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HAMILTONIAN CYCLES IN A CLASS OF RANDOM GRAPHS: ONE STEP FURTHER

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Let D_k denote the underlying graph of a random k-outregular digraph $\vec{D}_k$. We shall show that the probability that D_5 is hamiltonian tends to 1 as the number of its vertices tends to infinity.

1. Introduction

Let $\vec{D}_k=\vec{D}_k(n)$ be a digraph with the vertex set $V_n=\{1,2,...,n\}$ in which each vertex chooses randomly and independently k out-neighbours from V_n and let D_k be the underlying graph of $\vec{D}_k$. We shall say that D_k has some property a.s. (*almost surely*) if the probability that D_k has this property tends to 1 as $n\to\infty$. In this paper we shall consider the problem of the existence of a hamiltonian cycle in D_k. In [4] Fenner and Frieze showed that D_k is a.s. hamiltonian for $k\geqslant 23$ and in [5] it has been established algorithmically that D_k has this property for $k\geqslant 10$. On the other hand, D_2 a.s. contains

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[2]) Permanent position.

a vertex adjacent to three others of degree two, so three is the smallest possible candidate for D_k to a.s. have a hamiltonian cycle. This paper is a further small step toward the solution of this problem.

Main Theorem. *D_5 is a.s. hamiltonian. Moreover, there is an algorithm which a.s. finds a hamiltonian cycle in D_5 in polynomial time.*

2. Description of the Algorithm

The following result of Frieze [6] about matchings in D_2 will be needed for our arguments.

Lemma 1. *D_2 a.s. contains a matching of size* $\lfloor \frac{n}{2} \rfloor$. □

Remark. Actually [6] shows only that $D_2(n)$ with n even a.s. contains a perfect matching. But if n is odd, consider $\vec{D}_2(n+1)$. With probability $1-o(1)$, $D_2(n+1)$ contains a perfect matching and with probability $e^{-2}+o(1)$ vertex $n+1$ has indegree 0 in $\vec{D}_2(n+1)$. But conditional on this latter event, the first n vertices induce $\vec{D}_2(n)$ and so we can extend the result of [6] to n odd.

It is well known that the problem of finding a largest matching is solvable in polynomial time, Edmonds [2]. Furthermore, an algorithm with time complexity $O(n^{1.5})$ (when run on $D_2(n)$) is described by Micali and Vazirani [7]. Let us call this algorithm FINDMATCH. However, to obtain a "random" matching we shall modify this procedure slightly.

FUNCTION RFM ($\vec{D}_2$);
input $\vec{D}_2$ plus a random permutation σ of V_n;
$\tilde{D} := \{\{\sigma(i),\sigma(j)\} : (i,j) \in D_2\}$;
find matching $\tilde{M}$ in $\tilde{D}$ using FINDMATCH (or FAIL)
(FINDMATCH is allowed to take account of the orientation of the edges of $\tilde{D}$, but has no reason to do it. Also it is ignorant of σ);
output RFM $:= \{\{\sigma^{-1}(i),\sigma^{-1}(j)\} : (i,j) \in \tilde{M}\}$

Lemma 2. *Each possible matching of size* $\lfloor \frac{n}{2} \rfloor$ *is equally likely as output from* RFM.

Proof. For a matching M and permutation σ let $\sigma M=\{\{\sigma(i),\sigma(j)\} : (i,j) \in M\}$. Note that if M is fixed and σ ranges over all permutations then σM ranges over all matchings of size $|M|$ and each such matching arises the same number of times, μ say. Also for any fixed σ, $\tilde{D}$ above has the same distribution as D_2. Thus

$$\text{Prob}\{\text{RFM}=M^*\}=\sum_{\sigma*}\frac{1}{n!}\text{Prob}\{\tilde{M}=\sigma^*M^*|\sigma=\sigma^*\}$$

$$=\sum_{M}\frac{\mu}{n!}\text{Prob}\{\text{FINDMATCH finds } M \text{ in } \vec{D}_2\}$$

which is independent of M^*. □

The idea of the proof of the Main Theorem goes as follows. We split all arcs randomly into five groups $\vec{E}_1, \vec{E}_2,..., \vec{E}_5$ in such a way that every vertex is a head of one arc from each group and each arc is equally likely to be in each group. Since $E_1 \cup E_2$ and $E_3 \cup E_4$ are copies of D_2 we can apply RFM to find matchings M_1 and M_2 in them. Thus we obtain a 2-factor in $D_4=E_1 \cup E_2 \cup E_3 \cup E_4$ (to be precise we can also have some number of isolated edges and one other path when the number of vertices of D_4 is odd). Then we shall join all components of the 2-factor into one cycle, using edges from E_5 if necessary. To do it we shall need another procedure, known as *Pósa's transformation*, finding for a given path $P=v_0 v_1...v_l$ a family of paths Π and a set of vertices $X \subseteq V(P)$ such that for each $x \in X$ the family Π contains a unique path with endpoints v_0, x which uses all the vertices $V(P)$.

PROCEDURE ROTATION (P, v_0, Π)

input path $P=v_0 v_1...v_l, E$;

$\Pi_1 := \{(P, v_0, v_l)\}$;

$\Pi := \emptyset$;

$X := \{v_l\}$;

for $(P', v_0, w_l) \in \Pi_1$ **do**

begin

> **suppose** $P'=v_0...w_{l-1} w_l$;
>
> $\Pi := \Pi \cup \{(P', v_0, w_l)\}$;
>
> $Y := \{w_{i+1} : \{w_i, w_l\} \in E, w_{i+1} \notin X, 1 \leqslant i \leqslant l-2\}$;
>
> $\Pi_2 := \{(\bar{P}, v_0, w_{i+1}) : \bar{P}=v_0 w_1...w_i w_l w_{l-1}...w_{i+2} w_{i+1}$ and $w_{i+1} \in Y\}$;
>
> $\Pi_1 := \Pi_1 \cup \Pi_2 \setminus \{(P', v_0, w_l)\}$;

$X := (X \cup Y) \setminus \{w_i\}$;
end
return
end

Our algorithm consists of an initialization phase and a rotation phase.

Initialization Phase

input $\tilde{D}_5$;
find E_1, E_2, E_3, E_4, E_5;
$M_1 := \mathrm{RFM}(E_1 \cup E_2)$;
$M_2 := \mathrm{RFM}(E_3 \cup E_4)$;
$M := M_1 \cup M_2$;
$E := E_1 \cup E_2 \cup E_3 \cup E_4$;

We observe that the graph induced by M consists of a set of components which are even cycles $C_1, C_2, \ldots, C_r$ (some of which are merely double edges) and if n is odd there is a further component P_0 which is a path (may be just isolated vertex).

We now find a path P. If n is odd then $P = P_0$, otherwise take as P a path obtained from C_r by deleting an edge. Assume now that the initialization phase ends with a path P and remaining cycles $C_1, C_2, \ldots, C_s$.

Rotation Phase

for $t = 1$ **to** s **do**
begin
  ROTATION(P, v_0, Π) where v_0 is one endpoint of P;
  let $\Pi = \{(P_i, v_0, w_i) : i = 1, 2, \ldots, p\}$ and $q = \min\{p, \lceil 3 \log^3 n \rceil\}$;
  for $i = 1$ **to** q **do** ROTATION (P_i, w_i, Π_i);
  $\tilde{\Pi}_t := \bigcup_{i=1}^{q} \Pi_i$;
  if $\{v, v'\} \in E$ for some $(\tilde{P}, v, w) \in \tilde{\Pi}_t$, $v' \notin \tilde{P}$, $v' \in C' \subset M$ **then**
    $P :=$ a longest path with vertices in $\tilde{P} \cup C'$
  else do
  begin
    find $\{v, w\} \in E_5$ for some $(\tilde{P}, v, w) \in \tilde{\Pi}_t$;
    FAIL$_1$ if none found;
    $C := \tilde{P} + \{v, w\}$;
    if C is a hamiltonian cycle **then** HALT **else**
      find $\{v', w'\} \in E$, $v' \in C$, $w' \notin C$;
    FAIL$_2$ if none found;

```
        P:=a longest path with vertices in C∪M
        end
end
output C
stop
end
```

3. Proof of the Main Theorem

The assertion of the Main Theorem is an easy consequence of the fact that M has few components and the graph $D_4 = E_1 \cup E_2 \cup E_3 \cup E_4$ is so dense that each time the procedure ROTATION leads to a family Π either the set $W = \{w : (P, v, w)\}$ has neighbours outside P or $|\Pi| \geq 0.001\, n$. Note first that since D_4 is a.s. connected [3] the probability that we halt FAIL$_2$ tends to zero. So it is only the possibility of FAIL$_1$ that needs to be considered further.

Let us start with the following result.

Lemma 3. *Let X be a number of components of M. Then, $X \leq 3 \log n$ a.s.*

Proof. We shall only consider the case when n, the number of vertices of D_4, is even (for n odd, delete P_0 and note that what we have left is a random pair of matchings of the remaining vertices).

Consider the cycle C containing vertex 1. We show

(i) $\text{Prob}\{|C| \geq \frac{1}{2}n\} \geq \frac{1}{2}$

and observe that

(ii) deleting C from M leaves a random pair of matchings of the remaining vertices.

To prove (i) note that

$$\text{Prob}\{|C| = 2k\} = \prod_{i=1}^{k-1} \left(\frac{n-2i}{n-2i+1}\right) \frac{1}{n-2k+1} < \frac{1}{n-2k+1}.$$

Indeed, consider M_1-edge $\{1 = i_1, i_2\} \in C$ containing vertex 1. Let $\{i_2, i_3\} \in C$ be the M_2-edge containing i_2. $\text{Prob}\{i_s \neq 1\} = \frac{n-2}{n-1}$. Assume $i_3 \neq 1$ and let $\{i_3, i_4 \neq 1\} \in C$ be the M_1-edge containing i_3. Let $\{i_4, i_5\} \in C$ be the M_2-edge containing i_4. Then $\text{Prob}\{i_5 \neq 1 | i_3 \neq 1\} = \frac{n-4}{n-3}$ and so on.

Hence $\mathrm{Prob}\{|C|<\frac{1}{2}n\}<\sum_{k=1}^{[\frac{1}{2}n]}\frac{1}{n-2k+1}<\frac{1}{2}$ and (i) follows. Consider next the following experiment to determine the sizes of the cycles in M. Choose the size of the cycle containing vertex 1. Now choose the size of the cycle containing a particular vertex from the remaining $n-s$ vertices. Continue until the cycle chosen contains all remaining vertices. Then (i) and (ii) imply that whatever the currently chosen cycle sizes, with the probability at least 1/2 the size of remaining vertex set halves. It is now straightforward to show that it is unlikely that this process continues for $2\log_2 n < 3\log n$ iterations. □

We need also the result of Pósa [8].

Lemma 4. *Let a graph G have the property that for every set S of G, $|S|<m$, we have*

$$|S\cup N(S)|\geqslant 3|S| \tag{$*$}$$

where $N(S)$ denotes the set of all neighbours of S. Moreover, let Π be a family of paths in G obtained as a result of applying procedure ROTATION *to any path P. Then either for some $(P',v,w)\in\Pi$ we have $N(\{w\})\backslash P'\neq\emptyset$, or $|\Pi|\geqslant m$.* □

Now we shall show that for D_4 the assumption of Lemma 4 holds a.s. with $m=0.001n$. Indeed, let X be a random variable which counts all sets in D_4 for which $(*)$ fails. Then

$$EX\leqslant\sum_{k=1}^{m}\binom{n}{k}\binom{n}{2k}\binom{3k}{4}^{k}\binom{n-1}{4}^{-k}$$

$$\leqslant\sum_{k=1}^{m}\left(\frac{e^3n^3}{4k^3}\,\frac{81k^4}{n^4}\right)^k=\sum_{k=1}^{m}\left(\frac{81e^3k}{4n}\right)^k=o(1).$$

Hence, for any path P, the family Π either contains a triple (P',v,w) such that the vertex w has a neighbour outside P' and P' can be extended in this way, or $|\Pi|>0.001n$. In the last case set $W=\{w:(P',v,w)\in\Pi\}$. Then, from Lemma 4, for each vertex w from W we can find a family V_w of paths, beginning in w but with different ends, such that either one of the paths can be extended or $|V_w|\geqslant 0.001n$. We shall show that in the latter case for some w there is a.s. an edge from E_5 which joins w with some $v\in V_w$. Indeed, so far we have dealt only with edges from E, so the edges from E_5 are distributed independently of sets W and V_w. Thus, the probability that if we choose $\lfloor \log^2 n \rfloor$ arcs with tails in vertices $w_1, w_2, \dots, w_{\lceil 3\log^3 n \rceil}$ all of them miss appropriate V_{w_i} is less than $0.999^{\lfloor \log^2 n \rfloor} < n^{-2}$. So we can assume that in each step we use only $\lfloor \log^2 n \rfloor$ elements of E_5 and, due to Lemma 3, $s < 3\log n$, so we always find $\lfloor \log^2 n \rfloor$ vertices for which the arcs of E_5 with the head in them have not been used yet. Hence the probability that we shall halt in FAIL_1 in some $3\log n$ steps is at most

$$1-(1-n^{-2})^{3\log n} = o(n^{-1}).$$

Now we should only count the number of steps in the algorithm. The procedure RFM needs $O(n^{1.5})$ steps, and for each value of t procedure ROTATION appears at most $\lceil 3\log^3 n \rceil + 1$ times. Furthermore, the complexity of ROTATION is of the order of the number of all neighbours of vertices in P times the time per rotation. The first quantity is $O(n)$ and the second is $O(\log n)$ – see Angluin and Valiant [1]. Thus the complexity of ROTATION can be estimated as $O(n \log n)$. Since $s < 3\log n$ the most time consuming part of the algorithm is that dealing with matchings and it determines its complexity to be $O(n^{1.5})$. □

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