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ABSTRACT

Bondy and Fan recently conjectured that if we associate non-negative real weights to the edges of a graph so that the sum of the edge weights is W then the graph contains a path whose weight is at least $\frac{2W}{n}$. We prove this conjecture.

The celebrated double cycle conjecture [4] states that for any two edge-connected graph G there exists a set S of cycles of G (not necessarily distinct) such that each edge of G appears in precisely two of the cycles of S. Bondy [1] has conjectured that, in fact, in any two edge-connected graph with n vertices there exists such a set containing at most $n-1$ cycles. A necessary condition for this to be true is that in any two edge-connected graph with n vertices and E edges, there exists a cycle of length $\frac{2E}{n-1}$. Actually, Erdős and Gallai [3] proved the following stronger theorem.

Theorem 1. Let G be a graph on n vertices and E edges. Then

(i) G contains a path of length $\frac{2E}{n}$, and

(ii) if $E \geq n$, then G contains a cycle of length at least $\frac{2E}{n-1}$.

We shall consider the generalization of Theorem 1 to weighted graphs. We weight a graph G by assigning to each edge e of G a non-negative real weight $w(e)$. For ease of exposition, if $xy \notin E(G)$, we say that $w(xy) = 0$. A graph which has been weighted is called a *weighted graph.*

The weight of any subgraph H of a weighted graph G, denoted $w(H)$, is simply the sum of the weights of the edges of H. The weighted degree of a vertex x in G, $d_w(x)$, is just $\sum_{xy \in E(G)} w(xy)$. The average weighted degree, $\Delta_w(G)$ (or just Δ_w) of a graph G with n vertices and weight W is just $\frac{2W}{n}$. Bondy and Fan proposed the following two conjectures which generalize parts (i) and (ii) of Theorem 1.

Conjecture 1. Let G be a weighted graph with n vertices and weight W. Then G contains a path of weight at least $\frac{2W}{n}$ (equivalently, G contains a path of weight at least $\Delta_w(G)$).

Conjecture 2. Let G be a weighted two edge-connected weighted graph with n vertices and weight W. Then G contains a cycle of weight $\frac{2W}{n-1}$.

We note that if Bondy's conjecture about double cycle covers is to hold, so must conjecture 1.

In this paper, we prove Conjecture 1. We shall restrict our attention to integer weights. Having proved the conjecture for integer weights, we can prove it for rational weights using a simple scaling argument. Using an easy approximation argument, we can generalize our result from the rationals to the reals. Thus, we prove the following theorem.

Theorem 2. Let G be a weighted graph such that all the edge weights are integers. If G has weight W then there is a path in G of weight $\frac{2W}{n}$.

Proof. By a *minimal* counterexample to Theorem 2 we mean a counterexample

(i) of smallest cardinality

(ii) with no more edges than any other counterexample of the same cardinality

(iii) with no greater weight than any other counterexample with the same number of vertices and edges.

We prove this theorem in two steps. First we show that any longest path in a minimal counterexample is Hamiltonian. Then, we show that any path of length $n-k-1$ in a minimal counterexample has length less than $\Delta_w(G)-k$. This implies that given a minimal counterexample G, we can obtain a counterexample of lower weight simply by subtracting one from the weight of every edge of G. However, this contradicts the minimality of G.

Lemma 1. If G is a minimal cardinality counterexample to Theorem 2 then every path of maximum weight in G is Hamiltonian.

Proof. We begin by stating the following facts which we will use throughout the proof of Theorem 2.

Fact 1. Every vertex in any minimum cardinality counterexample H to Theorem 2 has weighted degree at least $\frac{\Delta_w(H)}{2}$. ■

Proof. If some vertex x of a minimum cardinality counterexample, H', contained a vertex x such that $d_w(x) < \frac{\Delta_w(H)}{2}$, then $\Delta_w(H-x) > \Delta_w(H)$. Thus, by the minimality of H, $H-x$ contains a path P such that $w(P) > \Delta_w(H)$. P is also a path in H, a contradiction.

Now, assume Lemma 1 is false and let G be a minimal cardinality counterexample to Theorem 2 which contains a non-Hamiltonian path of maximum weight. Let $P = \{x = p_0, p_1, p_2, \ldots, p_k = y\}$ be such a path.

Fact 2. $\sum_{z \notin P} w(xz) = 0$ and $\sum_{z \notin P} w(yz) = 0$.

Proof. P is of maximum weight. ■

Fact 3. For $1 \leq i \leq k$, if $xp_i \in E(G)$ then $w(xp_i) \leq w(p_{i-1}p_i)$.

Proof. If $i = 1$ xp_i is $p_{i-1}p_i$. Otherwise, consider the path $P' = \{p_{i-1}, p_{i-2}, \ldots, x, p_i, p_{i+1}, \ldots, y\}$. Since $w(P') \leq w(P)$, $w(xp_i) \leq w(p_{i-1}p_i)$. ■

Fact 4. For $0 \leq i \leq k-1$, if $yp_i \in E(G)$ then $w(yp_i) \leq w(p_i p_{i+1})$.

Proof. Analogous to the proof of Fact 3. ■

Fact 5. $\sum_{p \in P} w(xp) + \sum_{p \in P} w(yp) \geq w(P)$.

Proof. We know that since G is a counterexample to Theorem 1, $w(P) < \Delta(G)$. Thus, Fact 5 follows from Facts 1 and 2. ■

Fact 6. For some edge $p_i p_{i+1}$ of P, $xp_{i+1} \in E(G)$, $yp_i \in E(G)$, and $w(xp_{i+1}) + w(yp_i) > w(p_i p_{i+1})$.

Proof. Follows from Facts 3, 4, and 5 by summing over the edges of P.

Let $p_i p_{i+1}$ be an edge of P such that $xp_{i+1} \in E(G)$, $yp_i \in E(G)$ and $w(xp_{i+1}) + w(yp_i) > w(p_i p_{i+1})$. Let C be the cycle, $x, p_{i+1}, p_{i+2}, \ldots, y, p_i, p_{i-1}, \ldots, x$. Note that $E(C) = E(P) - p_i p_{i+1} + xp_{i+1} + yp_i$. Thus $w(C) > w(P)$. Now, consider a component A of $G - V(P)$. Let P' be a path of maximum weight in A with endpoints a and b. If either a or b sees a vertex of P then we will attempt to extend P to a heavier path by attaching P'. Otherwise A contains many long paths and we arrive at a contradiction by appending one of these paths to C. ■

Case 1. $\{N(a) \cup N(b)\} \cap P \neq \emptyset$.

Fact 7. For an edge $p_j p_{j+1}$ of P, $w(ap_j) + w(bp_{j+1}) \leq w(p_j p_{j+1})$. Furthermore, if $ap_j \in E(G)$ or $bp_{j+1} \in E(G)$ then $w(ap_j) + w(bp_{j+1}) \leq w(p_j p_{j+1}) - w(P')$.

Proof. If neither ap_j nor bp_{j+1} is an edge of G then there is nothing to prove. If both ap_j and bp_{j+1} are edges of G then consider the path P'' such that $E(P'') = E(P) - p_j p_{j+1} \cup \{ap_j\} \cup E(P') \cup \{bp_{j+1}\}$. Since P is of maximum weight, $w(P) \geq w(P'')$.) Thus, $w(ap_j) + w(bp_{j+1}) \leq w(p_j p_{j+1}) - w(P')$ as required.

If $ap_j \in E(G)$ and $bp_{j+1} \notin E(G)$ then we shall focus our attention on the cycle C. If $i \neq j$ (that is $p_j p_{j+1}$ is not the edge of P we removed when creating C), then $p_j p_{j+1} \in E(C)$. Consider the path P'' such that $E(P'') = E(C) - p_j p_{j+1} + ap_j + E(P')$. Since P has maximum weight, $w(P) \geq w(P'')$. Thus, since $w(C) \geq w(P)$, we have $w(ap_j) \leq w(p_j p_{j+1}) - w(P')$ as required. If $i = j$ then consider the path P'' such that $E(P'') = E(C) - yp_j + ap_j + E(P')$. As before, $w(C) \geq w(P)$ and $w(P) \geq w(P'')$. Thus, $w(ap_j) \leq w(yp_j) - w(P')$. Now, by Fact 4, $w(ap_j) \leq w(p_j p_{j+1}) - w(P')$ as required. An analogous proof yields the required result if $ap_j \notin E(G)$ but $bp_{j+1} \in E(G)$. ■

Fact 8. If $\{N(a) \cup N(b)\} \cap P \neq \emptyset$ then $\sum_{y \in P} w(ay) + \sum_{y \in P} w(by) \leq w(P) - 2w(P')$.

Proof. Note that $w(bx) = 0$ and $w(ay) = 0$ from Fact 2. Now, from Fact 7, $\sum_{i=0}^{k-1} (w(ap_i) + w(bp_{i+1})) \leq \sum_{i=0}^{k-1} w(p_i p_{i+1}) = w(P)$. Thus, $\sum_{y \in P} w(ay) + \sum_{y \in P} w(by) \leq w(P)$. In fact, if $|N_{(a)} \cap P| + |N(b) \cap P| \geq 3$ then from Fact 7, $\sum_{y \in P} w(ay) + \sum_{y \in P} w(py) \leq w(P) - 2w(P')$ as required. Furthermore, if $|N(a) \cap P| + |N(b) \cap P| = 2$ then the required result again follows from Fact 7, although we may have to interchange the roles of a and b. If $|N(a) \cap P| + |N(b) \cap P| = 1$ then we can again use Fact 7 to obtain the required result. To wit, if $N(a) \cap P = p_j$ and $N(b) \cap P = \emptyset$ then, from Fact 7, we have $w(p_j p_{j+1}) \geq w(ap_j) + w(P')$. Similarly, $w(p_{j-1} p_j) \geq w(ap_j) + w(P')$. Summing we obtain $w(P) \geq w(p_{j-1} p_j) + w(p_j p_{j+1}) \geq 2w(ap_j) + 2w(P')$. Thus, $\sum_{y \in P} w(ay) + \sum_{y \in P} w(by) \leq w(P) - 2w(P')$ as required. ■

Now, we consider the weighted degrees of a and b in G. Since A is a component of $G-P$, $d_w(a) + d_w(b) = \sum_{z \in P} w(az) + \sum_{z \in A} w(az) + \sum_{z \in P} w(bz) + \sum_{z \in A} w(bz)$. Since, P' is a longest path in A we know from Fact 2 that $\sum_{z \in A} w(az) = \sum_{z \in P'} w(az)$ and $\sum_{z \in A} w(bz) = \sum_{z \in P'} w(bz)$. Furthermore, by Fact 4, $\sum_{z \in P'} w(az) \leq w(P')$ and $\sum_{y \in P'} w(bz) \leq w(P')$. It follows that $d_w(a) + d_w(b) \leq \sum_{y \in P} w(ay) + \sum_{y \in P} w(by) + 2w(P')$. Thus, by Fact 8, $d_w(a) + d_w(b) \leq w(P)$. However, by Fact 1, $d_w(a) + d_w(b) \geq \Delta_w(G)$. This contradicts our assumption that G is a counterexample to Theorem 2.

Case 2. $\{N(a) \cup N(b)\} \cap P = \emptyset$.

In this case, we show that every edge of A is the endpoint of a fairly heavy path. This implies that there are few edges between A and P which lead us to a contradiction.

Fact 9. $w(P') \geq \dfrac{\Delta_w(G)}{2}$.

Proof. By assumption $\sum_{z \notin A} w(az) = 0$. Furthermore, since P' has maximum weight in A, by Fact 2, $\sum_{z \in A-P'} w(az) = 0$. Thus, $d_w(a) = \sum_{y \in P'} w(az)$. Now, by Fact 4, $\sum_{z \in P'} w(az) \leq w(P')$. Thus $d_w(a) \leq d_w(P')$ and by Fact 1, $w(P') \geq \dfrac{\Delta_w(G)}{2}$. ■

Fact 10. Every vertex z in A is the endpoint of a path P_z of length at least $\dfrac{\Delta_w(G)}{4}$.

Proof. Since A is connected for any $z \in A$ there is a z to a path P'' in A. Let p_i' be the first vertex of P' on this path. We can create a new path P_1 by following P'' from z to p_i' and then following P' from p_i' to b. Similarly, we can create a new path P_2 by following P'' from z to p' and then going from p' to a along P'. Clearly $w(P_1) + w(P_2) \geq w(P')$. Thus, by Fact 9, one of P_1 or P_2 has weight at least $\dfrac{\Delta_w(G)}{4}$.
■

Fact 11. At most two vertices of P are adjacent to elements of A. Furthermore, if two vertices of P are adjacent to elements of A then they are consecutive vertices of P.

Proof. Let u be a vertex of P which sees some $z \in A$. Let u^- and u^+ be the neighbours preceding and following u on the cycle C. Form paths P_1 and P_2 such that $E(P_1) = E(C) - uu^- + uz + P_z$ and $E(P_2) = E(C) - uu^+ + uz + P_z$. By the maximality of P, $w(P) \geq w(P_1)$ and $w(P) \geq w(P_2)$. By Fact 10, this implies $w(uu^-) \geq \frac{\Delta_w(G)}{4}$ and $w(uu^+) \geq \frac{\Delta_w(G)}{4}$. Also, clearly $u \neq x$, $u \neq y$. Now, by the construction of C, we know that both path edges incident to u have weight at least $\frac{\Delta_w(G)}{4}$. (In fact, if $v \neq p_i$ these are precisely the edges uu^- and uu^+). Since $w(P) \leq \Delta_w(G)$, we are done. ■

Now let $B = \{v \,|\, v \in P, vz \in E(G) \text{ for some } z \in A\}$. We know that $|B| = 1$ or 2. We shall consider these two possibilities separately.

Case 1. $|B| = 1$.

Let p_j be the unique element of B. Clearly $j \neq 0$ and $j \neq k$. Let z be a vertex of A such that $w(p_j z) = \max_{u \in A}(w(p_j u))$. Let $s = w(p_j z)$.

Fact 12. There is a path P_1 in $p_j \cup A$ which has p_j as an endpoint and has weight at least $\frac{\Delta_w(G)}{2}$.

Proof. If $s \geq \frac{\Delta_w(G)}{2}$ then let $P_1 = p_j z$ and we are done. Otherwise $w(G) \leq w(G-A) + w(A) + |A|s$. By the minimality of G, $w(G-A) \leq \frac{\Delta_w(G)}{2} \cdot (n-|A|)$ and thus $w(A) \geq (\frac{\Delta_w(G)}{2} - s) \cdot |A|$. Now, applying the minimality of G, we see that A contains a path P'' of weight at least $\Delta_w(G) - 2s$. Since A is connected there is a path from z to P''. It follows that z is the endpoint of a path P_z' in A which has weight at least $\frac{w(P'')}{2} = \frac{\Delta_w(G)}{2} - s$. Now set $P_1 = p_j z \cup P_z'$ and we are done. ■

Now, we can form a path from $C \cup P_1$ by deleting either of the edges of C incident to p_j. Since $w(P_1) \geq \frac{\Delta_w(G)}{2}$, it follows from the maximality of P that both edges of C incident to p_j have weight at least $\frac{\Delta_w(G)}{2}$. This implies that both ends of P incident to p_j have weight at least $\frac{\Delta_w(G)}{2}$. But, now we have $w(P) \geq \Delta_w(G)$, a contradiction.

Case 2. $|B| = 2$.

In this case $B = \{p_j, p_{j+1}\}$ with $j \neq 0$, $j \neq k-1$. Furthermore, as we saw in the proof of Fact 11, all of the three edges $p_{j-1}p_j$, p_jp_{j+1}, and $p_{j+1}p_{j+2}$ have weight at least $\frac{\Delta_w(G)}{4}$. We assume that $\max_{v \in A}(w(vp_j)) \geq \max_{v \in A}(w(vp_{j+1}))$. (If necessary we reverse the numbering on P.) Then let $z \in A$ be such that $w(p_jz) = \max_{v \in A}(w(p_jv))$ and let $s = w(p_jz)$.

Fact 13. There is a path P in $p_j \cup A$ which has p_j as an endpoint and has weight at least $\frac{3\Delta_w(G)}{8}$.

Proof. If $s \geq \frac{\Delta_w(G)}{8}$ then take $P_1 = p_jz \cup P_z$ and we are done. Otherwise, $w(G) \leq w(G-A) + w(A) + |A| \cdot 2s$. By the minimality of G, $w(G-A) \leq \frac{\Delta_w(G)}{2}(n - |A|)$. Thus, $w(A) \geq (\frac{\Delta_w(G)}{2} - 2s) \cdot |A|$. Now, by the minimality of G, A contains a path P'' of weight at least $\Delta_w(G) - 4s$. As before, we can walk from z to P'' and then to one of the endpoints of P''. This gives a path p_z' in A with z as an endpoint such that $w(p_z') \geq \frac{\Delta_w(G)}{2} - 2s$. Now, setting $P_1 = P_z' + p_jz$, we see that $w(P_1) \geq \frac{\Delta_w(G)}{2} - s \geq \frac{3\Delta_w(G)}{8}$ as required. □

Now, we can form a path from $C \cup P_1$ by deleting either edge of C incident to p_j. It follows that both these edges have weight at least $\frac{3\Delta_w(G)}{8}$. Thus, both $p_{j-1}p_j$ and p_jp_{j+1} have weight at least $\frac{3\Delta_w(G)}{8}$. But, this implies that

$$w(P) \geq w(p_{j-1}p_j) + w(P_jp_{j+1}) + w(p_{j+1}p_{j+2}) \geq \frac{3\Delta_w(G)}{8} + \frac{3\Delta_w(G)}{8} + \frac{\Delta_w(G)}{4}$$

$= \Delta_w(G)$, a contradiction.

This completes the proof of Lemma 1. By Lemma 1, all the heaviest paths in any minimum cardinality counterexample are Hamiltonian. We now show that in any minimal cardinality counterexample, the length of a path determines a bound on its weight.

Lemma 2. Let G be a minimal counterexample to Theorem 2. Let H be the weight of a heaviest path in G. Any path of weight $H-k$ has at least $n-k-1$ edges.

Proof. We prove Lemma 2 by induction on k. Lemma 1 simply says that Lemma 2 is true for $k = 1$. Assume Lemma 2 is false and let t be the smallest integer for which it fails. Now, let G be a minimal counterexample to Theorem 2 whose heaviest path has weight H. Furthermore, we insist that G contains a path P of weight $H-t$ which has length less than $n-t-1$. Let $P = \{x = p_a, p_1, \ldots, p_2 = y\}$ be such a path.

Fact 14. No edge of G has weight 0.

Proof. If e is an edge of G with weight 0 then $G-e$ is also a counterexample to Theorem 2. This contradicts the minimality of G. ∎

Fact 15. Every vertex of G has degree greater than $\frac{n}{2}$.

Proof. Assume some vertex z of G has degree at most $\frac{n}{2}$. Then, subtract 1 from each edge of G incident to z thereby obtaining a new weighted graph G'. Now $w(G') > w(G) - \frac{n}{2}$. Furthermore, since every heaviest path in G is hamiltonian, the heaviest path in G' has weight at most $H-1$. This contradicts the minimality of G. ∎

Fact 16. $N(x) \cup N(y) \subseteq V(P)$.

Proof. Otherwise we could extend P, contradicting the minimality of t. ∎

Fact 17. $|V(P)| > \frac{n}{2} + 1$.

Proof. Follows from Facts 15 and 16. ∎

Fact 18. We can find a cycle C on $V(P)$ such that $w(C) > w(P)$ and such that all but at most one of the edges of P are edges of C. Furthermore, if some edge p_jp_{j+1} of P is not an edge of P then $E(C) - E(P) = \{p_jy, p_{j+1}x\}$ where $w(p_jp_{j+1}) \geq w(p_jy)$ and $w(p_jp_{j+1}) \geq w(p_{j+1}w)$.

Proof. Analogous to the proof that such a cycle existed in Lemma 1. ∎

Now, let $A = G - V(P)$.

Fact 19. For $v \in A$, $|N(v) \cap P| \geq 3$.

Proof. Follows from Facts 15 and 17. ∎

Fact 20. For $v \in A$, $\sum_{z \in P} w(vz) \leq \frac{w(P)}{2}$.

Proof. Let p_jp_{j+1} be an edge of P. We claim that $w(ap_j) + w(ap_{j+1}) \leq w(p_jp_{j+1})$. Otherwise, if both ap_j and ap_{j+1} are edges of G we can construct a new path $P' = P - p_jp_{j+1} + ap_j + ap_{j+1}$. However, in this case $w(P') \geq w(P) + 1$ and $|P'| = |P| + 1$. This contradicts the minimality of t. If only one of ap_j or ap_{j+1} is an edge of P, we can obtain the same result by deleting the edge of C corresponding to p_jp_{j+1}. Now, to obtain Fact 20 we simply sum over the edges of P. ∎

Now, let $P' = \{a = p'_1, p'_2, \ldots, p'_k = b\}$ be a longest path in A with endpoints a and b.

Fact 21. $w(P') \geq 2t$.

Proof. $w(G) = w(G-A) + w(A) + \sum_{\substack{v \in A \\ z \in P}} w(vz)$. By Fact 20, $\sum_{\substack{v \in A \\ z \in P}} w(vz) \leq (\frac{\Delta_w(G) - t}{2}) \cdot |A|$. By the minimality of t, $w(G-A) \leq (\frac{\Delta_w(G) - t}{2})(n - |A|)$. It follows that $w(A) \geq t\,|A|$. Thus $w(P') \geq 2t$. ∎

Fact 22. For an edge p_jp_{j+1} of P, $w(p_jp_{j+1}) \geq w(ap_j) + w(bp_{j+1}) + 1$. Furthermore if either ap_j or bp_{j+1} is an edge of G then $w(p_jp_{j+1}) \geq w(ap_j) + w(bp_{j+1}) + w(P') - t$.

Proof. If neither ap_j or bp_{j+1} is an edge of G there is nothing to prove. If both ap_j and bp_{j+1} are edges of G then consider the cycle P'' with $E(P'') = E(P) - p_jp_{j+1} + ap_j + E(P') + bp_{j+1}$. Since $w(P'') \leq H$ and $w(P) = H-t$, $w(P'') < w(P) + t$. Thus, $w(p_jp_{j+1}) \leq w(ap_j) + w(bp_{j+1}) + w(P')-t$. If only one of ap_j or bp_{j+1} is an edge of G we obtain the same result by considering the path obtained from C and P' as in the proof of Fact 7.

Fact 23. $\sum_{z \in P} w(az) + \sum_{z \in P} w(bz) \leq w(P) - 2w(P') + 2t - (|P|-2)$.

Proof. Follows from Facts 19 and 22 by summing over the edges of P. ■

Fact 24. $\sum_{z \in A} w(az) + \sum_{z \in A} w(bz) \leq 2w(P')$.

Proof. Note first that $\sum_{z \in A} w(az) = \sum_{z \in P'} w(az)$ and $\sum_{z \in A} w(bz) = \sum_{z \in P'} w(bz)$ by the maximality of P'. Also by the maximality of P', $w(ap'_{j+1}) \leq w(p'_jp'_{j+1})$ and $w(bp_j) \leq w(p'_jp'_{j+1})$ for any edge $p'_jp'_{j+1}$ of P'. ■

Now, Facts 23 and 24 imply that $d_w(a) + d_w(b) \leq w(P) + 2t - (|P|-2)$. Thus, $d_w(a) + d_w(b) \leq \Delta_w(G) + t - (|P|-2)$. But $|P| > \frac{n}{2} + 1$ and $t \leq |A| \leq \frac{n}{2} - 1$. This implies that $d_w(a) + d_w(b) \leq \Delta_w(G)$, contradicting Fact 1.

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