extremely meticulous
orts is a better paper.

onal algebraic theories and
sium on Foundations of

tial algebra semantics and
. 68-95.
ges as mathematical objects,
ectures Notes in Computer
-101.
Recursion and iteration in
48-164.
d flow-diagram semantics,

eories, J. Algebra, 86 (1984),

the iteration equation and
. 9 (1980), pp. 26-45; Vector
980), pp. 525-540.
teration, J. Comput. System

ybernet., 8 (1987), pp. 1-4.
's, J. Comput. System Sci.,

omput. Sci., 25 (1983), pp.

, in Logic Colloquium '73,
. pp. 175-230; reprinted in

aic structure of rooted trees,

ments, IEEE Trans. Software

Springer-Verlag, New York,

omputational Linguistics and

(1983), pp. 291-303.
Acta Cybernet., 7 (1985), pp.

Proc. Nat. Acad. Sci. U.S.A.,

), pp. 67-94.
Sci., 35 (1987), pp. 163-191.
. pp. 187-216.

SIAM J. COMPUT.
Vol. 17, No. 5, October 1988

ON THE COMPLEXITY OF COMPUTING THE VOLUME OF A POLYHEDRON*

M. E. DYER† AND A. M. FRIEZE‡

Abstract. We show that computing the volume of a polyhedron given either as a list of facets or as a list of vertices is as hard as computing the permanent of a matrix.

Key words. volume, polyhedra, complexity, $\#P$-complete

AMS(MOS) subject classification. 68A20

1. Introduction. Recently there has been some interest in establishing the computational complexity of determining the volume of convex bodies in $\mathbb{R}^n$. Elekes [2] and Bárány and Füredi [1] have shown that it is even difficult to closely approximate volumes of convex bodies defined by certain types of oracles. These hardness results complement the approximation algorithm of Lovász [4].

Lovász [4] also enquires about the complexity of computing the volume of a rational polytope given either by listing its facets or by listing its vertices. He conjectures that these problems are hard. In this paper we confirm Lovász's conjecture, and show that both problems are as hard as computing the matrix permanent, (see Valiant [6]). We cannot quite describe the problems as $\#P$-complete, since they are not in the class $\#P$ as defined by Valiant [6]. However, since we have no wish to define yet another complexity class, we state our results relative to $\#P$.

Let us now be more specific. Consider first Problem 1.

Problem 1. Let $P = P(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leqq \mathbf{b}\}$ be a polyhedron. $A, \mathbf{b}$ have rational entries where $A = (a_{ij})$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, and $\mathbf{b} = (b_i)$, $i = 1, 2, \cdots, m$. We shall use the notation of Schrijver [5] to describe problem size. Thus if $x = p/q$ is rational where $p, q > 0$ are relatively prime integers, then

$$\operatorname{size}(x) = 1 + \lceil \log_2 (|p| + 1) \rceil + \lceil \log_2 (q + 1) \rceil$$

and

$$\operatorname{size}(A, \mathbf{b}) = m(n+1) + \sum_{i=1}^{m} \operatorname{size}(b_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{size}(a_{ij}).$$

$L = \operatorname{size}(A, \mathbf{b})$ will be used as our measure of problem size. Since we can (by linear programming) determine in polynomial time whether or not $\operatorname{vol}(P) = 0$ and whether or not $\operatorname{vol}(P) = \infty$, we can assume without loss of generality that $0 < \operatorname{vol}(P) < \infty$.

We shall prove two theorems. The first shows that computing $\operatorname{vol}(P)$ is $\#P$-hard.

THEOREM 1. *Computing* $\operatorname{vol}(P(A, \mathbf{b}))$ *is* $\#P$*-hard, even when A is totally unimodular.* □

There is a technical difficulty in stating the converse of this theorem. It may be encapsulated in the following problem.

Problem. Is size $(\operatorname{vol}(P))$ polynomially bounded in L? □

It is not too difficult to show that $\operatorname{vol}(P)$ is a rational, p/q, say. We can also show that $\operatorname{vol}(P)$ cannot be too large. The difficulty is that q does not appear to have a

* Received by the editors July 1, 1987; accepted for publication November 15, 1987.

of Computer Studies, University of Leeds, Leeds LS2 9JT, United Kingdom.

bound polynomial in L. The problem stated above thus seems to be an open (and possibly difficult) question. We show later that the difficulty disappears if we restrict ourselves to any class of polyhedra whose vertices can be scaled to become integer lattice-points by a polynomial size transformation. We conjecture that the answer to our problem is in the affirmative, but we are at present only able to state the following result in terms of approximation.

THEOREM 2. *Let $\varepsilon > 0$ be rational. Given a #P-complete oracle, in time polynomial in L and* size (ε), *we can compute $\hat{V}$ such that*

$$|\hat{V} - \text{vol}(P(A, \mathbf{b}))| < \varepsilon. \tag{1.1}$$ □

COROLLARY 1. *Suppose we restrict our attention in Theorem 2 to any class of polytopes for which* size (vol (P)) *is bounded by a polynomial in L. Then, using a #P-complete oracle,* vol (P) *can be computed exactly in time polynomial in L.* □

Let us now consider Problem 2.

Problem 2. Let $X = \{X_1, X_2, \cdots, X_m\}$ be a set of rational points in $\mathbb{R}^n$. Let $P(X)$ be the convex hull of X. We shall consider the problem of computing vol $(P(X))$ when X is given as an $n \times m$ matrix (x_{ij}), $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. □

THEOREM 3. *Computing* vol $(P(X))$ *is #P-hard.* □

This time we have no difficulty in stating the converse result because, as we will show, size (vol $(P(X))$) is polynomially bounded in size (X).

THEOREM 4. *Computing* vol $(P(X))$ *is #P-easy.* □

2. #*P*-hardness of Problem 1. Let $B = \{0, 1\}$ and $C = B^n$. We will consider a single linear inequality

$$\mathbf{a}^T\mathbf{x} \leqq b \tag{2.1}$$

in $\mathbb{R}^n$, with $\mathbf{a} > \mathbf{0}$ an integer vector. Define the polytope

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leqq b, \mathbf{0} \leqq \mathbf{x} \leqq \mathbf{e}\} \tag{2.2}$$

where $\mathbf{e}^T = (1, 1, \cdots, 1)$, and let

$$K = C \cap P. \tag{2.3}$$

We shall wish to regard (2.1) as being parameterised by b, in which case we will write $P(b)$ or $K(b)$ for emphasis. The following problem is known to be #*P*-hard (see [3]).

#KNAPSACK
Input. Positive integers $a_1, a_2, \cdots, a_n, b$.
Problem. Determine $N = |K|$.
We Turing reduce #KNAPSACK to computing the volumes of certain polyhedra. For $\mathbf{x} \in C$, let $|\mathbf{x}| = \mathbf{e}^T\mathbf{x}$ denote the number of 1's in $\mathbf{x}$.

Now consider the following problem, which may appear rather artificial, but will be required later.

#PARITY
Input. As for #KNAPSACK.
Problem. For $i = 0, 1$ let $N_i = |\{\mathbf{x} \in K : |\mathbf{x}| \equiv i \pmod 2\}|$. Determine

$$D = N_0 - N_1. \tag{2.4}$$

LEMMA 1. *#PARITY is #P-hard.*

Proof. Suppose we have a procedure for #PARITY. Let $M = \mathbf{e}^T\mathbf{a} + 1$, and consider the set of $(n+1)$ #PARITY problems, as above, corresponding to the inequalities

$$(\mathbf{a} + M\mathbf{e})^T\mathbf{x} \leqq b + rM \qquad (r = 0, 1, \cdots, n). \tag{2.5}$$

Let $N^{(r)} = |\{\mathbf{x} \in K : |\mathbf{x}| = r\}|$. It follows, by an easy analysis, that the rth #PARITY problem defined by (2.5) determines the value

$$D^{(r)} = (-1)^r N^{(r)} + \sum_{j=0}^{r-1} (-1)^j \binom{n}{j}.$$

A straightforward calculation now yields

$$N = 2^{n-1} + \sum_{r=0}^{n} (-1)^r D^{(r)}.$$

Thus a polynomial number of calls to the #PARITY procedure, plus polynomial additional time, would solve #KNAPSACK. □

We now turn to the proof of Theorem 1.

Proof of Theorem 1. Assume we have some procedure for determining vol (P). Let $\Delta = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leqq b, \mathbf{x} \geqq \mathbf{0}\}$. Note that Δ is bounded with vol $(\Delta) = b^n/(n! \prod_{j=1}^{n} a_j)$. Let U be the uniform measure on subsets of Δ such that $U(\Delta) = b^n$. Note that U is related to volume for a subset E of Δ by the equation vol (E)/vol $(\Delta) = U(E)/U(\Delta)$. Thus our procedure for determining vol (P) can easily be modified to determine $U(P)$, and we will assume this has been done. Let $E_j = \{\mathbf{x} \in \Delta : x_j \leqq 1\}$ for $j = 1, 2, \cdots, n$ and $\bar{E}_j = (\Delta - E_j)$. Then clearly $P = \bigcap_{j=1}^{n} E_j$. Now, for each $\mathbf{v} \in C$, define

$$\bar{E}_{\mathbf{v}} = \{\mathbf{x} \in \Delta : \mathbf{x} > \mathbf{v}\} = \bigcap_{v_i = 1} \bar{E}_i.$$

Note that $\bar{E}_{\mathbf{v}}$ is either empty or is an n-simplex (actually its closure is). The well-known inclusion-exclusion formula (see, e.g., [7]) now yields

$$U(P) = \sum_{\mathbf{v} \in C} (-1)^{|\mathbf{v}|} U(\bar{E}_{\mathbf{v}}).$$

It is easy to show that

$$\begin{aligned} U(\bar{E}_{\mathbf{v}}) &= 0 && \text{if } \mathbf{a}^T\mathbf{v} > b \\ &= (b - \mathbf{a}^T\mathbf{v})^n && \text{if } \mathbf{a}^T\mathbf{v} \leqq b. \end{aligned}$$

Thus

$$U(P) = \sum_{\mathbf{v} \in K} (-1)^{|\mathbf{v}|} (b - \mathbf{a}^T\mathbf{v})^n.$$

Suppose that b is an integer, and β real such that $b \leqq \beta < b+1$. The integrality of $\mathbf{a}$ then implies that $K(\beta) = K(b)$. Thus, writing $F(\beta) = U(P(\beta))$, it follows that

$$F(\beta) = \sum_{\mathbf{v} \in K(b)} (-1)^{|\mathbf{v}|} (\beta - \mathbf{a}^T\mathbf{v})^n. \tag{2.6}$$

Let β be a rational p/q with $p, q > 0$ relatively prime. It follows from (2.6) that $q^n F(\beta)$ is an integer such that $q^n F(\beta) \leqq (2p)^n$. Thus size $(F(\beta)) \leqq n$(size $(\beta) + 1)$. Thus the procedure for calculating $U(P)$ can be used to determine $F(\beta)$, provided β has size polynomial in L. Now, expanding the terms in (2.6), we see that

$$F(\beta) = \sum_{i=0}^{n} \alpha_i \beta^{n-i} \tag{2.7}$$

where

$$\alpha_i = (-1)^i \binom{n}{i} \sum_{\mathbf{v} \in K(b)} (-1)^{|\mathbf{v}|} (\mathbf{a}^T\mathbf{v})^i \qquad (i = 0, 1, \cdots, n). \tag{2.8}$$

Observe, in particular, that $\alpha_0 = \sum_{\mathbf{v} \in K(b)} (-1)^{|\mathbf{v}|} = D$, as defined in (2.4). This is #P-hard to compute (Lemma 1). It follows, a fortiori, that it is #P-hard to determine the coefficients of the polynomial F. But we may do this as follows:

(i) Determine $F(\beta)=U(P(\beta))=n!\prod_{j=1}^{n} a_j \operatorname{vol}(P(\beta))$ for $\beta=b+k/(n+1)$, $k=0,1,\cdots,n$.

(ii) Solve (in polynomial time) the resulting system of linear equations determined by (2.7).

We observe that none of the numbers involved in the computations is very large. For example, it follows from (2.8) that the α_i are integers such that $|\alpha_i|\leqq(4\mathbf{e}^T\mathbf{a})^n$.

There is one final point. The statement of Theorem 1 claims that it remains true when A is totally unimodular. This follows by making the substitution

$$y_j=a_jx_j \qquad (j=1,2,\cdots,n)$$

in the system (2.2) defining P. The constraints are then easily transformed into a totally unimodular system. The theorem now follows. □

Remark. The problem we have proved hard may be viewed in either of the following ways:

(i) If $\mathbf{X}$ is a point chosen randomly from the uniform probability distribution on the unit hypercube, then it is hard to compute the probability that $\mathbf{X}$ satisfies a single linear inequality.

(ii) Integrating the step-function

$$\begin{aligned}\psi(\mathbf{x})&=1 \qquad (b-\mathbf{a}^T\mathbf{x}\leqq 0)\\ &=0 \qquad (b-\mathbf{a}^T\mathbf{x}>0)\end{aligned}$$

over the unit hypercube is $\#P$-hard.

Note that the function ψ in (ii) is not even continuous. However, we may integrate ψ explicitly over k variables, using (2.6). This gives the piecewise polynomial function

$$\psi_k(\mathbf{x})=\sum_{\hat{\mathbf{v}}\in K(b-\mathbf{a}^T\mathbf{x})}(-1)^{|\hat{\mathbf{v}}|}(b-\mathbf{a}^T\mathbf{x}-\hat{\mathbf{a}}^T\hat{\mathbf{v}})^k$$

where $\hat{\mathbf{a}},\hat{\mathbf{v}}\in\mathbb{R}^k$ and $\mathbf{a},\mathbf{x}\in\mathbb{R}^{n-k}$. This function is readily shown to be of class C^{k-1}. Hence we have the conclusion that it is hard to integrate such a function over the unit hypercube. In fact, since the description of ψ_k continues to be of polynomial size for $k=O(\log n)$, the conclusion can be strengthened further.

3. $\#P$-easiness of Problem 1. We will go straight into the proof of Theorem 2.

Proof of Theorem 2. Since P is bounded, we have [5, Thm. 10.2]

$$P\subseteq\Gamma=\{\mathbf{x}\in\mathbb{R}^n: |x_j|\leqq D=2^{4n^2L}(j=1,2,\cdots n)\}. \tag{3.1}$$

Now let $s=\lceil mn^2 2^{4n^3L+n}/\varepsilon\rceil$, and divide Γ into s^n subcubes of side $\delta=2D/s$. These subcubes fall into three classes:

(i) *interior* (wholly) to P;

(ii) *exterior* (wholly) to P;

(iii) *boundary*.

We will estimate $V=\operatorname{vol}(P)$ by the total volume of the interior cubes, $\hat{V}$. Thus $\hat{V}=I\delta^n$, where I is the number of interior cubes. Clearly $V\geqq\hat{V}$, and the error $E=V-\hat{V}$ is bounded above by $J\delta^n$, where J is the number of boundary cubes. We show later that

$$J\leqq mn^2s^{n-1}, \tag{3.2}$$

and hence $J\delta^n\leqq\varepsilon$ as required.

The counting machine ϕ [6] which computes I works as follows. Each copy chooses one of the integer vectors $\mathbf{t}$, where $1\leqq t_j\leqq s$ $(j=1,2,\cdots,n)$ in time proportional to n size (s). It then tests, in polynomial time, whether the subcube

$$\Gamma'=\{\mathbf{x}\in\mathbb{R}^n: -D+(t_j-1)\delta\leqq x_j\leqq -D+t_j\delta, j=1,2,\cdots,n\}$$

is interior. Note that this is equivalent to

$$\frac{1}{2}\delta \sum_{j=1}^{n} |a_{ij}| \leqq b_i + \sum_{j=1}^{n} a_{ij}\{D-(t_j-\tfrac{1}{2})\}\delta \qquad (i=1,2,\cdots,m).$$

It is clear that the sizes of all numbers involved, in particular s, are polynomial in L. The output of ϕ is I, from which $\hat{V}$ is calculated.

It remains only to prove (3.2). If

$$P=\{\mathbf{x}\in\mathbb{R}^n: \mathbf{a}_i^T\mathbf{x}\leqq b_i (i=1,2,\cdots,m)\}$$

then let $H_i=\{\mathbf{x}\in\mathbb{R}^n: \mathbf{a}_i^T\mathbf{x}=b_i\}$. Call a subcube Γ' of Γ *intersected* if $H_i\cap\Gamma'\neq\varnothing$ for some $1\leqq i\leqq m$. Let K be the number of intersected subcubes. Clearly $K\geqq J$. Each subcube Γ' projects onto n *squares* Γ'_j in the coordinate planes, i.e., if $\Gamma'=\{\mathbf{x}\in\mathbb{R}^n: u_k\leqq x_k\leqq u_k+\delta\ (k=1,2,\cdots,n)\}$ then $\Gamma'_j=\{\mathbf{x}: x_j=0 \text{ and } u_k\leqq x_k\leqq u_k+\delta\ (k\neq j)\}$. The total number of distinct squares is clearly ns^{n-1}. Construct a mapping from the intersected cubes to the squares as follows.

For each intersected cube Γ' choose some i such that $H_i\cap\Gamma'\neq\varnothing$ (e.g., the least such i). Choose j so that

$$|a_{ij}|=\max_{1\leqq k\leqq n}|a_{ik}|.$$

Suppose two intersected cubes map to the same square with the same value of i. Then there exist points $\mathbf{x}, \mathbf{x}'$ within the two cubes such that

$$\mathbf{a}_i^T\mathbf{x}=\mathbf{a}_i^T\mathbf{x}'=b_i,$$

$$x_j\geqq x'_j \quad\text{and}\quad |x_k-x'_k|\leqq\delta \quad (k\neq j).$$

Thus $x_j-x'_j\leqq\sum_{k\neq j}|a_{ik}|\delta/|a_{ij}|\leqq(n-1)\delta$. Therefore at most n cubes can map onto the same square for a given i. Since there are only m values of i, at most mn cubes map onto the same square. Hence $K\leqq mn\times ns^{n-1}=mn^2s^{n-1}$, and (3.2) follows.

Proof of Corollary 1. Suppose we know that size (vol (P)) $\leqq p(L)$ for some polynomial $p(L)$. Take $\varepsilon=1/3p(L)^2$ in Theorem 2. We know that $V=a/b$ for integers $a, b\leqq 2^{p(L)}$, and $V-\hat{V}\leqq\varepsilon$. Having computed $\hat{V}$, we can use continued fractions to compute V exactly in polynomial time (see, e.g., [5, Cor. 6.3a]). □

Finally, we give a simple example of a class of polyhedra for which vol (P) has polynomial size.

LEMMA 2. *If P is integral, then* size (vol (P)) $=O(L^3)$.

Proof. Let $P=\{\mathbf{x}\in\mathbb{R}^n: \mathbf{a}_i^T\mathbf{x}\leqq b_i(i=1,2,\cdots,m)\}$ and be such that every vertex of P has integer coordinates. Now P has a triangulation using only its own vertices. If σ is a simplex of this triangulation, then

$$\operatorname{vol}(\sigma)=\frac{1}{n!}\det\begin{pmatrix}1 & 1 & & 1\\ \mathbf{v}_0 & \mathbf{v}_1 & \cdots & \mathbf{v}_n\end{pmatrix}=\frac{\nu_\sigma}{n!}, \quad \text{say},$$

where $\mathbf{v}_i (i=0,1,\cdots,n)$ are vertices of P. Now ν_σ is an integer. Hence vol $(P)=\nu/n!$, with $\nu=\sum_\sigma \nu_\sigma$, i.e.; $n!$vol (P) is an integer. Now (3.1) implies

$$\operatorname{vol}(P)\leqq 2^{8n^2L+n}.$$

Hence

$$\operatorname{size}(\operatorname{vol}(P))\leqq 8n^2L+n+\lceil\log_2(n!+1)\rceil+1,$$

and the result follows. □

4. $\#P$-hardness of Problem 2. We shall again reduce the counting problem #KNAPSACK to volume computations of the relevant form. We will make the following two additional assumptions for #KNAPSACK:

$$\text{(4.1)} \qquad \text{There are no 0–1 solutions to } \mathbf{a}^T\mathbf{x} = b.$$

We can ensure this simply by replacing a_j by $2a_j$ $(j = 1, 2, \cdots, n)$ and b by $2b+1$ in (2.1). This does not affect the value of N.

$$\text{(4.2)} \qquad b > \tfrac{1}{2}\mathbf{e}^T\mathbf{a}.$$

If this is not true on input, then we can add a variable x_{n+1} with $a_{n+1} = \mathbf{e}^T\mathbf{a} - b + 1$, and replace b by $b + a_{n+1}$. This adds 2^n to the value of N.

Now let $P_1 = P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leqq b, \mathbf{0} \leqq \mathbf{x} \leqq \mathbf{e}\}$. Then (4.2) implies that $\frac{1}{2}\mathbf{e} = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ is an interior point of P_1. Now, substituting $\mathbf{y} = 2\mathbf{x} - \mathbf{e}$, transform P_1 to

$$P_2 = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{y} \leqq b', -\mathbf{e} \leqq \mathbf{y} \leqq \mathbf{e}\},$$

where $b' = 2b - \mathbf{e}^T\mathbf{a} > 0$. Note that $\mathbf{x} \in \{0,1\}^n \cap P_1 \leftrightarrow \mathbf{y} \in \{-1,+1\}^n \cap P_2$. So we have reduced #KNAPSACK to computing $N = |Y|$, where

$$\begin{aligned} Y &= \{\mathbf{y} \in P_2 : y_j = \pm 1 \ (j = 1, 2, \cdots, n)\} \\ &= \{\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \cdots, \mathbf{y}^{(N)}\}, \text{ say.} \end{aligned}$$

The constraints defining P_2 can be written as

$$\text{(4.3)} \qquad \begin{aligned} &\mathbf{A}^T\mathbf{y} \leqq 1, \\ &-y_j \leqq 1, \quad y_j \leqq 1 \quad (j = 1, 2, \cdots, n) \end{aligned}$$

where $\mathbf{A} = \mathbf{a}/b'$. Now $\mathbf{0}$ is an interior point of P_2. Consider the *polar* P_2^* [5, Chap. 9], where

$$\text{(4.4)} \qquad \begin{aligned} P_2^* &= \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z}^T\mathbf{u} \leqq 1 \text{ for all } \mathbf{u} \in P_2\} \\ &= \operatorname{conv}\{\mathbf{A}, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, -\mathbf{e}_1, -\mathbf{e}_2, \cdots, -\mathbf{e}_n\}. \end{aligned}$$

The second equality in (4.4) merely states the well-known relationship between the facets of P_2 and the vertices of P_2^*.

We show that N can be computed from $\operatorname{vol}(P_2^*)$ and $\operatorname{vol}(\hat{P}_2^*)$, where $\hat{P}_2^*$ is the polyhedron obtained by using $(b+\frac{1}{2})$ in place of b in the definition of P_1 before transforming to P_2^*. Since P_2^*, $\hat{P}_2^*$ are defined as convex hulls, Theorem 3 will follow.

Now polarity yields a one-to-one correspondence between the (*nondegenerate*) vertices of P_2 not lying in facet $\{\mathbf{y} \in P_2 : \mathbf{A}^T\mathbf{y} = 1\}$ (i.e., members of Y) and the (*simplicial*) facets of P_2^* not containing the vertex $\mathbf{A}$. Observe that the facet corresponding to $\mathbf{y}^{(t)}$ has vertex set $\Lambda^{(t)} = \{y_i^{(t)}\mathbf{e}_i : i = 1, 2, \cdots, n\}$ for $t = 1, 2, \cdots, N$. Thus there is a decomposition of P_2^* into simplices $\sigma_1, \sigma_2, \cdots, \sigma_N$, where $\sigma_t = \operatorname{conv}(\{\mathbf{A}\} \cup \Lambda^{(t)})$. Hence

$$\begin{aligned} \operatorname{vol}(P_2^*) &= \sum_{t=1}^{N} \operatorname{vol}(\sigma_t) \\ &= \frac{1}{n!} \sum_{t=1}^{N} \left| \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{A} & y_1^{(t)}\mathbf{e}_1 & \cdots & \mathbf{y}_n^{(t)}\mathbf{e}_n \end{pmatrix} \right| \\ &= \frac{1}{n!} \sum_{t=1}^{N} |1 - \mathbf{A}^T\mathbf{y}^{(t)}| \quad \text{since } y_i^{(t)} = \pm 1 \\ &= \frac{2}{n!} \sum_{t=1}^{N} \left| \frac{b - \mathbf{a}^T\mathbf{x}^{(t)}}{2b - \mathbf{e}^T\mathbf{a}} \right| \end{aligned}$$

where $\mathbf{x}^{(t)} = \frac{1}{2}(\mathbf{e}+\mathbf{y}^{(t)})$ is the zero-one solution to (2.1) which corresponds to $\mathbf{y}^{(t)}$. Thus, using (4.2),

$$\frac{1}{2}n!\,\mathrm{vol}\,(P_2^*) = \sum_{t=1}^{N} \frac{b-\mathbf{a}^T\mathbf{x}^{(t)}}{2b-\mathbf{e}^T\mathbf{a}} = \frac{bN-S}{2b-\mathbf{e}^T\mathbf{a}}$$

where $S=\sum_{t=1}^{N}\mathbf{a}^T\mathbf{x}^{(t)}$. Furthermore

$$\frac{1}{2}n!\,\mathrm{vol}\,(\hat{P}_2^*) = \frac{(b+\frac{1}{2})N-S}{2(b+\frac{1}{2})-\mathbf{e}^T\mathbf{a}}$$

where we have used, in the notation of § 2, $K(b)=K(b+\frac{1}{2})$, so that N and S are unchanged.

Now N can be easily computed from $\mathrm{vol}\,(P_2^*)$ and $\mathrm{vol}\,(\hat{P}_2^*)$. Finally note that S is an integer with $0<S\leqq 2^n b$, so that the numbers involved are of polynomial size. Thus Theorem 3 has been proved. □

5. #*P*-easiness of Problem 2. Let us first show that size (vol $(P(X))$) is polynomially bounded. Indeed we can prove quite straightforwardly that, for $m>n\geqq 1$,

$$\tau = \mathrm{size}\,(\mathrm{vol}\,(P(X))) \leqq 3mn^2L \tag{5.1}$$

where $L=\mathrm{size}\,(X)$. To see this, let λ be the least common multiple of the denominators of entries of X. Then $\mathrm{size}\,(\lambda)\leqq mnL$. Now λX contains integer vectors and $\mathrm{vol}\,(P(\lambda X)) = \lambda^n\,\mathrm{vol}\,(P(X))$. Since $P(\lambda X)$ can be decomposed into simplices (see proof of Lemma 2), it follows that $n!\,\mathrm{vol}\,(P(\lambda X))$ is an integer. Hence

$$\mathrm{vol}\,(P(X)) = \frac{n!\lambda^n\,\mathrm{vol}\,(P(X))}{n!\lambda^n}$$

expresses vol (P) as a ratio of two integers. Noting that $\mathrm{vol}\,(P(X))\leqq 2^{nL}$, (5.1) follows.

The proof of Theorem 4 is similar to that of Theorem 2, and so we will only give an outline. Now

$$P(X)\subseteq \Gamma_1 = \{\mathbf{x}\in\mathbb{R}^n: |x_j|\leqq 2^L\,(j=1,2,\cdots,n)\}.$$

Let $s=\lceil m^n n^2 2^{6mn^2L+n(L+1)+2}\rceil$ and divide Γ_1 into s^n subcubes of size $\delta_1 = 2^{L+1}/s$. Our counting machine computes the number I_1 of subcubes which intersect $P(X)$. The estimate of volume is then $I_1\delta_1^m/m!$. Note that this time we use an overestimate, rather than an underestimate, of the volume.

We can now reduce the testing for the intersection of an individual subcube with $P(X)$ to the solution of a single linear program.

To establish this claim, for simplicity let us assume we translate and scale the problem to that of testing the intersection of the cube $\Sigma = \{\mathbf{x}\in\mathbb{R}^n: -1\leqq x_j\leqq 1\,(j=1,2,\cdots,n)\}$, with the polytope $P=\mathrm{conv}\,\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_m\}$. Then $\Sigma\cap P=\emptyset$ if and only if there exists a $\boldsymbol{\gamma}\in\mathbb{R}^n$ such that

$$\sum_{i=1}^{n}|\gamma_i|<1 \quad\text{and}\quad \boldsymbol{\gamma}^T\mathbf{x}_i\geqq 1 \quad (i=1,2,\cdots,m).$$

This can be tested by solving the linear program

$$\begin{aligned} &\text{minimise} && z=\mathbf{e}^T(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\\ &\text{subject to} && \mathbf{x}_i^T(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\geqq 1\,(i=1,2,\cdots,m)\end{aligned}$$

and by checking whether min $z<1$.

The error in the volume estimate is again bounded by the total volume of the subcubes which intersect the boundary of $P(X)$. Since $P(X)$ has less than m^n facets, this is at most

$$m^n n^2 s^{n-1} \delta_1^n$$

(see the proof of Theorem 2 with m replaced by m^n). By (5.1), this is at most $2^{-2(r+1)}$, and so vol $(P(X))$ can be computed exactly using continued fractions.

6. Remarks. Our results leave open two interesting questions. The first is the problem raised in § 1 concerning the size of description of polyhedral volumes. The second is as to whether it remains hard to approximate the volume in either Problems 1 or 2; i.e., for some given $\varepsilon > 0$, is it hard to obtain an estimate $\hat{V}$ of the volume V such that $(1-\varepsilon) < \hat{V}/V < (1+\varepsilon)$? Our methods appear to shed little light on this issue, but we conjecture that this approximation problem is also hard. Finally, we observe that determining the volume of a polyhedron in *fixed* dimension is easy. We simply determine the complete face-lattice of the polyhedron, triangulate it, and then use the formula for the volume of a simplex.

REFERENCES

[1] I. BÁRÁNY AND Z. FÜREDI, *Computing the volume is difficult*, in Proc. 18th Annual ACM Symposium on Theory of Computing, 1986, pp. 442–447.
[2] G. ELEKES, *A geometric inequality and the complexity of computing volume*, Discrete and Computational Geometry, to appear.
[3] M. R. GAREY AND D. S. JOHNSON, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman, San Francisco, 1979.
[4] L. LOVÁSZ, *An Algorithmic Theory of Numbers, Graphs and Convexity*, CBMS-NSF Regional Conference Series in Applied Mathematics 50, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1986.
[5] A. SCHRIJVER, *Theory of Linear and Integer Programming*, John Wiley, Chichester, 1986.
[6] L. G. VALIANT, *The complexity of enumeration and reliability problems*, SIAM J. Comput., 8 (1979), pp. 410–421.
[7] S. G. WILLIAMSON, *Combinatorics for Computer Science*, Computer Science Press, Rockville, MD, 1985.