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Department of Computing Science

K-GREEDY ALGORITHMS

by

A M FRIEZE

Department of Computer Science & Statistics

Queen Mary College, University of London

ABSTRACT K-Greedy algorithms are natural generalisations of the classic Greedy Algorithm.

In this paper we discuss their application to the problem of maximising a separable concave function over sets of non-negative integer vectors satisfying a simple closure property.

We describe necessary and sufficient conditions for these algorithms to be truly optimising and conduct a worst-case analysis of a particular case when they are not.

The results generalise well known results on matroids and polymatroids.

We also give some results on an interchange algorithm.

Key-words: Matroids, Independence Systems, Greedy Algorithms, Worst Case Analysis.

§1. Introduction

Let E be a finite set and let F be a non-empty family of subsets of E satisfying: if $I \varepsilon F$ and $J \subseteq I$ then $J \varepsilon F$. The members of F are called independent sets and (E,F) is called an independence system.

Many optimisation problems can be posed as: given w: E→R the set of real numbers

(1.1) Maximise $w(A) = \sum_{e \varepsilon A} w(e)$ subject to $A \varepsilon F$

The greedy algorithm for this problem is

(1.2a) $G: = \phi$

(1.2b) Order the elements of $E = \{e_1, \ldots e_n\}$ so that

$w(e_1) \geq w(e_2) \geq \ldots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \ldots \geq w(e_n)$

(1.2c) For $i=1,\ldots k$ do if $G \cup \{e_i\} \varepsilon F$ then $G := G \cup \{e_i\}$

(1.2d) Output the greedy solution G:

The performance of the greedy algorithm is well understood:

(PG_1) For a fixed independence system (E,F) the greedy algorithm produces a maximum weight independent set for arbitrary w if and only if (E,F) is a matroid - see Edmonds [2] or Lawler [11] or Welsh [12].

(PG_2) If (E,F) is not necessarily a matroid and for given w M is a maximum weight independent set then

(1.3) $w(G) \geq q(E,F) w(M)$

where q(E,F) is the rank quotient (defined later on). Furthermore for any (E,F) there exists w such that (1.3) holds with equality - see Jenkyns [7] or Korte and Hausmann [9].

The purpose of this paper is to generalise the greedy algorithm and (i) describe those systems for which the generalisation optimises and (ii) provide some worst-case analysis when it does not.

In §2 we describe a greedy approach to a generalisation of problem (1.1). The characterisation of when this approach works generalises PG_1.

In §3 we analyse the worst-case performance of a particular example of this method. This leads to a generalisation of PG_2.

The results of this paper generalise previous results given in Frieze [4] and are related to results given in Hausmann, Jenkyns and Korte [6] and Jenkyns [8].

In §4 we discuss an interchange heuristic.

§2. A Greedy Algorithm for a Generalised Independence System

Let m,n be positive integers and let $M=\{0,1,\ldots m\}$.

A Generalised Independence System, GIS, is a set $X \subseteq M^n$ satisfying: $\underline{x} \in X$ and $\underline{x} \geq \underline{y} \in M^n$ implies $\underline{y} \in M^n$.

(Vectors are compared component-wise).

When m=1 if we take $E=\{1,\ldots n\}$ and $F=\{I(\underline{x}):\underline{x} \in X\}$ where $I(\underline{x})=\{j:x_j=1\}$ then (E,F) is an independence system and conversely.

GIS's can for example be the solution sets of integer programs with non-negative constraint matrices. Greedy algorithms and GIS's are discussed in Edmonds [3] and Dunstan and Welsh [1].

The optimisation problem discussed here is the following: given concave functions $w_j:M \to R$ with $w_j(0)=0$ for j=1,...n

(2.1) Maximise $w(\underline{x}) = \sum_{j=1}^{n} w_j(x_j)$ subject to $\underline{x} \in X$

Note that when m=1 (2.1) 'reduces' to (1.1).

Notation

For j=1,...n $u_j(w) = \max(x \in M:w_j(x)>w_j(x-1))$

$w_j(-1)$ is defined to be $-\infty$.

For $\underline{x} \in X$ $|\underline{x}| = \sum_{j=1}^{n} x_j$

For $\underline{x} \in X$ $ADD(\underline{x},w,k) = \{\underline{y} \in X: |\underline{y}| = k$ and $\underline{x}+\underline{y} \in X$ and $x_j+y_j \leq u_j(w)$ for $j=1,\ldots n\}$.

The greedy algorithm to be described has a parameter K. For each positive integer t K(t) is a set of non-negative integers satisfying $\phi \neq K(t) \subseteq \{0,1,\ldots t-1\}$.

K-Greedy Algorithm GA(K)

The algorithm computes a set of vectors $\underline{g}^{(t)}$ where $|\underline{g}^{(t)}| = t$ in a 'greedy' manner.

(S0) $t:=0; \underline{g}^{(0)}:=\underline{0}$

(S1) $t:=t+1$

(S2) For $k\varepsilon K(t)$ let $S_k=ADD(\underline{g}^{(k)},w,t-k)$

If $S_k=\phi$ for all $k\varepsilon K(t)$ go to (S3)

else for each $S_k\neq\phi$ define $\underline{y}^{(k)}$ by

$$w(\underline{g}^{(k)}+\underline{y}^{(k)}) = \max(w(\underline{g}^{(k)}+\underline{y}):\underline{y}\varepsilon S_k)$$

Then define $\underline{g}^{(t)}$ by

$$w(\underline{g}^{(t)}) = \max(w(\underline{g}^{(k)}+\underline{y}^{(k)}):S_k\neq\phi)$$

go to (S1)

(S3) Output $\underline{g}$ where $w(\underline{g})=\max(w(\underline{g}^{(s)}):s<t)$

If $m=1$ and $K(t)=\{t-1\}$ then we have the greedy algorithm of §1. If $m\geq 1$ and K is the same then we have the greedy algorithm of Dunstan and Welsh (more or less).

We can give a complete characterisation of those GIS's for which GA(K) is always valid, only when $|K(t)| = 1$ for all t. If $|K(t)| \geq 2$ our characterisation is only sufficient. ($|A|$, where A is a set, denotes its cardinality.)

Note the somewhat 'premature' stopping rule in (S2) - it may be possible to continue with larger t. This rule is necessary to prove theorem 2.2 below.

A GIS X is said to be a K-matroid if the following holds:

(2.2) If $\underline{z}\varepsilon X$ and $|\underline{z}|=t$ and $\underline{x}^{(i)}$ $i\varepsilon K(t)$ satisfy $|\underline{x}^{(i)}|=i$ for $i\varepsilon K(t)$ then there exist $k\varepsilon K(t)$ and $\underline{y}$ such that

(a) $|\underline{y}|=t-k$ (b) $\underline{x}^{(k)}+\underline{y}\varepsilon X$ (c) $y_j>0\rightarrow y_j\leq z_j-x_j^{(k)}$

Thus if $m=1$ and $K(t)=\{t-1\}$ (2.2) is a 0-1 vector form of matroid axiom. If $m\geq 1$ and $K(t)=\{t-1\}$ then K-matroids are the integer points of integer polymatroids.

Theorem 2.1

If X is a K-matroid then GA(K) solves (2.1).

Proof

We show by induction on p that if $\underline{z}\varepsilon X$ and $|\underline{z}|=p$ and $z_j\leq u_j(w)$ for $j=1,\ldots n$ then $\underline{g}^{(p)}$ is defined and $w(\underline{z})\leq w(\underline{g}^{(p)})$. Since w_j is concave for $j=1,\ldots n$ no better $\underline{z}$ exists with $z_j>u_j(w)$.

The induction hypothesis is clearly true for p=0 and so assume it to be true for p<q. Let $\underline{z}\varepsilon X$ and $|\underline{z}|=q$. The inductive assumption implie $\underline{g}^{(t)}$ exists for $t\varepsilon K(q)$. From (2.2) there exists $\underline{y}$ and k with properties (a) - (c). Note that (c) implies that

$g_j^{(k)}+y_j\leq\max(g_j^{(k)},z_j)\leq u_j(w)$.

Thus $ADD(\underline{g}^{(k)},w,q-k)\neq\phi$ and so $\underline{g}^{(q)}$ exists. Now

$$w(\underline{g}^{(q)})\geq w(\underline{g}^{(k)}+\underline{y})$$

$$=w(\underline{g}^{(k)})+\sum_{j=1}^{n}(w_j(g_j^{(k)}+y_j)-w_j(g_j^{(k)}))$$

$$\geq w(\underline{z}-\underline{y})+\sum_{j=1}^{n}(w_j(z_j)-w_j(z_j-y_j))$$

by induction and the concavity of w_j for $j=1,\ldots n$

$$=w(\underline{z})$$

We can prove the following restricted converse

Theorem 2.2

If $|K(t)|=1$ for t>0 and GA(K) is guaranteed to solve 2.1 for arbitrary concave w then X is a K-matroid.

Proof

Let $K(t)=\{k(t)\}$ and $\underline{z},\underline{x}\varepsilon X$ satisfy $|\underline{z}|=p$ and $|\underline{x}|=k(p)$. Define w_j for $j=1,\ldots n$ by

$$w_j(\xi) = \begin{cases} \xi & 0\leq\xi\leq u_j=\max(x_j,z_j) \\ u_j & u_j\leq\xi \end{cases}$$

Note that $w(\underline{y})\leq|y|$ for all $\underline{y}\varepsilon X$. It follows that GA(K) could at some stage produce $\underline{g}^{(k(p))}=\underline{x}$. If the algorithm were to stop before t reached p then $w(\underline{g})<w(\underline{z})$. Similarly on t reaching p we must have $S=ADD(\underline{g}^{(k(p))},w,p-k(p))\neq\phi$. But for $y\varepsilon S$ $x_j+y_j\leq u_j(w)=u_j$ implies that $x_j+y_j\leq z_j$ whenever $y_j>0$ for $j=1,\ldots n$ and so (2.2) holds.

Theorems 2.1 and 2.2 can be seen to generalise known results on matroids and polymatroids by taking $K(t)=\{t-1\}$.

It is not difficult to construct K-matroids for different K that are not matroids or polymatroids.

For example consider the set X of solutions to

$$p\sum_{i=1}^{r} x_i + (p+1)\sum_{i=r+1}^{n} x_i \leq ap+k$$

$$0 \leq x_i \leq m \text{ and integer}$$

where a,p,k,r,n are positive integers and $mr, m(n-r) \geq a > k+1$. This is not a K-matroid for $K(t)=\{t-1\}$ as can be seen by choosing $\underline{x}^{(0)}, \underline{x}^{(1)} \varepsilon X$ with $|\underline{x}^{(0)}| = \sum_{i=1}^{r} x_i^{(0)} = a$ and with $|\underline{x}^{(1)}| = \sum_{i=r+1}^{n} x_i^{(1)} = a-1$.

However X is an M_k-matroid where $M_k(t) = \{\max(0,t-k)\}$. In this case $GA(M_k)$ is the k-greedy algorithm considered in [4].

When $K(t)=k\lfloor t/k \rfloor$ we have an 'enhancement' of the $\leq$k-greedy algorithm of [6].

Note that being a M_k-matroid implies being an $M_{k\ell}$-matroid for $\ell \geq 1$ and so matroids and polymatroids which are M_1-matroids form subclasses of this type of system.

It has not however been possible to find interesting K-matroids for which GA(K) provides a new efficient (polynomial time) algorithm other than of course for $K=M_1$.

This is rather disappointing but one lives in hope. One can however look at this from another viewpoint. For some functions K e.g. $K(t)=\{t-1,\ldots\max(t-k,0)\}$ for a fixed k the K-greedy algorithm is not at all an unnatural generalisation of the standard greedy algorithm. The preceding analysis indicates however that this is not likely to be an optimising algorithm for any interesting problem.

Finally in this section we note that theorem 2.1 is still true if we define $|\underline{x}| = \sum_{j=1}^{n} a_j x_j$ where $a_1, \ldots a_n$ are positive integers and X satisfies the extra axiom (trivial for $a_1 = \ldots = a_n = 1$)

(2.3) $|\underline{x}| = t$ and $k \varepsilon K(t)$ implies there exists $\underline{y} \leq \underline{x}$ such that $|\underline{y}| = k$.

The proof of theorem 2.1 is unchanged. The extra axiom is used to imply the existence of $\underline{g}^{(t)}$ for $t \varepsilon K(q)$.

As an example if $a_j \varepsilon \{1,2\}$ for $j=1,\ldots n$ and $X = \{\underline{x} \varepsilon M^n : \sum_{j=1}^{n} a_j x_j \leq L\}$ for some positive integer L then X is an M_2-matroid.

Finally for theorem 2.2 if we assume (2.3) we can show (2.2) assuming GA(K) always solves problem 2.1. However we cannot deduce (2.3) as opposed to assuming it.

§3. Worst-Case Analysis

For the remainder of the paper we restrict our attention to independence systems defined in §1 and revert to sets as opposed to vectors. We shall assume $E = \{1,2,\ldots n\}$ and in problem 1.1 use $w(j)$ and w_j interchangeably.

Notation

For $S \subseteq E$ a basis of S is a maximal (with respect to inclusion) independent subset of S.

$ur(S) = \max(|I| : I \text{ is a basis of } S)$ = upper rank of S

$lr(S) = \min(|I| : I \text{ is a basis of } S)$ = lower rank of S

The rank quotient $q(E,F)$ is defined by

$$q(E,F) = \min_{S \subseteq E}(lr(S)/ur(S))$$

Inequality (1.3) can be interpreted as 'the worst-case performance of the greedy algorithm occurs when $w_j = 1$ or 0'.

Equivalently for this algorithm we can say $w_j=1$ or $w_j<0$. We shall generalise this statement to a wider class of greedy algorithm.

Specifically now let K be a sequence $k_1, k_2, \ldots k_a$ of positive integers satisfying $k_1+\ldots+k_a \geq n$.

Let $w: E \to R$ be given. The K-greedy algorithm of this section is

(S0) $G_K := \phi$; $t:=0$

(S1) $t:=t+1$

(S2) Let $S=\{I : |I|=k_t,\ I \cap G_K = \phi,\ I \cup G_K \varepsilon F, w(I)>0$ and $w_j \geq 0$ for $j \varepsilon I\}$

If $S=\phi$ go to (S3)

else let $w(I^*)=\max(w(I) : I \varepsilon S)$

G_K: $G_K \cup I^*$

go to (S1)

(S3) Output G_K as solution.

Notation

For a fixed (E,F) and a given w and sequence $J=j_1,j_2,\ldots j_b$ where $j_1+\ldots+j_b\geq n$ let

$F_J(w)=\{I\varepsilon F : |I| \varepsilon \{j_1,j_1+j_2,\ldots j_1+\ldots j_b\}$ and $w_i\geq 0$ for $i\varepsilon I\}$

If $F_J(w) \neq \phi$ define $M_J(w)$ by $w(M_J(w)) = \max (w(I) : I\varepsilon F_J(w))$ otherwise $M_J(w) = \phi$.

Also let $F_d = \{I\varepsilon F : |I| \geq d\}$ for non-negative integer d.

Next let

WORST(J,K) = $\mathrm{Sup}(w(M_J(w))/w(G_K)$: w satisfies $w(G_K)>0$ for G_K chosen by K-greedy)

For $j_1<k_1$ we can have WORST(J,K)=∞ if there is some $I\varepsilon F$ with $|I| = j_1$ and an $\ell\varepsilon I$ such that $\ell\notin J$ for any $J\varepsilon F_{k_1}$. One simply makes w_ℓ large and $w_j=1$ for $j\neq\ell$.

The worst-cast result we have proved requires:

(3.1a) $j_1\geq k_1$

(3.1b) $j_r\leq k_s$ for any $r\geq 1$, $s\geq 2$.

The following lemma requires no proof:

Lemma 3.1

If $|J| = t$ and $|S|\geq t$ and $w(J)\geq w(I)$ for all $I\subseteq S$ with $|I| = t$ then $w(S)/|S| \leq w(J)/|J|$.

Furthermore the second and third inequalities can be reversed

Keeping J,K fixed now abbreviate WORST(J,K) by WST.

We ignore the moment the trivial case with $F_{k_1}=\phi$ and WST undefined. It follows from the lemma that $WST\leq n/k_1$ if $F_{k_1}\neq\phi$.

Elementary topological arguments can be used to show that there exist $\hat{w},\hat{M}=M_J(\hat{w})$ and K-greedy solution $\hat{G}$ such that WST = $\hat{w}(\hat{M})/\hat{w}(\hat{G})$. We assume that $|\hat{M}| + |\hat{G}|$ is as small as possible under these circumstances.

Now let $\hat{M}=\{m_1,\ldots m_a\}$ where $w(m_i)\geq w(m_{i+1})$ for $1\leq i<a$. If $a=j_1+\ldots+j_p$ we define $M_t=\{m_{\ell_{t-1}+1},\ldots m_{\ell_t}\}$ where $\ell_t=j_1+\ldots+j_t$ for $0\leq t\leq p$ assuming $\ell_p=a$ and then define $X_i=\hat{w}(M_i)$ for $1\leq i\leq p$.

Next let $\hat{G}=\bigcup_{i=1}^{q} G_i$ where $G_1,\ldots G_q$ are the q successive augmentations in step (S2) of the particular application that produced $\hat{G}$. Let $Y_i=\hat{w}(G_i)$ for $1\leq i\leq q$.

The following properties of $X_1,\ldots X_p,Y_1,\ldots Y_q$ follow from lemma 3. and the definition of K-greedy.

(3.2) $\quad X_t/j_t \geq X_{t+1}/j_{t+1} \quad$ for $\quad 1\leq t<p$

(3.3) $\quad Y_t/k_t \geq Y_{t+1}/k_{t+1} \quad$ for $\quad 1<t<q$

We next define the function h by

(3.4) $\quad h(r) = \min(s : Y_r/k_r\geq X_s/j_s)$

We show later that $h(q)\leq p$ and it follows that

(3.5) $\quad 1=h(1)\leq h(2)\leq\ldots\leq h(q)\leq p$

<u>Notation</u>

Let $\square$ represent a symbol used to identify some entity which is indexed by the positive integers. We use the notation $\square[i : j]$ to represent $\square_i + \square_{i+1}+\ldots \square_j$ e.g. $X[i:j]=X_i+\ldots+X_j$ Note that when $j<i$ we take $\square[i:j] = 0$.

The following lemma is the basis for our result.

<u>Lemma 3.2</u>

(a) $\quad Y_s/k_s\geq X_r/j_r$ or $WST\leq X[r+1:p]/Y[s:q]$ for $\quad 1\leq r\leq p$ and $2\leq s\leq q$

(b) $\quad Y_q/k_q\geq X_p/j_p$

<u>Proof</u>

(a) $\quad$ Let $S = \bigcup_{i=1}^{s-1} G_i$ and $T = S\cup \bigcup_{i=1}^{r} M_i$. If there exists $Z\subseteq T-S$ with with $|Z|=k_s$ and $S\cup Z\varepsilon F$ then we deduce from the definition of the algorithm that $Y_s\geq w(Z)\geq w(A)$ where $A=\{m_{\ell_r-k_s+1},\ldots m_{\ell_r}\}$. Lemma 3.1 plus the fact that $k_s\geq j_r$ implies $w(A)/k_s\geq X_r/j_r$. - note 3.1b.

On the other hand if no such Z exists then we can define $\tilde{w}$ by $\tilde{w}_i=\hat{w}_i$ for $i\varepsilon T$ and $\tilde{w}_i= -1$ otherwise. Thus using $\tilde{w}$ the algorithm could choose S as its solution. We deduce then that

(3.6) $\quad WST\geq X[1:r]/Y[1:s-1]$

The result follows as $WST=X[1:p]/Y[1:q]$

(b) If $q=1$ we can apply lemma 3.2 otherwise let $r=p$ and $s=q$ and argue as in (a) up to (3.6). Thus $Y_q/k_q \geq X_p/j_p$ or $WST \geq X[1:p]/Y[1:q-1]$. The latter inequality is impossible as $Y_q>0$.

The next two lemmas are concerned with bounding the ratio of two sums of real numbers satisfying the given properties of the X_i, Y_j.

Lemma 3.3

Let real non negative numbers $x_1,\ldots x_p, y_1,\ldots y_q$ satisfy

(3.7a) $x_1/j_1 \geq x_2/j_2 \geq \ldots \geq x_p/j_p$

(3.7b) $y_1/k_1 \geq y_2/k_2 \geq \ldots \geq y_q/k_q$

(3.7c) $y_1>0$

(3.7d) $x_{h(s)}/j_{h(s)} \leq y_s/k_s$

for positive integers $h(1),\ldots h(q)$ satisfying

(3.7e) $1=h(1) \leq h(2) \leq \ldots \leq h(q) \leq p$

If $\rho=x[1:p]/y[1:q]$ then

(3.8) $\rho \leq \max(j[1:t]/k[1:g(t)] : 1 \leq t \leq p)$

where $g(t)=\max(i:h(i) \leq t)$

Proof

For fixed values of $x_1,\ldots x_p$ ρ will be maximised by putting $y_s=k_s x_{h(s)}/j_{h(s)}$ for $1 \leq s \leq q$. We can scale these quantities to satisfy $y[1:q]=1$.

Then defining $z_t=x_t/j_t$ for $1 \leq t \leq p$ we have

$\rho \leq \max \quad j_1 z_1 + \ldots j_p z_p$

subject to

(3.9) $k_1 z_{h(1)} + \ldots k_q z_{h(q)} = 1$

$z_1 \geq z_2 \geq \ldots \geq z_p \geq 0$

Equation (3.9) can be restated as

$k[1:g(1)]z_1 + k[g(1)+1:g(2)]z_2 + \ldots k[g(p-1)+1:g(p)]z_p = 1$

Thus defining $u_p=z_p$ and $u_t=z_t-z_{t+1}$ for $1\leq t<p$ we obtain

$$\rho\leq \max \quad j_1u_1+j[1:2]u_2+\ldots+j[1:p]u_p$$

subject to

$$k[1:g(1)]u_1+k[1:g(2)]u_2+\ldots+k[1:g(p)]u_p=1$$

$$u_1,\ldots u_p\geq 0$$

The solution to the above linear program is well known and gives the required result.

Lemma 3.4

Suppose that the conditions of lemma 3.3 are satisfied and further that $h(t)=1$ or

(3.10) $\rho\leq x[h(t):p]/y[t:q]$

holds for $1\leq t\leq q$. Then

(3.11) $\rho\leq j[1:p]/k[1:q]$

Proof

We shall prove first using a backward induction that $h(t)>1$ implies

(3.12) $\rho\leq j[h(t):p]/k[t:q]$

If $h(q)>1$ then (3.12) is obvious from (3.7) and (3.10). Suppose then there exists $s<q$ such that $h(s)>1$ and that (3.12) holds for all $t>s$. Applying lemma 3.3 to RHS of (3.10) with $t=s$ gives

(3.13) $\rho\leq\max(j[h(s):u]/k[s:g(u)]:h(s)\leq u\leq p)$

Let u^* denote a value of u maximising the RHS of (3.13). If $u^*=p$ then (3.13) is (3.12) with $t=s$.

So assume $u^*<p$ as u^* maximises the RHS of (3.13) we must have $g(u^*+1)>g(u^*)$ else u^*+1 is better than u^*. But this implies

(3.14) $h(g(u^*)+1)=u^*+1$

Thus

(3.15) $\rho\leq j[h(s):h(g(u^*)+1)-1]/k[s:g(u^*)]$

Combining (3.15) and (3.12) with $t=g(u^*)+1>s$ gives (3.12) with $t=s$ and completes the induction.

We now use this result in (3.8). Thus let t^* be a value of t maximising the RHS of (3.8). If $t^*=p$ we have (3.11) and so assume $1\leq t^*<p$. We argue as before that (3.14) holds this time with u^* replaced by t^*. Thus

(3.16) $\rho\leq j[1:h(g(t^*)+1)-1]/k[1:g(t^*)]$

Now $h(g(t^*)+1)>1$ by (3.14) and so (3.12) holds with $t=g(t^*)+1$

Combining this with (3.16) gives $\rho\leq j[1:p]/k[1:q]$

We have already shown that the X_i,Y_j and h as defined prior to lemma 3.2 satisfy the conditions of lemma 3.4. Our next task is simply to put this result in a more convenient form.

Notation

Let $S\subseteq E$ and $D=d_1,\ldots d_s$ be a sequence of positive integers with $d[1:s]\geq n$. Let $S(D)=\{d_1,d[1:2],\ldots d[1:s]\}$. Let $F_D=\{I\varepsilon F:|I|\varepsilon S(D)\}$

A D-basis of S is a set I satisfying

(1) $I\subseteq S$ (2) $I\varepsilon F_D$ and (3) no J strictly containing I satisfies (1) and (2).

Let $ur_D(S)=\max(|I|:I$ is a D-basis of $S)$

$lr_D(S)=\min(|I|:I$ is a D-basis of $S)$

where $ur_D(S)=0$ and $lr_D(S)=1$ if S contains no D bases.

For the given sequences J,K define $W(J,K)=\max(ur_J(S)/lr_K(S):S\subseteq E)$

Theorem 3.1

If J,K satisfy (3.1) then

(3.17) $WST=WORST(J,K)=W(J,K)$

Proof

Equation (3.17) is trivial if F_J or $F_K=\emptyset$ and so we assume $F_J,F_K\neq\emptyset$

(a) $WST\geq W(J,K)$

Let T,U,V be such that $W(J,K)=ur_J(T)/lr_K(T)$ and $ur_J(T)=|U|$ and $lr_K(T)=|V|$ where U is a J-basis of T and V is a K-basis of T. Define w by $w(e)=1$ for $e\varepsilon U\cup V$ and $w(e)=-1$ otherwise. Then $M_J(w)=U$ and G_K could be V giving $WST\geq|U|/|V|$.

(b) $WST\leq W(J,K)$

Let $\hat{G},\hat{M}$ be as before and let $\hat{T} = \hat{M}\cup\hat{G}$. Now $X_p>0$ else $|\hat{M}|$ can be made smaller. Thus $\hat{G}$ is a K-basis of T for if there exists $Z\subseteq\hat{M}-\hat{G}$ with $\hat{G}\cup Z\varepsilon F$ and $|Z| =k_{q+1}$ then $w(Z)>0$ as $k_{q+1}\geq j_p$. But this implies the K-greedy algorithm stopped prematurely. Thus

$$W(J,K)\geq|\hat{M}|/|\hat{G}|$$

$$\geq WST \qquad \text{by lemma 3.4}$$

Note that PG_2 of §1 follows by taking J=K=1,1,1...1

We used inequality 3.1 in our proof of lemma 3.2. If 3.1 does not hold then theorem 3.1 can fail to hold. Consider the following example:

$E=\{e_1,\dots e_5,f_1,\dots f_7\}$. The maximal elements of F are
$I_e=\{e_1,\dots e_5\},I_f=\{f_1,\dots f_7\},\{e_i,e_j,e_k,f_6,f_7\},$
$\{e_i,e_j,f_k,f_6,f_7\},\{e_i,f_j,f_k,f_6,f_7\}$
where $1\leq i,j,k\leq 5$

Let J=1,3,3,3,... and K=1,2,2,2,... . One can check that $W(J,K)=7/5$ but putting $w(g)=1$ for $g\varepsilon E-\{e_4,e_5,f_6,f_7\}$ and $w(g)=1/2$ otherwise gives $w(I_f)=6$ and $w(I_e)=4$. Thus $WST\geq 6/4$ as I_e is a possible K-greedy solution.

We next consider the case where J and K are of the form $Z(i,k)=i,k,k,k,\dots$ for some $i\leq k$. If $J=Z(i_1,k)$ and $K=Z(i_2,k)$ where $i_1\geq i_2$ then theorem 3.1 applies. If we apply the Z(i,k)-greedy algorithm for i=1,...k and then take best of the k solutions generated we can use theorem 3.2 to show that this will be valid for arbitrary w if and only if (E,F) is a M_k-matroid.

We can give no non-trivial worst case results for the greedy algorithm of §2. In the case of an independence system where $|K(t)|=1$, although it can be split into a number of applications of the algorithm of this section the rules for termination are incompatible. If the termination rules are made to coincide the best we have been able to do is to apply theorem 3.1 separately. This is too weak even to determine if the bound q(E,F) for the ordinary greedy algorithm is achieved.

§4. Interchange Algorithms

We consider here a nice property of M_k-matroids which generalises a known property of matroids.

Theorem 4.1

Let (E,F) be an M_k-matroid. If $I^*\varepsilon F$ is such that

(4.1) $w(I^*)\geq w(I)$ for all $I\varepsilon F$ with $|I| = |I^*|$ and $|I-I^*|\leq k$

then $w(I^*)\geq w(I)$ for all $I\varepsilon F$ with $|I|=|I^*|$.

(i.e. a 'local' optimum is also a global optimum over independent sets of a fixed size.)

Proof

Suppose I^* satisfies 4.1 and $I^*=\{e_1,\ldots e_p\}$ where $w(e_i)\geq w(e_{i+1})$ for $1\leq i<p$. Let $q=p \bmod k$ and let $I_0=\{e_1,\ldots e_q\}$, $I_t=\{e_{q+(t-1)k+1},\ldots e_{q+tk}\}$ for $t=1,\ldots r=[p/k]$.

Let $J=\{f_1,\ldots f_p\}\varepsilon F$ where $w(f_i)\geq w(f_{i+1})$ for $1\leq i<p$ and let $J_0,J_1,\ldots J_r$ be defined similarly. We show $w(I^*)\geq w(J)$ by showing that $w(I_t)\geq w(J_t)$ for $t=0,1,\ldots r$. Suppose to the contrary that $w(I_s)<w(J_s)$. Let

$$A = \bigcup_{t=0}^{s-1} I_t \quad (=\emptyset \text{ if } s=0).$$

Let $B = \bigcup_{t=1}^{s} J_t$. Now either trivially if s=0 or from axiom 2.2 if s>0 there exists $Z\subseteq B-A$ such that $|Z|=|I_s|$ and $C=A\cup Z\varepsilon F$. Note that $w(Z)>w(I_s)$. Let D be obtained by successively using 2.2 to augment C,k elements at a time from I^*-C until a set of size p is reached. We need only note that $w(D)>w(I^*)$ and yet $|D-I^*|\leq k$.

We paraphrase 4.1 by saying that 'in an M_k-matroid k-optimality $\rightarrow$ optimality. When k=1 there is the converse result that 1-optimality implies optimality only in matroids. For $k\geq 2$ this is not the case. We have given a great deal of (unsuccessful) thought to the problem of characterising when theorem 4.1 holds.

For k=2 we note that theorem 4.1 holds when $F=F_1\cap F_2$ where (E,F_1) is an arbitrary matroid and $F_2=\{I\subseteq E:|I\cap S|\leq p\}$ for some fixed p and $S\subseteq I$ - a particularly simple matroid intersection. See Glover and Klingman [5]

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