Anti-Ramsey Properties of Random Graphs

Alan Frieze

Carnegie Mellon University

General Topic

The edges of graph G are coloured, under some suitable restrictions. The aim is to study the following question.

Does G contain a Rainbow copy of graph H.

A rainbow copy of H is one in which every edge has a distinct colour.

This is Anti-Ramsey in some sense.

Erdős, Simonovits, Sós (1973).

They introduced the following problem: Given a graph H let $f(n, H)$ be the maximum number of colours that you can use on the edges of K_n without creating a rainbow copy of H.

Theorem

Let $d + 1 = \min\{\chi(H - e) : \ e \in E(H)\}$.

$$f(n, H) \sim \binom{n}{2}\left(1 - \frac{1}{d}\right)$$

Lower Bound.

Suppose $d+1=\chi(H_1)$ and

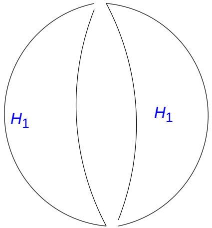
$$m_0 = ext(n, H_1) \sim \binom{n}{2}\left(1-\frac{1}{d}\right)$$

be the maximum number of edges in an H_1-free subgraph of K_n.

Use m_0 edges of a distinct colour to create a copy of an extremal graph for H_1 and then fill in the rest of K_n with a single colour.

Upper Bound

Take 2 copies of H_1 and let $e = (x_1, y_1)$ in one copy and let $e = (x_2, y_2)$ in the other copy. Form G by identifying x_1 with x_2 and y_1 with y_2.



$\chi(G) = \chi(H_1)$ and so $m_1 = ext(n, G) \sim \binom{n}{2}\left(1 - \frac{1}{d}\right)$ as well.

Note that $f(n, H) \leq ext(n, G)$.

b-bounded colourings.

An edge colouring is b-bounded if no colour is used more than b times.

Define

$$AR(G,H,b) = \begin{cases} 1 & \text{Every } b\text{-bounded colouring of } G \text{ contains a rainbow copy of } H \\ 0 & \text{Otherwise} \end{cases}$$

This function has been studied by a number of authors:

$G = K_n, H = K_m$:

Let $\alpha(n, b) = \min\{m : \ AR(K_m, K_n, b) = 1\}$.

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Given an b-bounded colouring of K_{b+2} that does not have a rainbow triangle. Let C be the largest (in number of vertices), connected subgraph spanned by edges of the same colour. C has at most $b+1$ vertices. Thus there exists $v \notin C$.

The edges from v to C must all have the same colour, contradicting the definition of C.

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Simple proof of upper bound: Let $m = 10bn^2$ and let $C_1, C_2, \ldots, C_M$ be the colour classes of an edge colouring of K_m where $|C_i| = b_i \leq b$ for $i = 1, 2, \ldots, M$.

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Let $p = 2n/m$ and choose a random subset of S by putting each vertex of K_m into S with probability p. Let $\mathcal{A}_i$ be the event that S contains two edges of colour i.

$$\begin{aligned}
\textbf{Pr}\left(\bigcap_{i=1}^{M}\overline{\mathcal{A}_i}\right) &\geq \prod_{i=1}^{M}(1-b_i^2p^3/2) \\
&\geq \exp\left\{-\sum_{i=1}^{M}(b_i^2p^3/2+b_i^4p^6)\right\} \\
&= (1-o(1))\exp\left\{-\sum_{i=1}^{M}b_i^2p^3/2\right\} \\
&\geq (1-o(1))e^{-2bn^3/m}.
\end{aligned}$$

Here we have used $\sum_{i=1}^{M} b_i^2 \leq b\sum_{i=1}^{M} b_i \leq m^2b/2$.

So,

$$\textbf{Pr}\left(\bigcap_{i=1}^{M}\overline{\mathcal{A}_i} \wedge |S| \geq n\right) \geq (1-o(1))e^{-2bn^3/m} - e^{-n/4} > 0.$$

Complexity Issues Fenner, Frieze (1984)

Given an edge colouring it is generally NP-hard to determine the existence of a rainbow copy of anything.

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NP-hard to determine whether or not there is a rainbow rooted arborescence in an edge coloured digraph – bad news for the Greedoid Intersection Problem.

Hamilton Cycles

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Complete Graph: Albert, Frieze, Reed (1995) (Correction by Rue)

Every $n/64$-bounded edge colouring of K_n contains a rainbow Hamilton cycle.

Proof: Choose a random Hamilton cycle and apply the (lop-sided local lemma).

Theorem

Cooper, Frieze *(1995)*

If $m = n(\log n + (2k-1)\log\log n + c_n)/2$ *and* $\lambda = e^{-c}$*, then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \in \mathcal{AR}_k) = \begin{cases} 0 & c_n \to -\infty \\ \sum_{i=0}^{k-1} \frac{e^{-\lambda}\lambda^i}{i!} & c_n \to c \\ 1 & c_n \to \infty \end{cases} \quad (1)$$

$$= \lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \in \mathcal{B}_k),$$

$\mathcal{AR}_k = \{G :$ any k-bounded colouring of G contains a rainbow Hamilton cycle

$\mathcal{B}_k = \{G : G$ has at most $k-1$ vertices of degree less than $2k\}$.

Proof: Throw away edges where a colour is used more than once and show that the remaining graph is Hamiltonian.

Random Graphs: Bohman,Frieze,Pikurhko,Smyth (2006)

We try to estimate

$$\lim_{n \to \infty} \mathbf{Pr}(AR(G_{n,p}, H, b) = 1)$$

for various b, H.

Simplest non-trivial case

Theorem

Let $p = \frac{c_n}{n^{2/3}}$. *Then*

$$\begin{aligned}\lim_{n\to\infty} \mathbf{Pr}(AR(G_{n,p}, K_3, 2)) &= \begin{cases} 0 & c_n \to 0 \\ 1 - e^{-c^6/24} & c_n \to c \\ 1 & c_n \to \infty \end{cases} \\ &= \lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \text{ contains no } K_4).\end{aligned}$$

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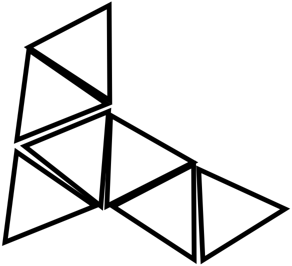
We argue that except for a very few cycles, which can easily be handled, Γ_H is a forest.



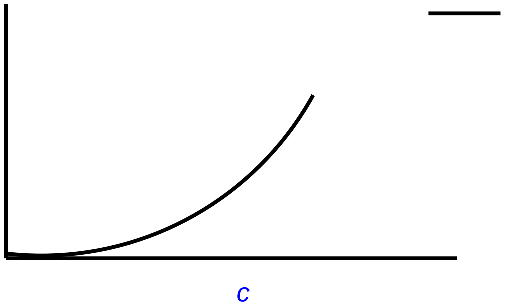
The next simplest example is

Theorem

Let $p = \frac{c}{n^{1/2}}$. *Then,*

$$\lim_{n \to \infty} \mathbf{Pr}(AR(G_{n,p}, K_3, 2)) = \begin{cases} 1 - e^{-c^{10}/120} & c < 1/\sqrt{2} \\ 1 & c > \sqrt{2} \end{cases}$$

$\mathbf{Pr}(AR(G_{n,p}, K_3, 3))$



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Suppose that $c > \sqrt{2}$. **Whp** $G_{n,p}$ has $(1 + o(1))cn^{3/2}/2$ edges, $(1 + o(1))c^3 n^{3/2}/6$ triangles and $o(n^{3/2})$ copies of K_4.

Suppose that we have a 3-bounded colouring and A_i is the set of colours that are used i times and $a_i = |A_i|$ for $i = 1, 2, 3$. Thus,

$$a_1 + 2a_2 + 3a_3 = (1 + o(1))cn^{3/2}/2.$$

Suppose that there are no rainbow triangles. Then each triangle T contains a pair of edges of the same colour $c(T)$.

For colour x let $t(x)$ be the number of triangles T such that $c(T) = x$.

So $t(x) = 0$ for $x \in A_1$, $t(x) \leq 1$ for $x \in A_2$ and $t(x) \leq 2$ for $x \in A_3$, unless x is used to colour an edge of a copy of K_4.

These latter colourings are relatively rare and so we have

$$a_2 + 2a_3 \geq (1 + o(1))c^3 n^{3/2}/6.$$

and since

$$a_1 + 2a_2 + 3a_3 = (1 + o(1))cn^{3/2}/2$$

we have

$$\frac{c^3}{4} \leq \frac{c}{2} \text{ or } c \leq \sqrt{2}.$$

Now lets consider general H.

We let

$$m_H = \frac{e_H - 1}{v_H - 2}$$

and

$$m_H^* = \max_{\substack{H' \subseteq H \\ v_{H'} \geq 3}} m_{H'}.$$

Theorem

Suppose that H is connected and not a tree and that b is sufficiently large. Then there exist $c_1 = c_1(b, H)$ and $c_2 = c_2(b, H)$ such that if $p = cn^{-1/m_H^*}$ then

$$\lim_{n \to \infty} \mathbf{Pr}(AR(G_{n,p}, H, b)) = \begin{cases} 0 & c \leq c_1 \\ 1 & c \geq c_2 \end{cases}.$$

Assuming that $m_H = m_H^*$, when $p = cn^{-1/m_H^*}$ the expected number of copies of H sitting on a fixed edge of $G_{n,p}$ is $O(c^{e_H - 1})$.

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Small c

Thinking in terms of branching processes and the size of the components of Γ_H, if c is small then these components will be small (polylog(n)).

It will be possible to order the vertices of a component $v_1, v_2, \ldots$ so that each v_i has at most C_H neighbours in $v_1, v_2, \ldots, v_{i-1}$.

So if $b \geq C_H$ then we can avoid rainbow copies of H.

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In a b-bounded colouring, the number of isolated copies of H that are not rainbow is at most

$$|E(G_{n,p})|b \leq 2bcn^{2-1/m_H} \ll X_H.$$

Trees

Whp $G_{n,p}$, $p \gg n^{-k/(k-1)}$ contains a copy of every tree with k vertices or less.

Threshold question reduces to evaluating, for a fixed tree T and integer b, the value of

$$s(b,T) = \min\{k : \exists \text{ tree } T_1 \text{ with } k \text{ vertices such that } AR(T_1, T, b) = 1\}.$$

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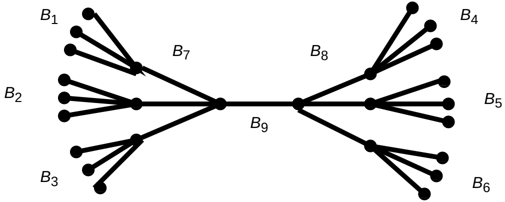
For example if $T = P_l$ a path of length l then

$$s(b, P_l) = \begin{cases} 1 + (b+1)\sum_{i=0}^{k-1} b^i & l = 2k \\ 2 + 2\sum_{i=1}^{k} b^i & l = 2k+1 \end{cases}.$$

$s(3, P_5) = 26$



Break edges into 9 bundles, 8 of size 3 and one of size 1.

Hall's Theorem shows that for any 3-bounded colouring, there is a set of distinct (colour) representatives for the bundles.

Using this one gets a rainbow P_5.