty} p_X(t)>0$. Recall that we did prove the latter for $m=1$. To transfer this earlier result to $m>1$, we need to establish some kind of monotonicity with respect to m. A coupling to the rescue!

For the B-R model, with loops allowed at every vertex, the following coupling between $G_{m,0}(t)$ and $G_{1,0}(mt)$ was discovered by Bollobás and Riordan [7]. Start with the $\{G_{1,0}(t)\}$ random process and let the vertices be $v_1, v_2, \dots$. To obtain the random graph process $\{G_{m,0}(t)\}$ from $\{G_{1,0}(mt)\}$,

(1) collapse the first m vertices $v_1,\dots,v_m$ into the first vertex w_1 of $G_{m,0}(t)$, the next m vertices $v_{m+1},\dots,v_{2m}$ into the second vertex w_2 of $G_{m,0}(t)$, and so on;
(2) keep the full record of the multiple edges and loops formed by collapsing the blocks $\{v_{(i-1)m+1},\dots,v_{im}\}$ for each i.

Doing this collapsing indefinitely we get the jointly defined Bollobás-Riordan graph processes $\{G_{m,0}(t)\}$ and $\{G_{1,0}(mt)\}$. The beauty of the δ-extended Bollobás-Riordan model is that similarly this collapsing operation applied to the process $\{G_{1,\delta/m}(mt)\}$ delivers the process $\{G_{m,\delta}(t)\}$, [26]. (See Appendix for the explanation.)

Remark 3.4. It follows from this coupling that $G_{m,-m}(t)$ is the star centered at vertex 1. This follows from the fact that $G_{1,-1}(mt,1)$ is a star and confirms the claim in Remark 2.1. In the coupling, each additional vertex is joined to vertex 1 by m parallel edges.

Lemma 3.5. *For the processes $\{G_{m,\delta}(t)\}$ and $\{G_{1,\delta/m}(mt)\}$ coupled this way, we have $X_{m,\delta}(t,r)\geq m^{-1}X_{1,\delta/m}(mt,mr)$.*

Proof. Let us simply write G_1 and G_m for the two graphs $G_{1,\delta/m}(mt)$ and $G_{m,\delta}(t)$, respectively. Similarly, write T_1 and T_m, respectively, for the descendant tree in $G_{1,\delta/m}(mt)$ rooted at mr and the descendant tree in $G_{m,\delta}(t)$ rooted at r. If $v_a\in T_1$, i.e. v_a is a descendant of mr, then for $b=\lceil a/m\rceil$ we have $w_b=\{v_{m(b-1)+i}\}_{i\in[m]}\ni v_a$, implying that w_b is a descendant of r in G_m, i.e. $w_b\in T_m$. (The converse is generally false: if w_b is a descendant of r, it does not mean that every $v_{m(b-1)+i}$, $(i\in[m])$, is a descendant of mr.) Therefore

$$X_{m,\delta}(t,r)=|V(T_m)|\geq m^{-1}|V(T_1)|=m^{-1}X_{1,\delta/m}(mt,mr). \qquad \square$$

Thus, to complete the proof of the theorem, i.e. for $\delta > -m$, we **(a)** use Theorem 2.2, to assert that for the process $\{G_{1,\delta/m}(t)\}$, a.s. $\lim p_X(t) > 0$; **(b)** use Lemma 3.5, to assert that a.s. $\liminf p_X(t) > 0$ for $\{G_{m,\delta}(t)\}$ as well. □

4. Proof of Theorem 2.4

Recall that the greedy algorithm generates the increasing sequence $\{M(t)\}$ of partial matchings on the sets $[t]$, with $M(1) = \emptyset$. Given $M(t)$, let

$X(t) :=$ number of unmatched vertices at time t,

$Y(t) :=$ total degree of unmatched vertices at time t,

$U(t) :=$ number of unmatched vertices selected by $t+1$ from $[t] \setminus M(t)$,

$x(t) := X(t)/t$,

$y(t) := Y(t)/(2mt)$.

We want to prove that, for any $\delta > -m$ and $\alpha < 1/3$, almost surely,

$$\lim_{t\to\infty} t^{\alpha} \max\{0, x(t) - \rho_{m,\delta}\} = 0,$$

where $\rho_{m,\delta}$ is the unique root in $(0,1)$ of

$$h(z) = h_{m,\delta}(z) := 2\left[1 - \left(\frac{m+\delta}{2m+\delta}\right) z\right]^m - z - 1. \tag{4.1}$$

(Note that, for $\delta > -m$, the function $h(z)$ is decreasing on $(0,1)$ and $h(z) = 0$ has a unique solution in the same interval.) We will prove this first for a slightly different model that does not allow any loops other than the first vertex. In this model, vertex 1 has m loops, and the i-th edge of vertex $t+1$ attaches to $u \in [t]$ with probability

$$\frac{d_{t,i-1}(u) + \delta}{2mt + 2(i-1) + t\delta}.$$

We will need the following Chernoff bound. (See e.g. [24, Theorem 2.8].)

Theorem 4.1. *If $X_1, \ldots, X_n$ are independent Bernoulli random variables, $X = \sum_{i=1}^{n} X_i$, and $\lambda = \mathbb{E}[X]$, then*

$$\mathbb{P}(|X - \lambda| > \varepsilon\lambda) < 2\exp\left(-\varepsilon^2\lambda/3\right) \quad \forall \varepsilon \in (0, 3/2).$$

Proof of Theorem 2.4 for "loops only at vertex 1". Let $\varepsilon = \varepsilon_t := t^{-1/3}\log t$. We will show

$$\mathbb{P}(x(t) > \rho + \varepsilon) \le \exp\left(-\Theta\left(\log^3 t\right)\right). \tag{4.2}$$

Once we show (4.2), the Borel-Cantelli lemma gives

$$\mathbb{P}(x(t) - \rho > t^{-1/3}\log t \quad \text{infinitely often}) = 0,$$

which gives what we want. Let us prove (4.2).

Since each degree is at least m, we have $Y(t) \geq mX(t)$ and hence $y(t) \geq x(t)/2$. Also, since

$$X(t+1) = \begin{cases} X(t)+1 & \text{if } U(t)=0 \\ X(t)-1 & \text{if } U(t)>0, \end{cases}$$

we have

$$\mathbb{E}[X(t+1)|\circ] = X(t) + \mathbb{P}(U(t)=0|\circ) - \mathbb{P}(U(t)>0|\circ). \tag{4.3}$$

Since $\mathbb{P}(\text{vertex } t+1 \text{ has some loop}) = O(t^{-1})$, using $Y(t) \geq mX(t)$ in the last step below, we get

$$\begin{aligned}
\mathbb{P}(U(t)=0|\circ) &= \mathbb{P}(U(t)=0 \text{ and vertex } t+1 \text{ has no loop}|\circ) + O\left(t^{-1}\right) \\
&= \left(1-O\left(t^{-1}\right)\right)\frac{(2mt - Y(t) + \delta t - \delta X(t))^{(m)}}{(2mt+\delta t)^{(m)}} + O\left(t^{-1}\right) \\
&= \frac{(2mt + \delta t - Y(t) - \delta X(t))^m}{(2mt+\delta t)^m} + O\left(t^{-1}\right) \\
&= \left(1 - \frac{2m}{2m+\delta}y(t) - \frac{\delta}{2m+\delta}x(t)\right)^m + O\left(t^{-1}\right) \qquad (4.4)\\
&\leq \left(1 - \frac{m+\delta}{2m+\delta}x(t)\right)^m + O(t^{-1}).
\end{aligned}$$

Now using (4.3) and (4.4) gives

$$\begin{aligned}
\mathbb{E}[x(t+1)|\circ] &\leq x(t) + \frac{1}{t}\left[2\left(1 - \frac{m+\delta}{2m+\delta}x(t)\right)^m - x(t) - 1\right] + O\left(t^{-2}\right) \\
&= x(t) + h(x(t)) + O\left(t^{-2}\right), \qquad (4.5)
\end{aligned}$$

where $h(z)$ is as defined in (4.1).

We know that $x(1)=0$. For $T<t$, let $\mathcal{E}_T$ be the event that $x(t) > \rho+\varepsilon$ and $T \in [1,t)$ be the last time such that $x(\tau) \leq \rho + \varepsilon/2$, that is,

$$x(T) \leq \rho + \varepsilon/2; \quad x(\tau) > \rho+\varepsilon/2, \ \forall \tau \in (T,t); \quad x(t) > \rho+\varepsilon.$$

Now

$$\begin{aligned}
X(T) + t - T \geq X(t) > t(\rho+\varepsilon) &\Longrightarrow Tx(T) + t - T > t(\rho+\varepsilon), \\
&\Longrightarrow T(\rho+\varepsilon/2) + t - T > t(\rho+\varepsilon),
\end{aligned}$$

implying, with a bit of algebra, that

$$t - T \geq (1+O(\varepsilon))\frac{t\varepsilon}{2(1-\rho)}.$$

We conclude that

$$\{x(t)>\rho+\varepsilon\}\subseteq\bigcup_{T=1}^{s}\mathcal{E}_T,\quad s=s(t):=t-\left\lceil\frac{t\varepsilon}{3(1-\rho)}\right\rceil.$$

Now let us fix a $T\in[1,s]$ and bound $\mathbb{P}(\mathcal{E}_T)$. The main idea of the proof is that, as long as $x(\tau)>\rho$, by Equation (4.5), the process $\{x(\tau)\}$ has a negative drift.

Let us say that we have a "failure" at step $i+1$ when $X(i+1)=X(i)+1$. On the event $\mathcal{E}_T$ we have $X(\tau)\geq\tau(\rho+\varepsilon/2)$, $\tau\in(T,t]$, intuitively meaning that there are many failures between steps $T+1$ and t, despite negativity of expected shift. And this should make the event $\mathcal{E}_T$ rather unlikely. To prove it rigorously, let ξ_j denote the indicator of the event $\{x(j-1)>\rho+\varepsilon/2$ and $X(j)=X(j-1)+1\}$. Let $\mathcal{Z}_T:=\xi_{T+2}+\cdots+\xi_t$. On the event $\mathcal{E}_T$, the sum $\mathcal{Z}_T$ counts the total number of upward unit jumps ($X(j)-X(j-1)=1$, $j\in[T+2,t]$) and therefore

$$X(T+1)+\mathcal{Z}_T-\big[(t-T)-\mathcal{Z}_T\big]=X(t)\geq t(\rho+\varepsilon).$$

Since $X(T+1)=X(T)+1\leq T(\rho+\varepsilon/2)+1$, we see that

$$\frac{Z_T-[(t-T)-Z_T]}{t-T}>\rho+\varepsilon,\quad Z(T):=1+\mathcal{Z}_T,$$

or equivalently,

$$Z_T>(t-T)(1+\rho+\varepsilon)/2.$$

On the other hand, for $\tau\geq T+1$, using (4.4) and conditioning on the full record (up to and including time τ), such that $x(\tau)>\rho+\varepsilon/2$, we have

$$\begin{aligned}\mathbb{P}(\xi_{\tau+1}=1|\circ)&\leq\mathbb{P}(X(\tau+1)=X(\tau)+1\,|\,\circ)\\&\leq\left(1-\left(\frac{m+\delta}{2m+\delta}\right)\left(\rho+\frac{\varepsilon}{2}\right)\right)^m+O\left(\tau^{-1}\right).\end{aligned}$$

Hence, the sequence $\{\xi_\tau\}$ is stochastically dominated by the sequence of *independent* Bernoulli random variables B_τ with parameters $\min\big(\mu+O(\tau^{-1}),1\big)$, where

$$\mu:=\left(1-\left(\frac{m+\delta}{2m+\delta}\right)\left(\rho+\frac{\varepsilon}{2}\right)\right)^m=\left(1-\left(\frac{m+\delta}{2m+\delta}\right)\rho\right)^m+O(\varepsilon).$$

Consequently, Z_T is stochastically dominated by $1+\sum_{j=T+2}^t B_j$, and

$$\lambda:=\sum_{j=T+2}^{t}\mathbb{E}[B_j]=\mu(t-T)+O(\log t).$$

Note that, since

$$\left(1-\left(\frac{m+\delta}{2m+\delta}\right)\rho\right)^m=\frac{\rho+1}{2},$$

we have

$$\begin{aligned}\frac{\rho+1+\varepsilon}{2} &= \frac{\rho+1}{2}\left(1+\frac{\varepsilon}{\rho+1}\right)\\ &= \left(1+\frac{\varepsilon}{\rho+1}\right)\left(1-\left(\frac{m+\delta}{2m+\delta}\right)\rho\right)^m \geq \left(1+\frac{\varepsilon}{2}\right)\mu.\end{aligned}$$

Thus, by Theorem 4.1, we have

$$\begin{aligned}\mathbb{P}(Z_T > (t-T)(1+\rho+\varepsilon)/2) &\leq \mathbb{P}\Big(1+B_{T+2}+\cdots+B_t > (t-T)(\rho+\varepsilon+1)/2\Big)\\ &\leq \mathbb{P}\left(1+B_{T+2}+\cdots+B_t > \left(1+\frac{\varepsilon}{2}\right)(t-T)\mu\right)\\ &\leq \exp\left(-\Theta(\varepsilon^2(t-T))\right) \leq e^{-\Theta(\log^3 t)}.\end{aligned}$$

Using the union bound on T finishes the proof of (4.2) and the theorem. □

Loops allowed everywhere. The above analysis is carried over to this more complicated case via an argument similar to the one for the descendant trees in the subsections 1.1. Here is a proof sketch. First, the counterpart of (4.4) is:

$$\begin{aligned}&\mathbb{P}(\{U(t)=0\}\cap\{\text{no loops at } t+1\}|\circ)\\ &= \Pi_m(t)\prod_{j=0}^{m-1}\left(\frac{2mt-Y(t)+\delta t-\delta X(t)+j}{2mt+2j+1+\delta t+(j+1)\,\delta/m}\right)\\ &\leq \Pi_m(t)\left[\left(1-\frac{m+\delta}{2m+\delta}\,x(t)\right)^m + O(t^{-1})\right]\\ &= \left(1-O(t^{-1})\right)\left[\left(1-\frac{m+\delta}{2m+\delta}x(t)\right)^m + O(t^{-1})\right]\\ &= \left(1-\frac{m+\delta}{2m+\delta}\,x(t)\right)^m + O(t^{-1});\end{aligned}$$

see (3.7) for $\Pi_m(t)$. Therefore we obtain again the equation (4.5). The rest of the proof remains the same.

Remark 4.2. Let $r = r_{m,\delta} := 1-\rho_{m,\delta}$, where $\rho_{m,\delta}$ is the unique root in $(0,1)$ of

$$h(z) = h_{m,\delta}(z) := 2\left[1-\left(\frac{m+\delta}{2m+\delta}\right)z\right]^m - z - 1.$$

Then, r is the unique root in $(0,1)$ of

$$f(z) = f_{m,\delta}(z) := 2-z-2\left(\frac{m}{2m+\delta}+\frac{m+\delta}{2m+\delta}z\right)^m.$$

Thus, by Theorem 2.4, we have

$$\liminf(1-x(t)) \geq r$$

almost surely, where $1 - x(t)$ is the fraction of the vertices in $L(t)$. See (2.3) for various r values when $\delta = 0$.

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Appendix

In order to show that the coupling described in Section 3 really works, we can compute the probability that the i-th edge of vertex w_{t+1} connects to vertex w_x in the coupling and compare it with the probability in (2.1). Let us denote by $\{G'_{m,\delta}(t)\}$ the process obtained by collapsing the vertices of $\{G_{1,\delta/m}(mt)\}$. Note that $(mt+i)$-th edge of the $\{G_{1,\delta/m}(mt)\}$-process becomes the i-th edge of w_{t+1} after the collapsing. Hence the i-th edge of vertex w_{t+1} connects to w_x ($x \leq t$) if and only if the $(mt+i)$-th edge of $\{G_{1,\delta/m}(mt)\}$-process connects v_{mt+i} with one of the vertices $v_{m(x-1)+1}, \ldots, v_{mx}$. Let us denote by $d_{mt+i-1}(v_y)$ the degree of v_y ($y \leq mt+i$) just before the $(mt+i)$-th edge of $\{G_{1,\delta/m}\}$-process is drawn. Also, let $D_{t,i-1}(w_x)$ denote the degree of w_x at the exact same time. Hence, by definition,

$$D_{t,i-1}(w_x) = \begin{cases} \displaystyle\sum_{y=mx-m+1}^{mx} d_{mt+i-1}(v_y), & x \leq t \\ \displaystyle\sum_{y=mt+1}^{mt+i} d_{mt+i-1}(v_y), & x = t+1. \end{cases}$$

By (2.2), for $x \leq t$, the probability that v_{mt+i} connects to one of the vertices $v_{m(x-1)+1}, \ldots, v_{mx}$ (equivalently, the probability that the i-th edge of w_{t+1}

connects to w_x) is

$$\frac{\sum_{y=mx-m+1}^{mx}\ (d_{mt+i-1}(v_y)+\delta/m)}{(2+\delta/m)(mt+i-1)+1+\delta/m}=\frac{\delta+\sum_{y=mx-m+1}^{mx}\ d_{mt+i-1}(v_y)}{\delta(t+i/m)+2mt+2i-1}$$
$$=\frac{\delta+D_{t,i-1}(w_x)}{\delta(t+i/m)+2mt+2i-1}.$$

Similarly, the probability that v_{mt+i} selects one of $v_{mt+1},\dots,v_{mt+i}$ (equivalently, the probability that the i-th edge of w_{t+1} is a loop) is

$$\frac{1+i\delta/m+\sum_{j=1}^{i-1} d_{mt+i-1}(v_{mt+j})}{\delta(t+i/m)+2mt+2i-1}=\frac{1+i\delta/m+D_{t,i-1}(w_{t+1})}{\delta(t+i/m)+2mt+2i-1}.$$

Note that the two probabilities above are the same as those in (2.1) if we replace $D_{t,i-1}(w_x)$ with $d_{t,i-1}(x)$. Moreover, the two processes, $\{G'_{m,\delta}(t)\}$ and $\{G_{m,\delta}(t)\}$ as defined by (2.1), both start with m loops on the first vertex, which implies $d_{1,0}(\cdot)=D_{1,0}(\cdot)$. This gives us that $\{G_{m,\delta}(t)\}$ and $\{G'_{m,\delta}(t)\}$ are equivalent processes, that is, at every stage, they produce the same random graph.

Department of Mathematics, Drexel University, Philadelphia, PA 19104
E-mail address: huseyin.acan@drexel.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213
E-mail address: alan@random.math.cmu.edu

Department of Mathematics, Ohio State University, Columbus, OH 43210
E-mail address: bgp@math.ohio-state.edu