Line of Sight Networks

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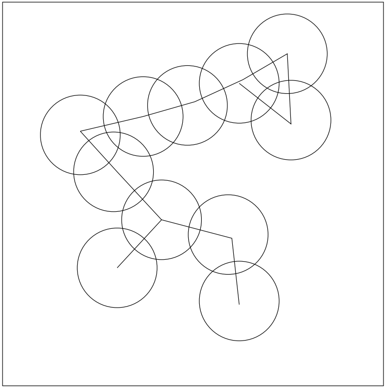
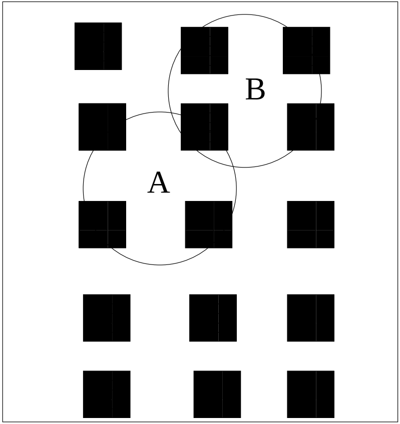
Modelling Wireless Networks



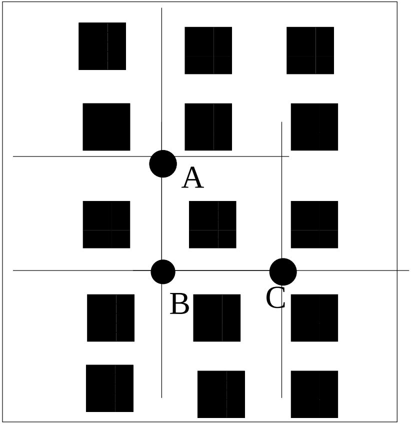
Sensors modelled as discs of a fixed size placed randomly in $[0, 1]^2$. Two discs can "communicate" if they overlap.

Suppose that there are obstacles.



Processors A, B cannot communicate. Need another model.

LINE OF SIGHT MODEL



Sensors are at centres of crosses and can communicate with sensors lying on their arms.

A, B can communicate, but A, C cannot.

$T = \{0, 1, \ldots, n-1\}^2$ is a toroidal grid.

Distance:

$$d((x,y),(x',y')) = \min(|x-x'|, n-|x-x'|) + \min(|y-y'|, n-|y-y'|).$$

Two points are mutually visible if they are in the same row or column and within distance ω of each other.

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We study the random graph G that results if, for some *placement probability* $p > 0$, we locate a node at each point of T independently with probability p, and then connect those pairs of nodes that are mutually visible.

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If $\omega = 1$ then G is a site percolation model.
If $\omega = n$ then G is the line graph of a random bipartite graph with edge probability p.

Connectivity

Theorem

Suppose that $\omega/\ln n \to \infty$ *where* $\omega = n^{\delta}$, $\delta \leq 6/(8k+7)$.

Let $k \geq 1$ *be a fixed positive integer and let* $p = \frac{(1-\frac{1}{2}\delta)\ln n + \frac{k}{2}\ln\ln n + c_n}{2\omega}$. *Then*

$$\lim_{n\to\infty} \mathbf{Pr}(G \text{ is } k\text{-connected}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-\lambda_k} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

where

$$\lambda_k = \frac{2^{k-2}(1-\frac{1}{2}\delta)^k e^{-2c}}{(k-1)!}.$$

Note that if $\omega = o(\ln n)$ and $p = x/\omega$ then the expected number of isolated vertices is

$$n^2 p \left(1 - \frac{x}{\omega}\right)^{4\omega} = n^2 p \exp\left\{-4x\left(1 + \frac{x}{2\omega} + \frac{x^2}{3\omega^2} + \cdots\right)\right\}.$$

So unless $n^2 p \to 0$ or x/ω is very close to one, this expectation tends to infinity. In which case a second moment calculation will show isolated vertices exist **whp**.

To summarize: We need to consider $\omega = \Omega(\ln n)$ to get any sensible results.

Giant Component

G will **whp** contain $\sim n^2p$ vertices. A giant component is therefore one with $\Omega(n^2p)$ vertices.

Theorem

(a) *If* $p = \frac{c}{\omega}$ *where* $c > 1$ *and* $\omega \to \infty$ *then* **whp** G *contains a unique component with* $(1 - o(1))(1 - x_c^2)n^2/\omega$ *vertices, where* x_c *is the unique solution in* $(0, 1)$ *of* $xe^{-x} = ce^{-c}$.

(b) *If* $p = \frac{c}{\omega}$ *where* $c < 1/(4e)$ *and* $\omega \to \infty$ *then* **whp** *the largest component in* G *has size* $O(\ln n)$.

Since (a) is valid for arbitrary $\omega \to \infty$, we can get a result about the existence of a giant component assuming only that ω is sufficiently large.

Finding Paths Between Nodes

Theorem

Let $p = C \ln n/\omega$ for a constant $C \geq 3$. There is a decentralized algorithm that **whp***, given nodes s and t, constructs an s-t path with $O(d(s,t)/\omega + \ln n)$ edges while involving $O(d(s,t)/\omega + \omega \ln n)$ nodes in the computation.*

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This bound is nearly optimal, since $\Omega(d(s,t)/\omega)$ is a simple lower bound on the number of edges and the number of nodes involved in any s-t path.

Relay Placement: An Approximation Algorithm

Relay Placement Problem: Given a set of nodes on a grid, we would like to add a small number of additional nodes (Steiner Set) so that the full set becomes connected.

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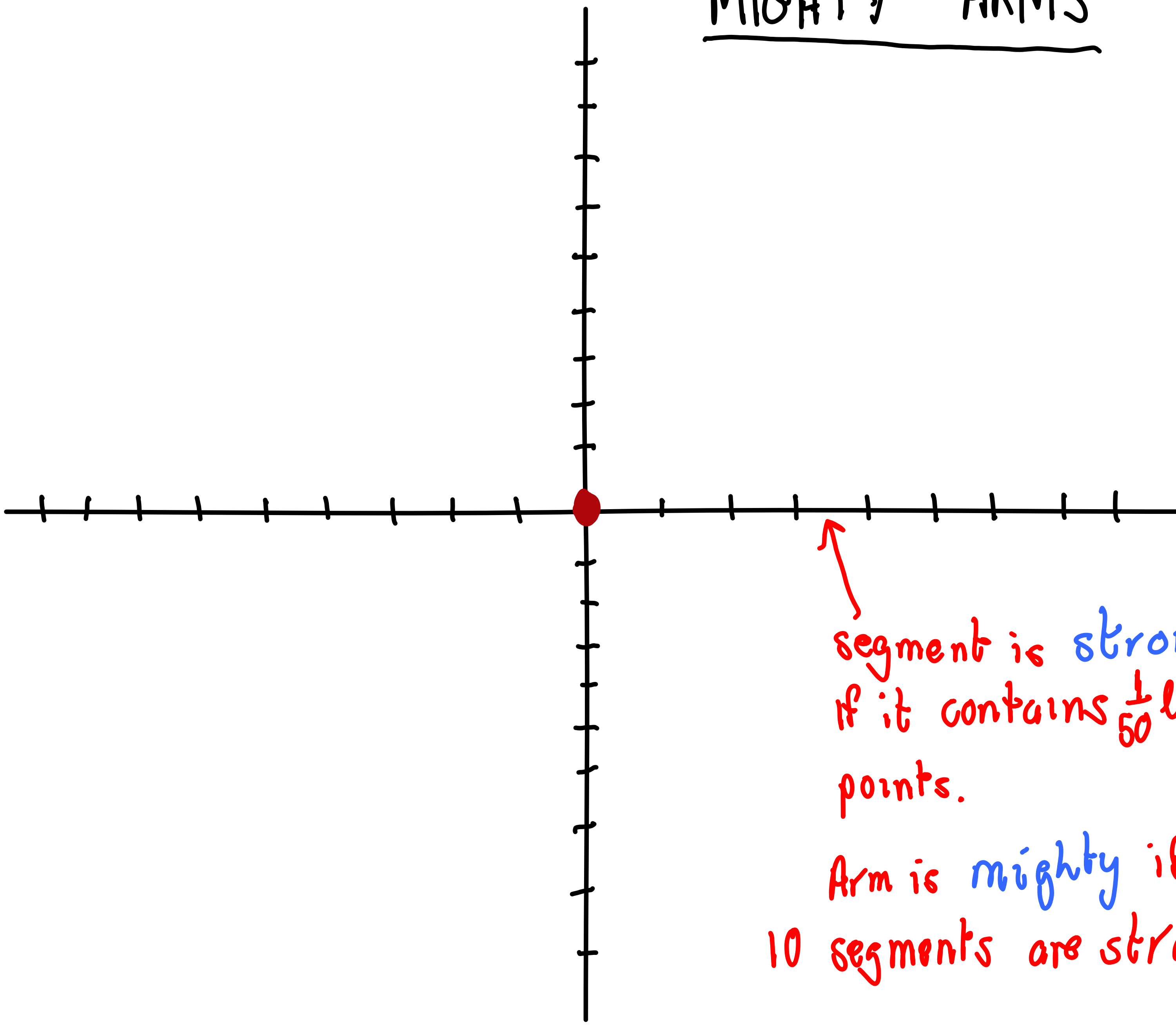
In a general graph, there is an $\Omega(\log n)$ hardness of approximation result for this problem and this is matched by a corresponding upper bound, Klein and Ravi.

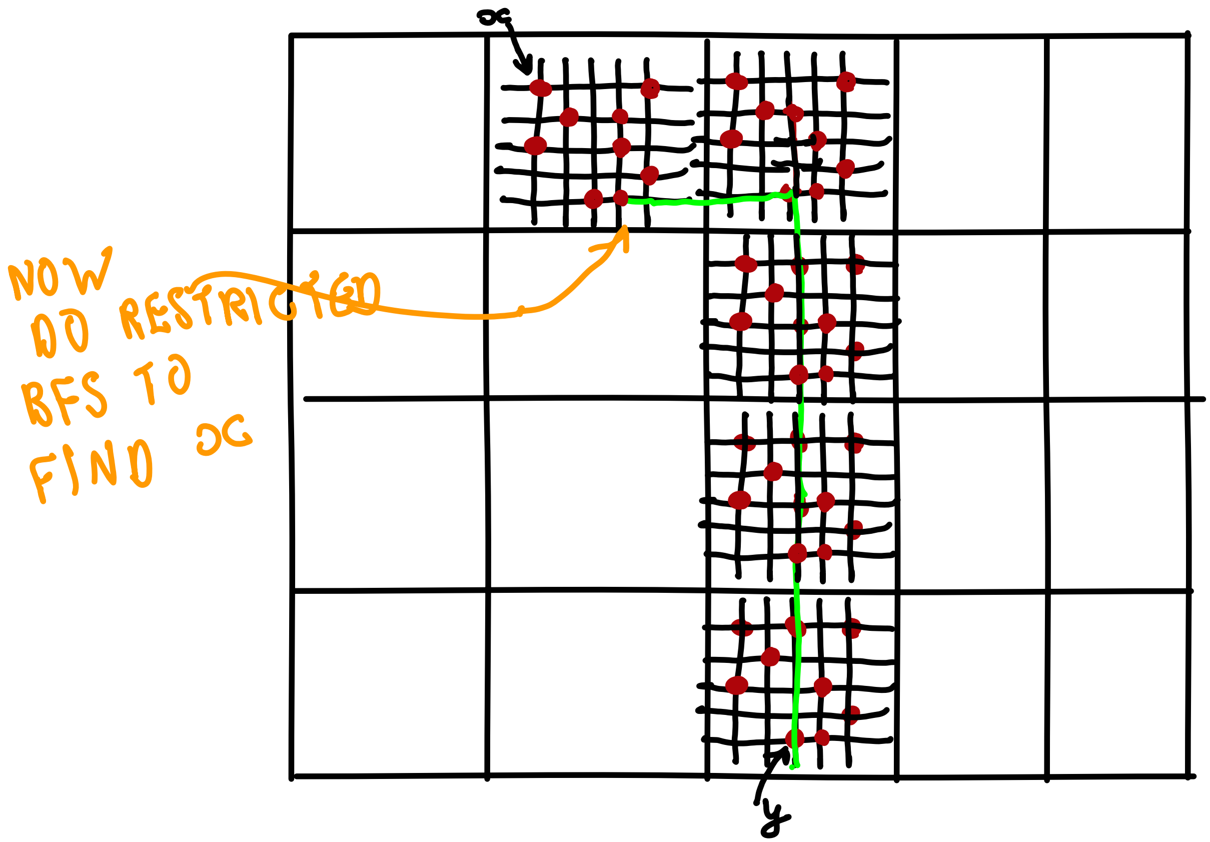
MIGHTY ARMS



segment is strong if it contains $\frac{1}{50}\ln n$ points.

Arm is mighty if all 10 segments are strong.

FINDING PATHS BETWEEN VERTICES - x & y.



Since now $p = K\log n / w$

(i) Each H_i is connected; (ii) Each H_i has diameter $O(\log n)$.

(iii) Each vertex v of G has 4 mighty arms.

No Giant Component

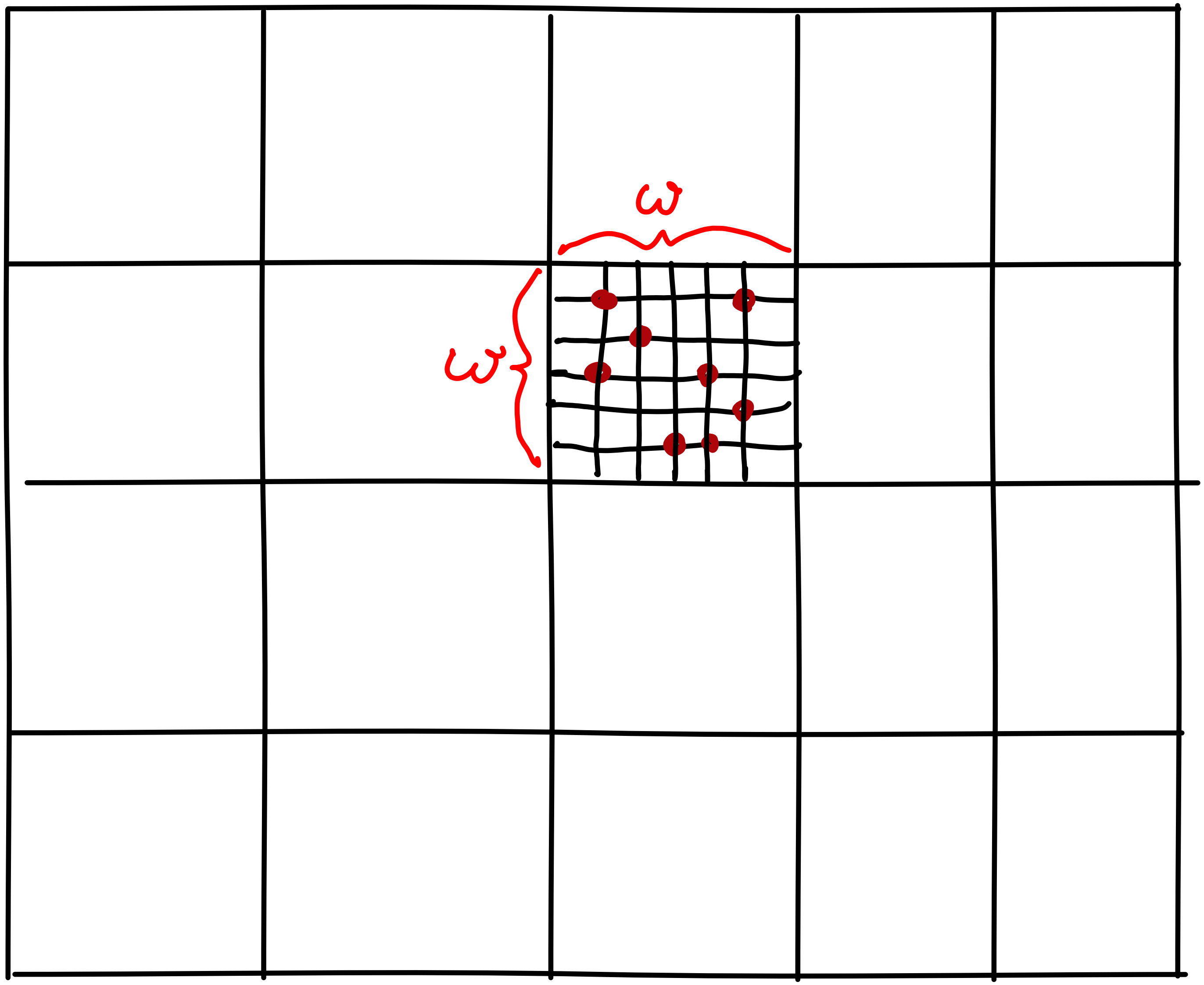
We note that an r-regular, N-vertex graph contains at most $N(er)^{k-1}$ trees with k vertices.

Thus the expected number of k-vertex trees in G is bounded by

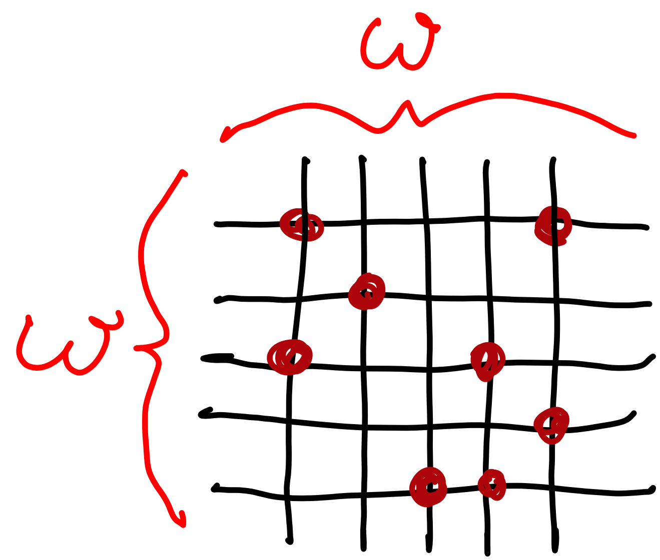
$$n^2(4e\omega p)^{k-1} = n^2(4ec)^{k-1} = o(1)$$

if $c < 1/(4e)$ and $k \geq A \ln n$ and A is sufficiently large.

Partition torus into n^2/w^2 $w \times w$ subsquares.



Subsquare S_i induces a subgraph H_i of G_v



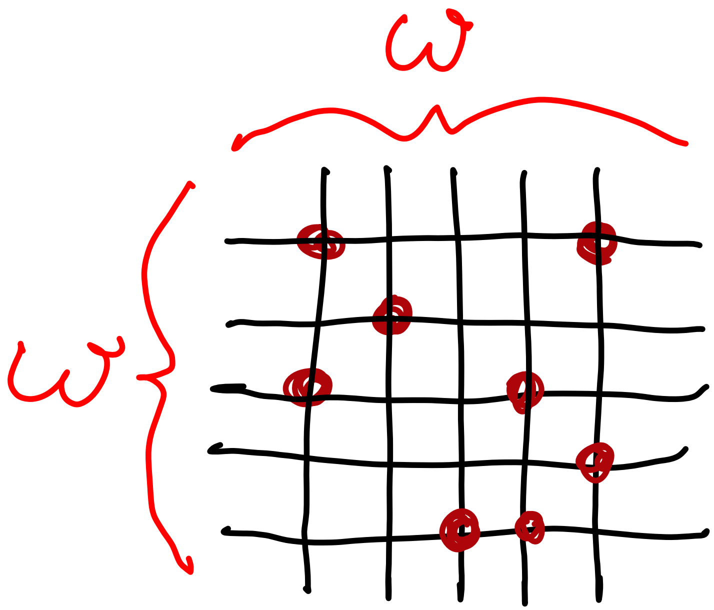
Γ_i: one edge for each point $\bullet$.

Defines a random bipartite graph Γ_i with $w+w$ vertices and edge density c/w, $c>1$.

Fact 1: whp Γ contains a giant component K with $\geq (1-o(1))(1-x_c^2)\,w^2$ edges.

$x_c e^{-x_c} = c e^{-c}$.

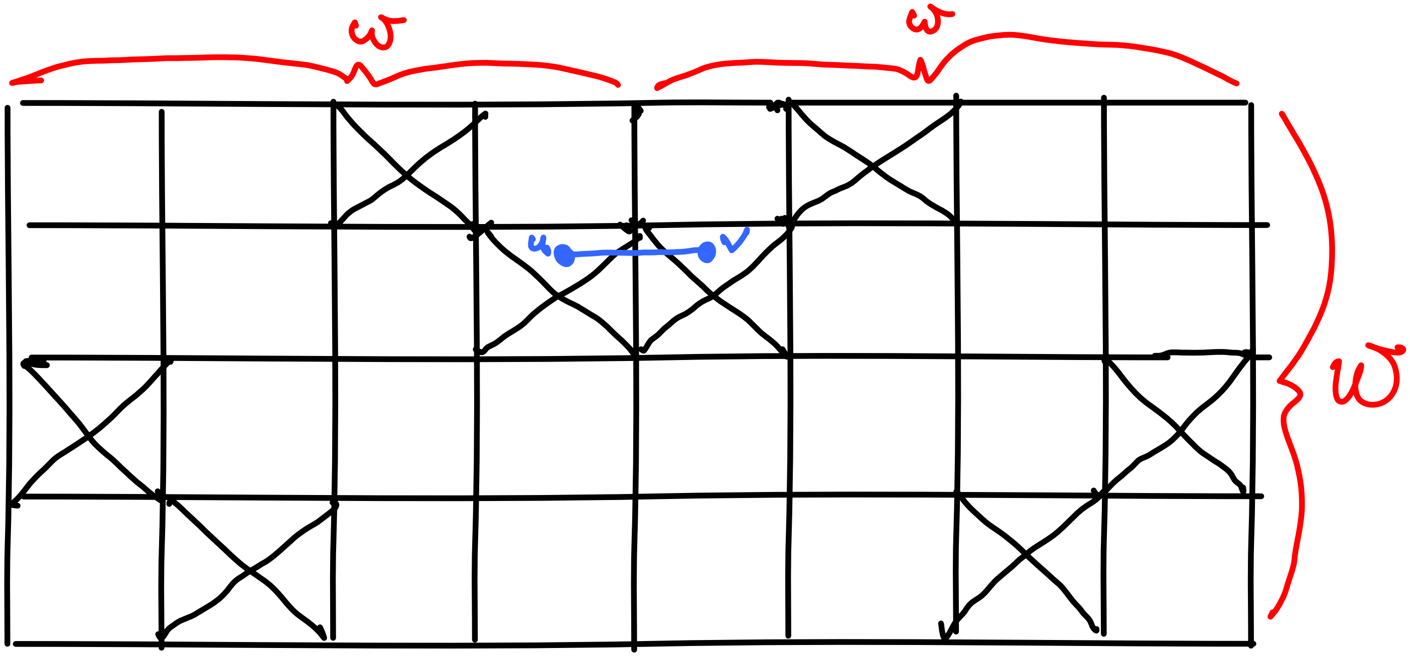
Fact 2: whp K contains $\geq (1-o(1))\, x_c w$ vertices on each side of the partition.



Facts imply whp

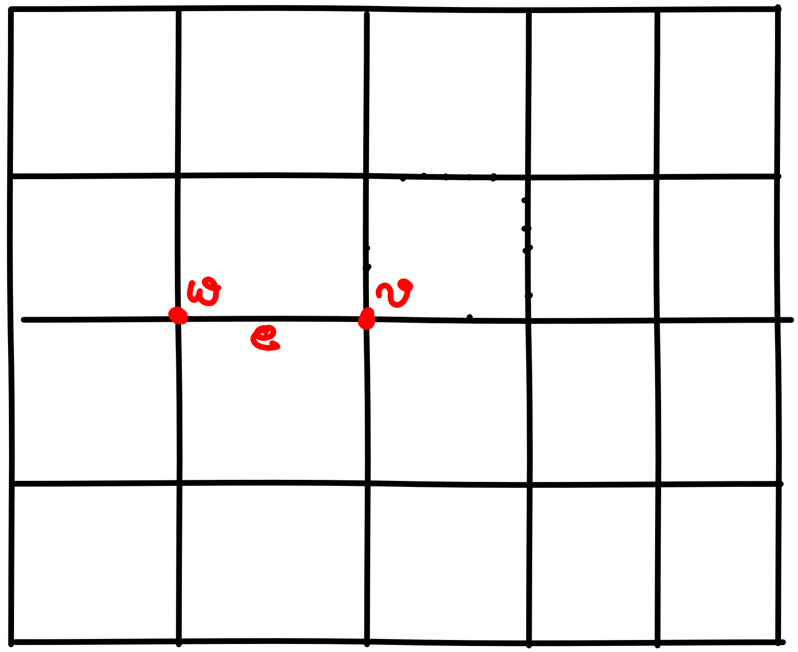
(i) H_i contains a connected component K_i with $\geq (1-o(1))(1-x_c^2)w^2$ vertices.

(ii) K_i contains a vertex in $\geq (1-o(1))\, x_0 w$ rows and columns.



whp $\exists$ u,v connecting the two giants together.

Can argue that these connections between w×w squares are independent.



Consider mixed percolation on $n/w \times n/w$ lattice where

$p_v = \Pr(\text{site open}) = \Pr(H_v \text{ has giant}) = 1 - o(1)$.

$p_e = \Pr(\text{edge } e \text{ open}) = \Pr(K_v, K_w \text{ connected by edge}) = 1 - o(1)$.

Whp there is a cluster of size $(1-o(1))\, n^2/w^2$

$\Rightarrow$ whp G contains component of size $\geq (1-o(1))\, n^2/w^2 \times (1-o(1))\, x_c n$.

$\square$

Connectivity

Assume that

$$p = \frac{(1 - \frac{1}{2}\delta)\ln n + \frac{k}{2}\ln\ln n + c}{2\omega}.$$

Let X_l denote the number of vertices of degree $0 \leq l < k$.

$$\mathbf{E}(X_l) \sim \begin{cases} 0 & l \leq k-2 \\ \lambda_k & l = k-1 \end{cases}.$$

For $t = O(1)$.

$$\mathbf{E}((X_{k-1})_t) \sim \lambda_k^t$$

($(a)_t = a(a-1)\cdots(a-t+1)$).

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$((a)_t = a(a-1)\cdots(a-t+1))$.
So **whp** there are no vertices of degree $\le k-2$ and

$$\mathbf{Pr}(\delta(G) = k-1) \sim 1 - e^{-\lambda_k}.$$

We condition on $\delta(G) \geq k$.

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We write $G = G_1 \cup G_2$ where G_i is defined using p_i where $p_1 = p - \frac{1}{2\omega \ln n} = (1 - o(1))p$ and $1 - p = (1 - p_1)(1 - p_2)$.

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G_1 defines the red nodes and G_2 defines the blue nodes.

The following hold **whp**:

- No red node has an arm α on which we can find 1000 red vertices each having an arm orthogonal to α which is not mighty.

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- There is no red vertex with at most $k-1$ red neighbours and at least one blue neighbour.
- There is no blue node with fewer than k red neighbours.

Assume that the previous properties hold.

Let L be the set of points in T with coordinates (i, j), where each of i and j is a multiple of ω.

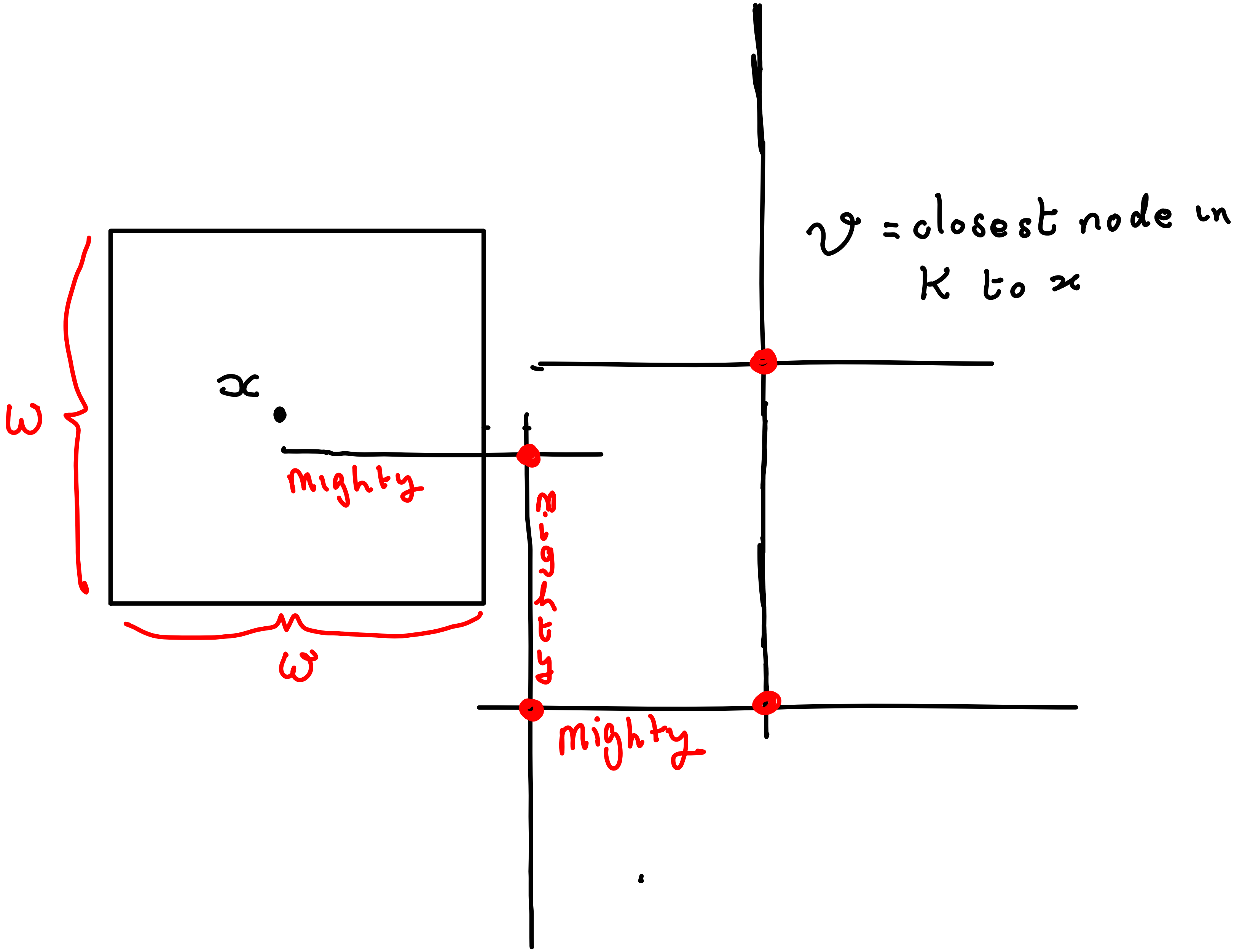
Suppose S is a set of $k-1$ red nodes and let $G_S = G_1 - S$.

For each connected component K of H_S, and for each point $x \in L$, let v_{Kx} denote the node in K that is closest to x in L_1 distance. We claim

Lemma

v_{Kx} lies within the $\omega \times \omega$ box B_x centered at x.

Simplest Case



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For each component J, K and $\omega \times \omega$ box with centre x there is a point $z(J, K, x)$ which is a neighbour of a point in J and K.

The probability that there is no blue node at $z(J, K, x)$ is $(1 - p_2)^{n^2/\omega^2}$ and so the probability that J, K do not get merged into one component is at most $n^2 e^{-n^2 p_2/\omega^2} \leq n^2 e^{-\Omega(n^2/(\omega^3 \ln n))}$ which is small enough to handle all the $\leq n^k$ choices for S.

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So, if we remove any set of $k - 1$ vertices S then there is a component of $G - S$ containing all of the red vertices.

Each blue node has at least k red neighbours and so if we remove any set S of $k - 1$ vertices the remaining graph $G - S$ is connected.

Relay Placement

Problem: Given $c_v \geq 0$ for $v \in T$ and a set $X \subseteq T$ find Y such that $X \cup Y$ is connected and $c(Y)$ is small.

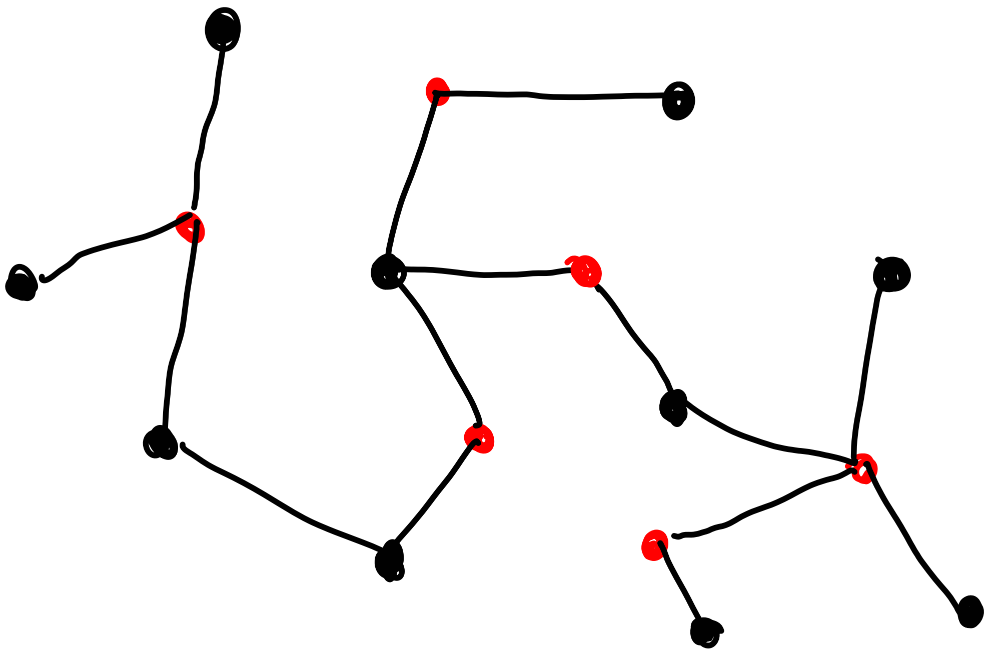
Define $c_v^X = \begin{cases} 0 & v \in X \\ c_v & v \notin X \end{cases}$ and for an edge $e = \{v, w\}$ let its weight be $w(e) = \max\{c_v^X, c_w^X\}$.

Let Y^* be a Steiner set for X of minimum cost, and let Λ^* be a Steiner tree for X of minimum total edge weight.

A Steiner tree Λ' whose edge weight is within a constant factor $\gamma \leq 1.55$ of optimal can be computed in polynomial time – Robins and Zelikovsky.

RELAY REPLACEMENT

$X = \{ \bullet \}$ $\qquad$ $Y = \{ \bullet \}$



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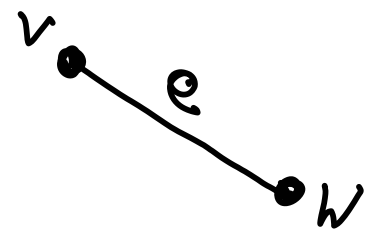

$w(e) = \max\{c_v, c_w\}$

(i) $Y^* =$ min. cost Y.

(ii) $\Lambda^* =$ min. weight Steiner tree.

(iii) $\Lambda' = 1.55$ approx. to Λ^*

(iv) $Y' = \{$Steiner nodes of $\Lambda'\}$

(a) Λ^* has max. degree 4.

$\Downarrow$

(b) $w(\Lambda^*) \leq 4c(Y^*)$

(c) $c(Y') \leq w(\Lambda') \leq 1.55\, w(\Lambda^*) \leq 6.2\, c(Y^*)$.

Let Y' be the Steiner nodes of Λ'.

$$c(Y') \leq w(\Lambda') \leq \gamma w(\Lambda^*) \leq 4\gamma c(Y^*).$$

Open Questions

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- Study problems associated with the points of G moving (randomly).

THANK YOU