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ON THE COMPLEXITY OF PARTITIONING GRAPHS INTO CONNECTED SUBGRAPHS

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This paper is mainly concerned with the computational complexity of determining whether or not the vertices of a graph can be partitioned into equal sized subsets so that each subset induces a particular type of graph. Many of the NP-completeness results are for planar graphs. These are proved using a planar version of 3-dimensional matching.

1. Introduction

This paper is mainly concerned with the computational complexity of the problem of determining whether or not the edges or vertices of a graph can be partitioned into equal sized subsets so that each subset induces a particular type of graph.

Notation

For graph $G=(V,E)$ we denote induced subgraphs as follows:

$$\text{for } S\subseteq V \quad G[S]=(S,E_S) \quad \text{where } E_S=\{\{v,w\}\in E:\{v,w\}\subseteq S\},$$

$$\text{for } S\subseteq E \quad G(S)=(V_S,S) \quad \text{where } V_S=\bigcup_{e\in S} e.$$

For positive integer k and finite set X we define a k-partition of X as a partition $X=X_1\cup X_2\cup\cdots\cup X_p$ where $|X_i|=k$ for $i=1,2,\ldots,p$. For a graph property π we define a π-k-partition of the vertices or edges to be one in which each subset induces a graph with property π. Thus for example a *connected-k*-partition of V or E is a k-partition of V or E for which each subset induces a connected subgraph. The abbreviation for the problem of deciding whether a graph has a π-k-partition of its vertices (resp. edges) will be V$k(\pi)$ (resp. E$k(\pi)$).

Planar V$k(\pi)$ denotes V$k(\pi)$ restricted to planar graphs etc.

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Vertex set partitioning

A connected 2-partitioning of V corresponds to a perfect matching and so one can decide in polynomial time whether or not one exists (Edmonds [2]).

For $k \geq 3$ it has been known for some time that Vk(connected) is NP-complete (Garey & Johnson [3], Hadlock [5], Kirkpatrick & Hell [10]). We are able to show that Planar Vk(connected) is NP-complete. The main tool in the proof is the NP-completeness of Planar 3DM – see Section 2 and Dyer & Frieze [1].

We also prove the NP-completeness of V(n/k)(connected) for $k \geq 2$ fixed ($n = |V|$). The complexity of this problem was left open in Perl & Schach [14] who considered weighted generalisations of this problem when G is a tree. They refer to applications of this problem in information and library processing [15] and paging and overlaying [16].

It is of interest to note the following result of Gyori [4] and Lovász [11] which provides sufficient conditions for the existence of connected partitions in terms of connectivity.

Theorem 1.1. *Let $G = (V, E)$ be a k-connected graph. Let $n = |V|$, $v_1, v_2, \ldots, v_k \in V$ and let $n_1, n_2, \ldots, n_k$ be positive integers satisfying $n_1 + n_2 + \cdots + n_k = n$. Then there exists a partition of V into $V_1, V_2, \ldots, V_k$ satisfying $v_i \in V_i$, $|V_i| = n_i$ and $G[V_i]$ is connected for $i = 1, 2, \ldots, k$.*

To balance the NP-completeness results we have looked for classes of graphs for which these problems are polynomially solvable. Trees are an obvious case but we have also managed to prove some results for series-parallel graphs.

All of the above results plus some related ones are discussed in Section 2.

Edge set partitioning

It is straightforward to specialise Theorem 1.1 to line-graphs in order to have a result on connected edge-set partitioning. In addition Junger, Reinelt & Pulleyblank [7] proved

Theorem 1.2. (a) *If G is k-edge connected, then G has a connected $(k+1)$-partition but not necessarily a connected $(k+2)$-partition for $k = 1, 2, 3$.*

(b) *If G is 4-edge connected, then G has a connected k-partition for all k.*

The question of the complexity of Ek(connected) was left open in the above paper. We have been able to prove the NP-completeness of this problem for planar graphs when $k \geq 3$ is fixed and for general graphs when m/k is fixed, $m = |E|$.

These results are discussed in Section 3.

2. Vertex set partitioning

We first define *Planar 3-Dimensional Matching* (Planar 3DM).

Instance I. Disjoint sets R, B, Y with $|R| = |B| = |Y| = m$ and a set of triples $T \subseteq R \times B \times Y$ such that the bipartite graph $G_I = (T \cup R \cup B \cup Y, E_I)$ is *planar*, where (see Fig. 1)

$$E_I = \bigcup_{t=(r,b,y)\in T} \{\{r,t\},\{b,t\},\{y,t\}\}.$$

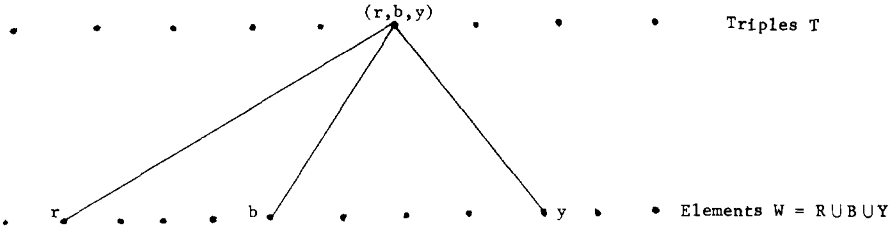

Fig. 1.

Question. Does there exist a matching $M \subseteq T$, i.e. $|M| = m$ and each element of $W = R \cup B \cup Y$ occurs in exactly one triple of M?

It is well known (e.g. Karp [9]), that 3DM is NP-complete when the restriction that G_I is planar is removed. The NP-completeness of Planar 3DM is proved in Dyer & Frieze [1] using Lichtensteins result on Planar 3SAT [8].

It is important to note that Planar 3DM is NP-complete under the restriction

$$\text{each element of } W \text{ appears in 2 or 3 triples only.} \tag{2.1}$$

Theorem 2.1. *Each of the following problems is NP-complete for any fixed* $k \geq 3$.

(a) Planar bipartite Vk(connected).
(b) Planar bipartite Vk(tree).
(c) Planar bipartite Vk(path).

Proof. By transformation from Planar 3DM – see Fig. 2.

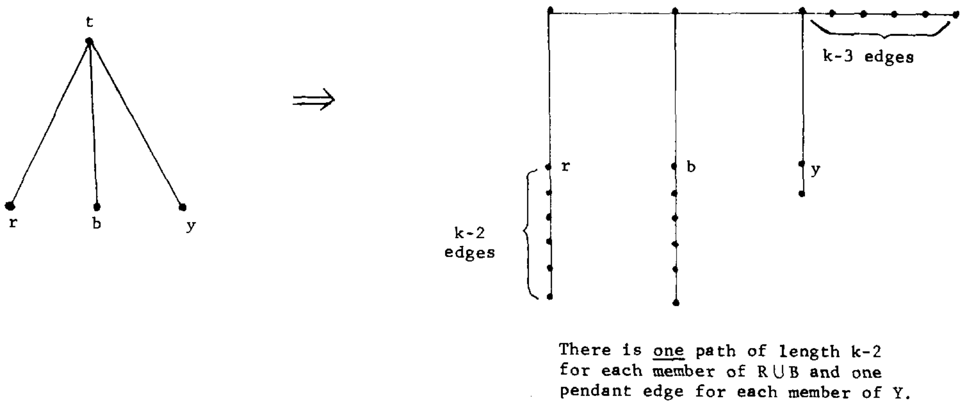

Fig. 2.

The resulting graph is clearly bipartite planar if G_I is. This construction is only new in the sense that we can assume G_I is planar. Therefore we only state

(i) I has a matching implies G has a k-partition into paths.

(ii) G has a connected k-partition implies I has a matching. □

We note that $|E(G)| \leq (3k-1)|T|$. We can therefore allow k to vary with the size of our given graph G, as long as $k = O(n^{1-\varepsilon})$ and $\varepsilon > 0$ ($n = |V(G)|$ as usual) and the conclusions of Theorem 2.1 will still hold.

We next consider the case where n/k rather than k is bounded.

Theorem 2.2. *If* $n_k = n/k$, *then* Bipartite Vn_k(connected) *is NP-complete for any fixed* $k \geq 2$.

Proof. We prove this first for $k=2$ and then indicate the simple modifications needed for arbitrary k. The reduction is again from 3DM but this time our transformation does not preserve planarity.

Given an instance of 3DM we define a graph $G=(V,E)$ as follows (see Fig. 3).

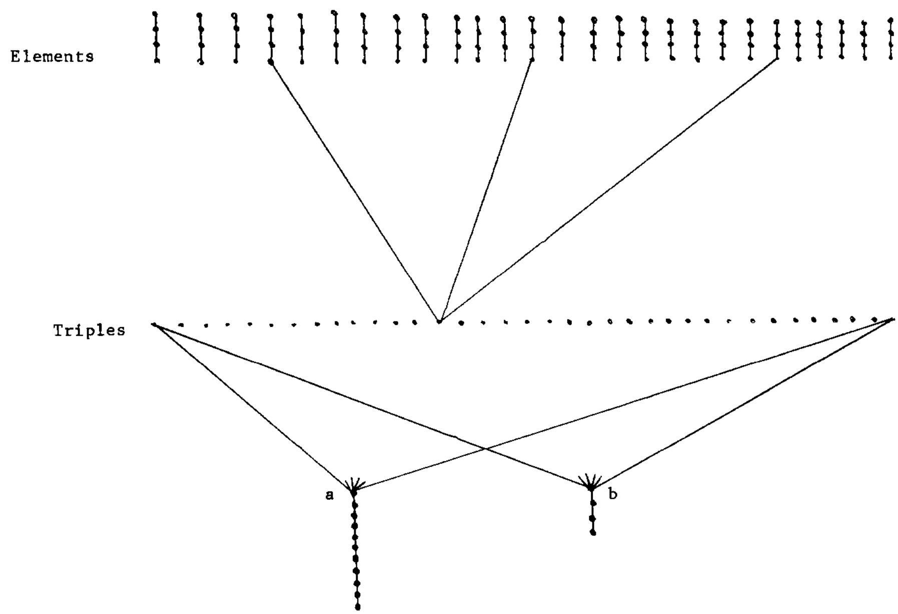

Fig. 3.

Let $W^+ = W \cup \{a,b\}$,

$$n_a = (3m+1)m^3 + 5m - |T| \quad \text{and} \quad n_b = m^3.$$

Let $n_\sigma = n_b$ for $\sigma \in W$.

$$V = W^+ \cup T \cup \bigcup_{\sigma \in W^+} P_\sigma \quad \text{where } P_\sigma = \{(\sigma, t) : t = 1, \ldots, n_\sigma\} \quad \text{for } \sigma \in W^+.$$

$$E=\{\{\sigma,t\}:\sigma\in\{a,b\}\text{ and }t\in T\}\cup\bigcup_{\sigma\in W^+}E_\sigma\cup E_T$$

where

$$E_\sigma=\{\{\sigma,(\sigma,1)\}\}\cup\{\{(\sigma,t),(\sigma,t+1)\}:1\le t<n_\sigma\}\quad\text{for }\sigma\in W^+,$$
$$E_T=\bigcup_{t\in T}E_t\quad\text{where if }t=\{x,y,z\}\quad E_t=\{\{t,x\},\{t,y\},\{t,z\}\}.$$

Note that

$$n=|V|=2+3m+|T|+n_a+(3m+1)n_b=2(n_a+1+|T|-m).$$

We show that G can be partitioned into 2 connected subgraphs $G[S]$, $G[\bar{S}]$, $|S|=n/2$ if and only if T contains a matching.

Suppose first that T contains a matching M. Let $S=\{a\}\cup P_a\cup(T-M)$. It is straightforward to check that $|S|+n/2$ and that $G[S]$, $G[\bar{S}]$ are both connected (are both in fact trees).

Conversely if such an S exists we can assume $a\in S$. It follows that $P_a\subseteq S$. Now $|S-(P_a\cup\{a\})|=|T|-m<n_b$ and as $\sigma\in S\cap(W\cup\{b\})$ implies $P_\sigma\subseteq S$ we have $S\cap(W\cup\{b\})=\emptyset$. Thus $S-(P_a\cup\{a\})\subseteq T$. Let $M=T-S$. Now $|M|=m$ and M must be a matching as $W\subseteq\bar{S}$ means that M 'covers' W. □

To deal with the case in which G must be partitioned into k connected subgraphs we replace n_a above by $n_a+(k-2)n/2$ and note that $|V|=kn/2$ now. Any partition into k connected subgraphs of equal size must consist of 2 subgraphs of the original G plus $k-2$ subpaths of P_a of length $n/2$.

It is again possible to vary the value of k with the size of G as long as $k=O(n^{1-\varepsilon})$. It follows from Theorem 2.1 and 2.2 that splitting a graph into k equal sized connected subgraphs is a hard problem for all $k\le n/3$.

The question of whether it is still a hard problem when k is fixed and G is planar is left open and on present evidence we conjecture that the problem is NP-complete.

We note that the proof of Theorem 2.2 also supports a proof that $\mathrm{V}n_k$(tree) is NP-complete.

We note also that McDiarmid & Papacostas [12] show that deciding whether the vertices of a planar graph can be partitioned into 2 sets, of arbitrary size, each inducing a tree, is NP-complete.

It is important to look for restricted classes of graphs on which our problems are polynomially solvable. We consider $\mathrm{V}k$(connected) when n/k is fixed. This is known to be polynomially solvable on trees [14] and we have conjectured that it is NP-complete for planar graphs. We show, in outline, that it is polynomially solvable for *series-parallel* graphs – see for example Valdes, Tarjan & Lawler [17].

Series-parallel graphs (SPG's) have the following inductive definition:

Basis: An edge $\{a,b\}$ is an SPG with *source* $\sigma=a$ and *sink* $\tau=b$.

Suppose now that G_1, G_2 are SPG's with disjoint vertex sets.

Series-construction. We define $G=G_1\circ G_2$ by identifying $\tau_1=\tau(G_1)$ with $\sigma_2=\sigma(G_2)$ and defining $\sigma=\sigma(G)=\sigma_1=\sigma(G_1)$ and $\tau=\tau(G)=\tau_2=\tau(G_2)$. See Fig. 4.

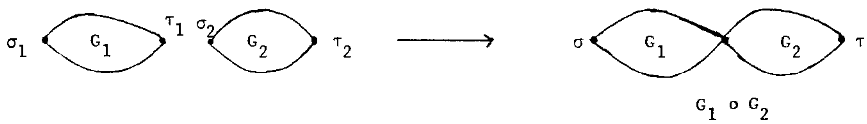

Fig. 4.

Parallel-Construction. We define $G=G_1 \parallel G_2$ by identifying $\sigma(G_1)$ with $\sigma(G_2)$, $\tau(G_1)$ with $\tau(G_2)$ and defining $\sigma(G)=\sigma(G_1)$ and $\tau(G)=\tau(G_1)$. See Fig. 5.

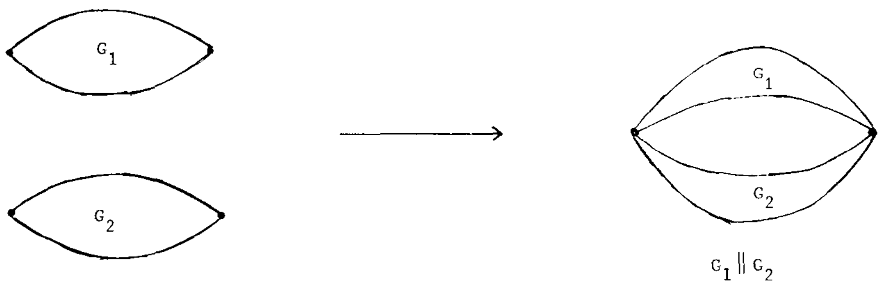

Fig. 5.

We shall use a dynamic programming approach based on the inductive construction of the given graph G. This can be found in linear time [17] if it is not given a priori.

For simplicity we consider the problem of deciding whether the vertices V of a given SPG G can be partitioned into 2 sets V_1, V_2 of equal size such that $G[V_1]$ and $G[V_2]$ are both connected.

For an SPG G define

$$X(G)=\{(a_1,a_2,a_3):\ \exists \text{ partition of } V(G) \text{ into } V_1, V_2, V_3 \text{ such that}$$
$$\text{(i) } |V_i|=a_i \text{ for } i=1,2,3 \text{ and } G[V_i] \text{ is connected,}$$
$$\text{(ii) } \sigma(G)\in V_1,$$
$$\text{(iii) } a_3=0\to\tau(G)\in V_1 \text{ and } a_3\neq 0\to\tau(G)\in V_3\}.$$

Note then that G has a connected $(n/2)$-partition if and only if

$$\{(n/2,n/2,0),(n/2,0,n/2)\}\cap X(G)\neq\emptyset.$$

The reader can easily check that $X(G)$ can be computed from $X(G_1)$ and $X(G_2)$ given that $G=G_1\circ G_2$ or $G_1\parallel G_2$. There is no room to go into details. The time taken is $O(|X(G_1)|\,|X(G_2)|)$ making an overall time bound of $O(n^5)$ where $n=|V(G)|$.

The idea generalises in several ways, i.e. splitting into an arbitrary *fixed* number of sets; putting weights on vertices and trying to find partitions with the weights of the sets being equal, here we only have a pseudo-polynomial time algorithm; and finally to partitioning graphs whose 2-connected components are series-parallel.

3. Edge set partitioning

We consider first the problem of finding a connected k-partition of the set of edges of a graph. For $k=2$ this can be solved using matching techniques – see [7]. However for $k \geq 3$ we have

Theorem 3.1. Planar bipartite Ek(connected) *is NP-complete for any fixed* $k \geq 3$.

Proof. We consider first the case $k \neq 4$. Let $G_1=(V_1, E_1)$ be the bipartite planar graph associated with an instance of Planar 3DM as defined in Section 2. Attach to each element which appears in d triples, say, a set of $(d-1)$ independent paths of $(k-1)$ edges. Additionally attach to each element in R a $(k-3)$ path. See Fig. 6.

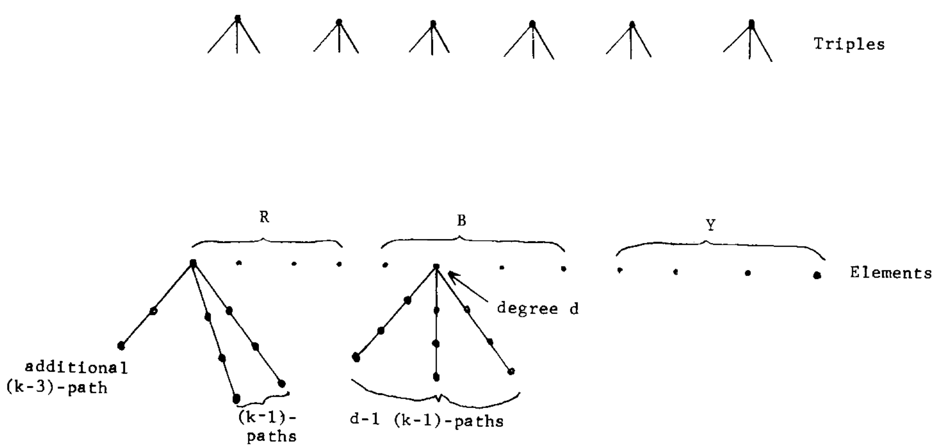

Fig. 6.

This construction gives us a graph $G=(V,E)$ which is clearly planar and bipartite, and constructible in polynomial time from G_1. We now claim that G decomposes into k-edge components if and only if the Planar 3DM instance contains a matching. First we note that for each element, the $(d-1)$ $(k-1)$-paths must each be in a different component. Thus they must form a component with some other edge incident with the element. Removing these components from G, to leave G_2 say, gives exactly one edge incident to each B, Y vertex, and an edge and a $(k-3)$-path incident to each R vertex. Because $k \neq 4$ this must be the $(k-3)$-path which we attached, since then $(k-1)+(k-3) \neq k$. Thus each R element is incident in G_2 with a unique triple and an isolated $(k-3)$-path and edge incident to an R vertex, and an edge incident to each of a B and a Y vertex. This decomposition induces a matching, since each component only contains edges incident to one triple. This argument can be reversed to show that any matching induces a k-edge decomposition of G.

The case $k=4$ must be considered separately. The above argument breaks down since the $(k-3)$-path is then a single edge, and can be combined with one of the $(k-1)$-paths. To obtain this case we use the same construction for the B, Y elements, i.e. we attach $(d-1)$ 3-paths, where d is the degree of the vertex in G_1. However a more complicated construction is required for the R vertices. Recall that we may assume that its degree d in G_1 is 2 or 3. Then replace this vertex and its (2 or 3) incident edges by the configurations shown in Fig. 7. This construction clearly preserves planarity and bipartiteness. This gives the graph G. Removing the 4-edge components for each 3-path attached to a B or Y vertex will mean that the T vertices in Fig. 7 have either 0, 1, 2 other edges incident from B or Y vertices. It may be verified that the graph can now be split into connected 4-edge components if and only if exactly one of these T vertices has 2 such incident edges. This then induces a three-dimensional matching. The argument is again obviously reversible. □

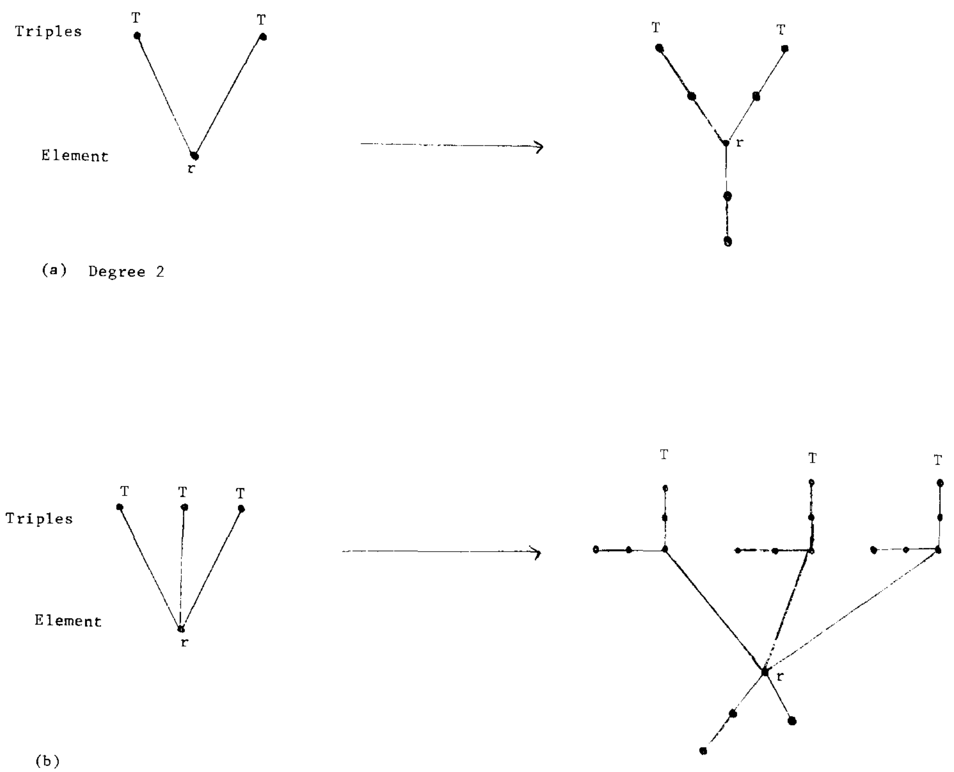

Fig. 7.

We may also put tighter degree bounds on the graphs. For example we can show that partition into 3-edge components remains NP-complete if all vertices have degree 2 or 3. (These bounds are best possible for planar bipartite graphs.) It may also be observed that the above proof remains valid as k grows with the number of vertices n in G. This only requires that the size of the matching problem, and hence the number of k-components into which the graph is partitioned, is not strongly bounded. This is clearly the case if $k=O(n^{1-\varepsilon})$ for some $\varepsilon>0$. We now consider the case where $k=\Omega(n)$, i.e. we wish to partition G into a number of equal sized components.

Theorem 3.2. *If* $m_k=m/k$, *then* Bipartite Em_k(connected) *is NP-complete for any fixed* $k\geq 2$.

Proof. We consider first the case $k=2$ and subsequently generalise the proof. We reduce from 3DM, but our construction does not preserve planarity, although it does preserve bipartiteness. Construct the graph G illustrated in Fig. 8. Each triangle represents a path, of the indicated number of edges, attached to the underlying bipartite graph for the 3DM instance. The new vertex v is joined to every triple. (This is the non-planar element in the construction.)

The numbers n_a, n_b, n_c, n_d, n_e are such that

$$n_d,\ n_e \gg n_c \gg n_b \gg n_a \gg 1.$$

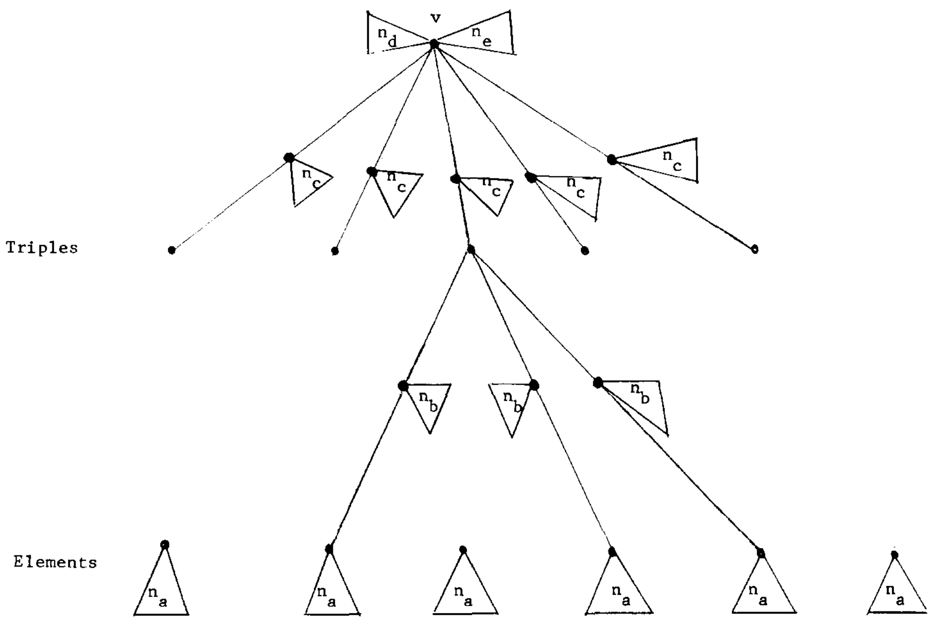

Fig. 8.

For example we may choose $n_a=m^4$, $n_b=m^8$, $n_c=m^{12}$ and $n_d=M^{16}$, where q is the size of the 3DM instance. Then n_e is chosen as follows. If $t=|T|\leq m^3$, then clearly

$$|E| = n_d+n_e+t(n_c+2+3(n_b+2))+3mn_a.$$

We choose n_e to satisfy

$$|E| = 2(n_e+m(n_c+2+3(n_b+2)+3n_a)). \tag{3.1}$$

It may be verified that this implies n_e, $n_d > m^{16}$, and $n_x < \frac{1}{2}|E|$ for $x = a, \dots, e$.

In what follows we will use the terms *a*- or *e*-path to denote the attached paths of length $n_a, \dots, n_e$ respectively, and *b*- or *c*-fork to denote an attached *b*- or *c*-path together with the two edges of G to which it is incident. Now suppose T contains a matching. Let E_1 consist of the *e*-path, the m *c*-forks joining v to triples in the matching plus the $3m$ *b*-forks joining these triples to the elements, plus all the *a*-paths. It follows from (3.1) that $|E_1| = \frac{1}{2}|E|$. It is also clear that E_1 and $E_2 = E - E_1$ are both connected.

Conversely suppose G is partitionable into E_1, E_2 with $|E_1| = |E_2|$. Without loss let us assume the *e*-path is in E_1 and the *d*-path in E_2. (Clearly these paths are too large to both be in the same component.) It follows from (3.1) that E_1 must contain *exactly* m *c*-paths and $3m$ *b*-paths and $3m$ *a*-paths. Consider an arbitrary *a*-path. Since E_1 is connected this must be incident with at least one *b*-fork in E_1. But this implies there must be exactly one such *b*-fork since E_1 contains $3m$ *b*-paths. Consider further the path in E_1 connecting this *a*-path to v. This must go directly through a *b*-fork and a *c*-fork, otherwise E_1 would have to contain at least two *b*-forks incident to some other element, which we know to be impossible. Thus there are exactly m *c*-forks in e_1. It now follows that the triples to which these are incident induce a three-dimensional matching. □

We now modify this construction to general k. Suppose $p = |E|$ in the above. By adding $(k-2)$ paths of length p to v to give a graph G^1, we obtain the NP-completeness of $|E^1|/k$ partition. Here again k could be polynomial in m and thus this result overlaps that of Theorem 2.1.

It is unfortunate that our proof is non-planar. The NP-completeness in the planar case is, as far as we know, an open problem. (Though our proof shows it is sufficient to consider $k = 2$.) We suspect that it is NP-complete.

We might inquire for which types of graph, if any, the problem Ek (connected) is polynomial. It appears that for *trees*, the problem is polynomial for k fixed or m/k fixed. We will sketch the method in each case for $k = 3$ and $k = m/2$. First consider $k = 2$ restricted to trees with n vertices and hence $m = (n-1)$ edges. Clearly if $n \neq 1$ (mod 3), there is no 3-edge decomposition. Otherwise, if G contains any configuration of the type shown in Fig. 9(a), remove it since it must form a component in any 3-edge decomposition. When all such components are removed, then T must have configurations of the type shown in Fig. 9(b).

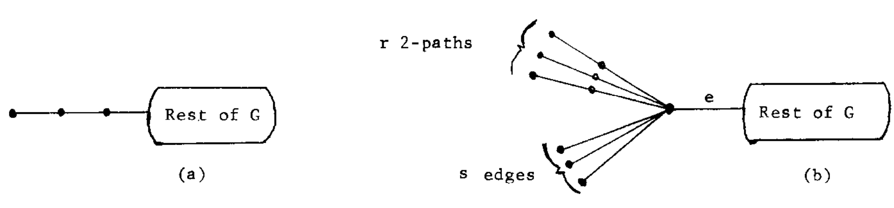

Fig. 9.

The 2-paths must each select an edge incident to v. If $r > s+1$, there is clearly no decomposition. Otherwise remove the r 2-paths and r edges incident to v (possibly including e). This will leave $(s+1-r)$ single edges incident to v, and e must be amongst these if $(s+1-r) \neq 0 \pmod 3$. These edges must now be decomposed into $[(s+1-r)/3]$ 3-stars (i.e. graphs isomorphic to $k_{1,3}$) leaving either no edges, e alone or e and one other edge incident with v. We now continue iteratively until we either find G cannot be decomposed further or we have obtained a 3-edge decomposition.

Now consider $k = m/2$. For each vertex we simply count the number of edges in each of its sub-trees. Now G can be decomposed into two equal components if and only if it has a vertex whose subtrees can be partitioned into two sets such that the sums of the cardinalities of all the subtrees in each set are equal. For each vertex this gives an instance of the problem PARTITION [3, p. 223], but with number of size bounded by n. This can be solved by dynamic programming in polynomial time. This is not suggested as being an efficient algorithm, but merely to demonstrate that the problem has a polynomial solution.

The assumption that k is fixed is essential here since

Theorem 3.3. Ek(connected) *is NP-complete with G restricted to be a tree when k can be selected as part of the problem instance.*

Proof. By reduction from 3-PARTITION [3, p. 224]. This problem is NP-complete is the strong sense. We recall its form

Instance. Set A of $3m$ elements, integer B and an integer $S(a)$ for each $a \in A$ such that $\frac{1}{4}B < s(a) < \frac{1}{2}B$ and $\sum s(a) = mB$.

Question. Can A be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{a \in A_i} s(a) = B$ for $i = 1, 2, \ldots, m$?

For a 3-PARTITION instance we construct a tree G as follows. It has a single vertex v to which are attached paths of length $s(a)$ for each $a \in A$. (This graph is constructible in polynomial time since we are dealing with a strongly NP-complete problem.) We now see that solving the 3-PARTITION problem is equivalent to solving EB(connected) on G. □

We will now consider decomposing the edges of a graph into subgraphs isomorphic to some given fixed graph. Holyer [6] has considered this problem in the case where the fixed graph is a complete graph or circuit. We consider first the problem where the fixed graph is a path of k edges, we have

Theorem 3.4. Ek(path) *is NP-complete for bipartite planar graphs provided* $k \geq 3$.

Proof. Reduction from Planar 3DM. We construct a graph G having a vertex for each element in R, B, Y and the configuration shown in Fig. 10 for each triple. Here r, b, y are any integers such that $0<r, b, y<k/2$ and $r+b+y=k$.

We will initially exclude the case $k=4$, since this is the only k for which these relationships cannot be satisfied. We now attach to each R element having degree d in this graph a set of $(d-1)$ independent r-paths, similarly to the proof of Theorem 3.1. We also attach $(d-1)$ b-paths to each B, and $(d-1)$ y-paths to each Y vertex to give G. Since $r, b, y \neq k/2$ it follows that each of these attached paths must form a k-path with one of the 'vertical' paths illustrated in Fig. 10. Thus, when these paths are removed from G, each R, B, Y vertex is incident to one such vertical path. Thus for each triple we must partition a configuration like that of Fig. 10 but with some or all of the vertical paths omitted. It is easy to see that this can be partitioned into k paths if and only if either none or all of the vertical paths are present. (In the first case there is a single k-path consisting of the 'horizontal' paths, in the latter there are three obvious k-paths. This obviously induces a three-dimensional matching.)

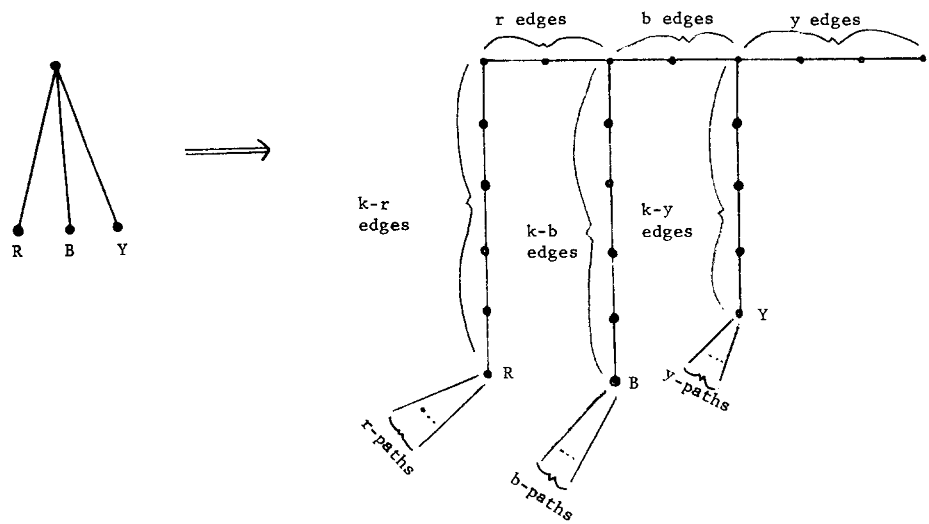

Fig. 10.

For the case $k=4$, we take $r=2$, $b=y=1$ and we construct the same graph as above, except that we attach paths to the R vertices in a slightly different manner. We may assume that the degree $d=2$ or 3 for such vertices. We attach a single 2-path if $d=2$, but no path if $d=3$. The proof now goes through, since it follows easily that two of the three 2-paths incident to each R vertex must combine to form a 4-path. For if they were all in different 4-paths, we would have three B and three Y vertices 'matched' with a single R vertex. But this would mean that some other R vertex was 'unmatched', i.e. incident to three 2-paths none of which is part of any other 4-path. This can obviously not be partitioned, since it has six edges.

Reversing the argument in both cases establishes the partition corresponding to any given matching. The construction is clearly planar and bipartite. □

We now examine the problem of decomposing the edges of G into 'stars' of k edges (i.e. $K_{1,k}$ isomorphs).

Theorem 3.5. Ek(star) *is NP-complete for all* $k \geq 3$, *and it is NP-complete for planar graphs in the case* $k = 3$.

Proof. We reduce from kDM. This will establish the planar result for $k = 3$ since our construction preserves planarity. Let G_1 be the bipartite graph corresponding to a kDM instance. Thus the vertex corresponding to each k-triple has degree k. We may also assume that the vertex corresponding to each element has degree $d = 2$ or 3. We now attach $(k - d + 1)$ independent additional edges to each of these element vertices. For each such vertex these edges must clearly be part of k-star by taking $(d - 1)$ of the edges incident to the vertex in G_1. Thus there will be exactly one edge left unaccounted for at each element vertex. It now follows that the only way the remainder of the graph, G_2, say, can be partitioned into k-stars is for each k-triple vertex to have degree 0 or k in G_2. This obviously corresponds to a matching and again we can reverse the argument. □

We conjecture that the problem remains NP-complete in the planar cases for larger values of k, but our proof would require the NP-completeness of Planar kDM, which is not available. However, in Theorems 3.4 and 3.5 we have shown that the problems of partitioning into both 3-paths and 3-stars are NP-complete in the planar case. Now the only other graph on 3 edges is a triangle, and we might ask whether this problem is also NP-complete in the planar case, since Holyer [6] has shown it to be NP-complete in general. It is easy to see that it is not, and we will sketch the method.

We assume that we start with a given *fixed* embedding of our graph $G = (V, E)$ in the plane. We note first that if E has a partition into triangles $T_1, T_2, \dots, T_p$ and T is any triangle of G, then for each i

$$T_i - T \text{ is contained entirely inside of } T \text{ or entirely outside of } T. \quad (3.2)$$

We call a triangle T *decomposing* if T is not a face of G and the number of edges inside T is divisible by 3.

It is clear from (3.2) that only faces and decomposing triangles can be used in a partition. Moreover

Lemma 3.6. *Suppose that T is decomposing and X is the set of edges inside T in our embedding. Then E is partitionable into triangles if and only if*

(a) *X and $E - X$ are partitionable into triangles, or*

(b) *X is not partitionable into triangles but $X \cup T$ and $E - (X \cup T)$ are partitionable into triangles.*

Furthermore, the conclusions of the Lemma remain true if X denotes the set of edges outside T.

Proof. If (a) or (b) hold, then clearly E is partitionable. Conversely, suppose $T_1, T_2, \dots, T_p$ is a partition of E into triangles. Suppose that $T_i \cap X \neq \emptyset$ for $i = 1, 2, \dots, q$ only. It follows from (3.2)

$$T_i \subseteq T \cup X \quad \text{for } i = 1, 2, \dots, q. \tag{3.3}$$

If $T_i \cap T = \emptyset$ for $i = 1, 2, \dots, q$, then (a) holds with $T_1, T_2, \dots, T_q$ partitioning X. If X is not partitionable, then for some $i \leq q$, $T_i \cap T \neq \emptyset$ and then $T \subseteq Y = \bigcup_{i=1}^{q} T_i$ and (b) holds as $3q = |Y| = |X| + |Y \cap T|$ by (3.3).

Finally, if X now denotes the set of edges outside T, then we verify (a) and (b) in an analogous manner. □

We show next that we can decide in polynomial time whether or not we can partition E into facial triangles (this coupled with Lemma 3.6 will provide a recursively defined algorithm for the whole problem).

Let $E_k = \{e \in E$: e lies on k triangular faces$\}$, $k = 0, 1, 2$. Clearly $E = E_0 \cup E_1 \cup E_2$ and this partition is constructible in linear time. If $E_0 \neq \emptyset$, then there is no facial triangular partition. If $e \in E_1$ we remove e and the unique triangle $T \ni e$ from the problem. It remains to consider the case where each edge lies on exactly 2 triangular faces i.e. G is a triangulation.

Lemma 3.7. *The edges of a triangulation G can be partitioned into facial triangles if and only if its dual graph G^* is bipartite.*

Proof. A set $\{T_1, T_2, \dots, T_p\}$ of facial triangles partitions E if and only if

(i) $T_i \cap T_j = \emptyset$, $i \neq j$, i.e. $T_1, T_2, \dots, T_p$ from a stable set in G^*.

(ii) $\bigcup T_i = E$, i.e. all edges of G^* are covered by $T_1, T_2, \dots, T_p$.

Clearly a graph is bipartite if and only if it has a stable set covering all its edges. □

Now let

$$\pi(E) = \text{true}, \quad \text{if } E \text{ can be partitioned into triangles,}$$
$$= \text{false}, \quad \text{otherwise}$$

and

$$\hat{\pi}(E) = \text{true}, \quad \text{if } E \text{ can be partitioned into facial triangles,}$$
$$= \text{false}, \quad \text{otherwise.}$$

We now give a recursive algorithm for computing $\pi(E)$. It is straightforward to amend it so that it produces a partition if $\pi(E) = \text{true}$.

Computation of $\pi(E)$

begin
    if G has a decomposing triangle T with inside X and outside Y
    then begin
        $Z := X$; **if** $|X| \geq |Y|$ **then** $Z := Y$;
        if $\pi(Z)$ **then** $\pi(E) := \pi(E - Z)$
                else $\pi(E) := \pi(Z \cup T) \Lambda \pi(E - (Z \cup T))$

end
else $\pi(E) := \hat{\pi}(E)$
end

It remains to show that the above algorithm runs in polynomial time. We note first that if G has M edges, then we can enumerate all triangles of G in $O(m)$ time – see Papadimitriou and Yannakakis [12]. Thus we can certainly check for the existence of a decomposing triangle in $O(m^2)$ time (possibly $O(m)$ time?). We can certainly check whether G^* is bipartite in $O(m)$ time and so if

$$g(m) = \text{maximum execution time of the algorithm on a planar graph with } m \text{ or fewer edges,}$$

then

$$g(m) \le cm^2 + \max_{3 \le k \le (m-3)/2} (g(k) + g(k+3) + g(m-k))$$

from which $g(m) = O(m^3)$ follows.

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