Random regular graphs

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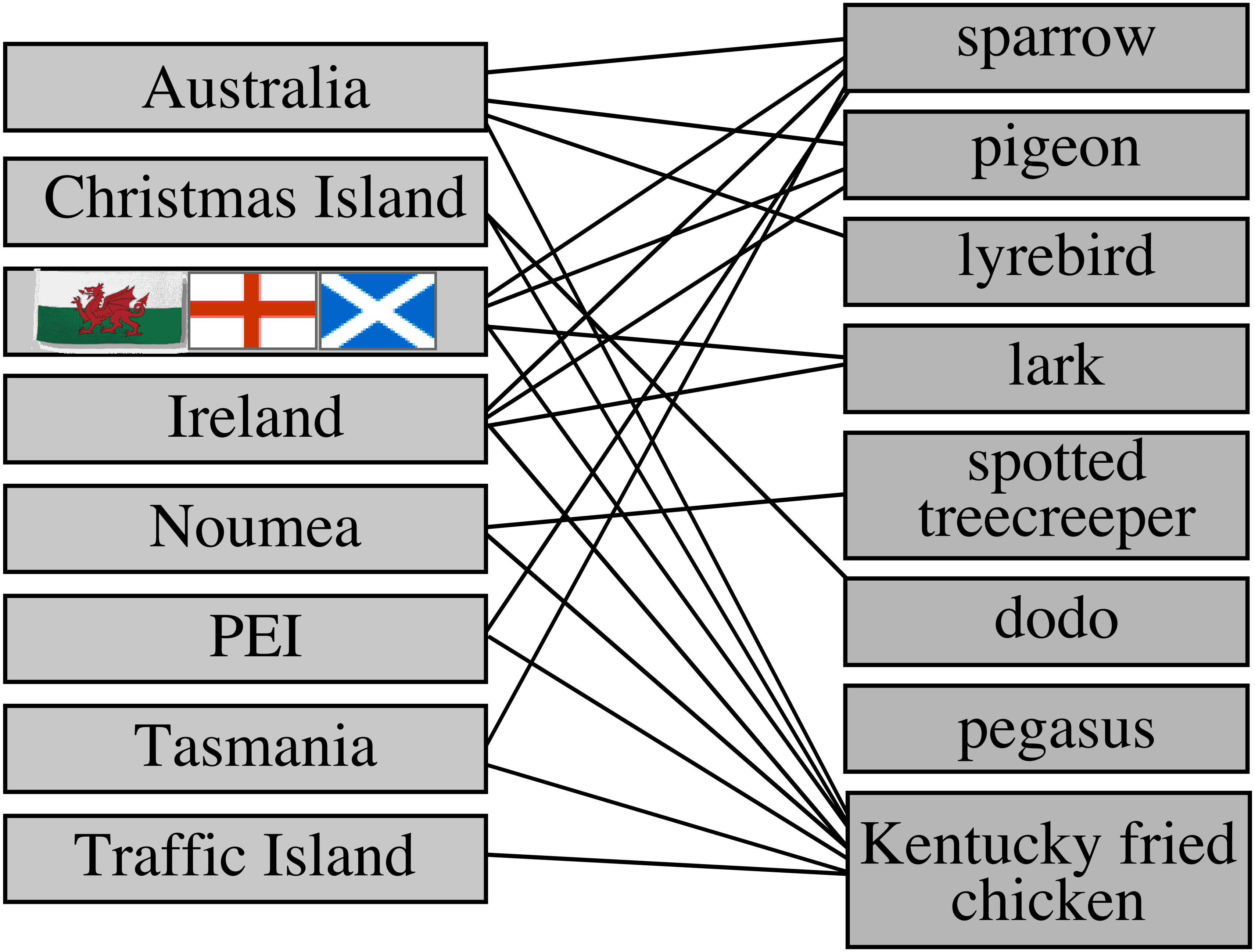
Random regular graphs

- What are they (models)
- Typical and powerful results
- Handles for analysis
- One or two general techniques
- Some open problems

Regular graphs

A vertex has degree d if it is incident with d edges.

A d-regular graph has all vertices of degree d.


Australia
Christmas Island
Ireland
Noumea
PEI
Tasmania
Traffic Island
sparrow
pigeon
lyrebird
lark
spotted treecreeper
dodo
pegasus
Kentucky fried chicken

Random regular graphs
What are they?

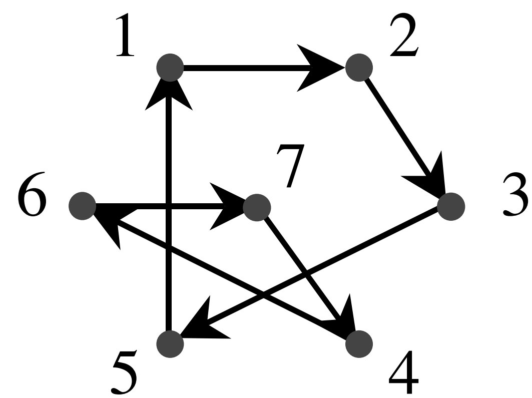
Faultily faultless, icily regular,
splendidly null, dead perfection;
no more.

- Lord Alfred Tennyson

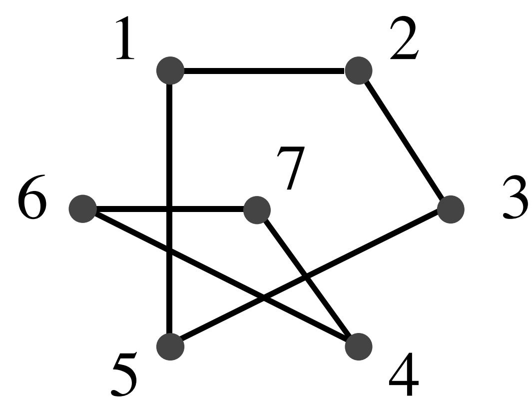
Random permutation

Choose a permutation of $\{1, \ldots, n\}$ uniformly at random.

e.g. (1 2 3 5) (4 6 7)



Then forget orientations



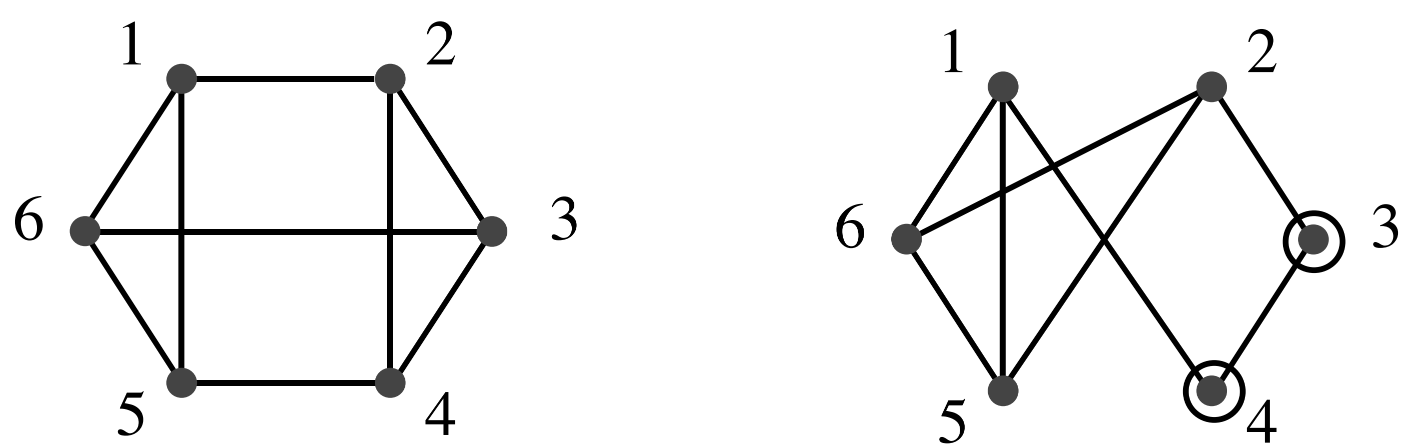
Uniform model

$\mathcal{G}_{n,d}$: Probability space, elements are the d-regular graphs on n vertices.
Each has the same probability: $\dfrac{1}{|\mathcal{G}_{n,d}|}$

But $|\mathcal{G}_{n,d}|$ is not known exactly. Hard to analyse.

Algorithmic models

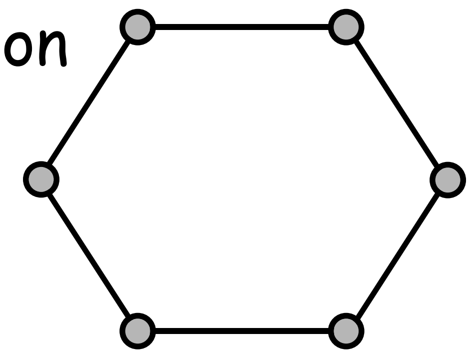
e.g. Degree-restricted process:
add edges to random places, keeping all vertex degrees at most d.



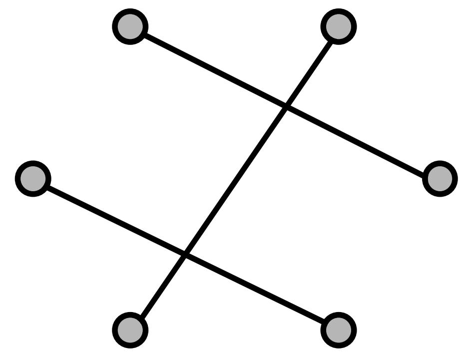
Superposition models

Random Hamilton cycle

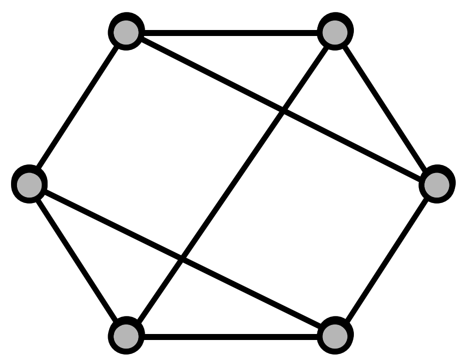
$\in \mathcal{H}_6$



$\in \mathcal{G}_{6,1}$



Take one member from each of two models, and superimpose.



$\in \mathcal{H}_6 \oplus \mathcal{G}_{6,1}$

Throw it away if a multiple edge is created.

a.a.s.

A property Q holds asymptotically almost surely (a.a.s.) in a random graph model if

$$\mathrm{P}(G \text{ has } Q) \to 1 \text{ as } n \to \infty$$

Questions - uniform model

Do graphs in $\mathcal{G}_{n,d}$ a.a.s. satisfy the following?

- ♠ connected, and moreover d-connected
- ♠ contain a perfect matching (for n even)
- ♠ hamiltonian (have cycle through all vertices)
- ♠ trivial automorphism group

... and how are the following distributed?

- ♠ subgraph counts
- ♠ chromatic number
- ♠ eigenvalues
- ♠ independent & dominating set sizes

Some answers for $\mathcal{G}_{n,d}$

Many things are known. Some examples:

For $3 \leq d \leq n-4$: a.a.s. d-connected and hamiltonian (Bollobas, Wormald, Frieze, Robinson, Cooper, Reed, Krivelevich, Sudakov, Vu),
trivial automorphism group (B, McKay, W, K, S, V),

For fixed d: distribution of eigenvalues (McKay),
second eigenvalue a.a.s. $< 2\sqrt{d-1} + \epsilon$ (Friedman)
Chromatic number bounds known (Achlioptas & Moore, Shi & Wormald)

Contiguity

Two sequences of models $\mathcal{G}_n$ and $\mathcal{F}_n$ are contiguous if for any sequence of events A_n

$$\boxed{\begin{array}{c} A_n \text{ is a.a.s. true in } \mathcal{G}_n \\ \text{if and only if} \\ A_n \text{ is a.a.s. true in } \mathcal{F}_n \end{array}}$$

Notation: $\mathcal{G}_n \approx \mathcal{F}_n$

Contiguity of Superposition models

Thm 1 (~Robinson & W; Janson) For $d \geq 3$

$$\mathcal{G}_{n,d-1} \oplus \mathcal{G}_{n,1} \approx \mathcal{G}_{n,d} \quad (n \text{ even}).$$

Thm 2 (Janson; Molloy, Reed, Robinson & Wormald)

$$\mathcal{G}_{n,1} \oplus \mathcal{G}_{n,1} \oplus \mathcal{G}_{n,1} \approx \mathcal{G}_{n,3} \ (n \text{ even}).$$

i.e. $\quad 3\,\mathcal{G}_{n,1} \approx \mathcal{G}_{n,3}$

Thm 3 (Robalewska) For $d \geq 3$.

$$\mathcal{G}_{n,d-2} \oplus \mathcal{G}_{n,2} \approx \mathcal{G}_{n,d}$$

Arithmetic of contiguity

Example with n even:

$$\mathcal{G}_{n,9} \approx \mathcal{G}_{n,7} \oplus \mathcal{G}_{n,2} \qquad \text{(Thm 3)}$$

$$\approx \mathcal{G}_{n,5} \oplus 2\mathcal{G}_{n,2} \qquad \text{(Thm 3)}$$

$$\approx \mathcal{G}_{n,4} \oplus \mathcal{G}_{n,1} \oplus 2\mathcal{G}_{n,2} \qquad \text{(Thm 1)}$$

$$\approx \mathcal{G}_{n,3} \oplus 2\mathcal{G}_{n,1} \oplus 2\mathcal{G}_{n,2} \qquad \text{(Thm 1)}$$

$$\approx 3\mathcal{G}_{n,1} \oplus 2\mathcal{G}_{n,1} \oplus 2\mathcal{G}_{n,2} \qquad \text{(Thm 2)}$$

$$= 5\mathcal{G}_{n,1} \oplus 2\mathcal{G}_{n,2}$$

1 + 1 is not 2

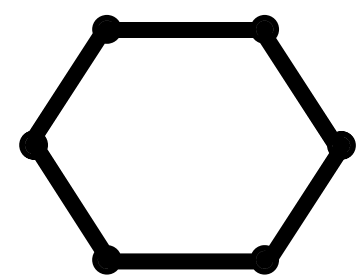
In general all such equations with $\mathcal{G}_{n,d}$ and respecting degree sums are true (n even):

$$\mathcal{G}_{n,d_1} \oplus \mathcal{G}_{n,d_2} \oplus \cdots \oplus \mathcal{G}_{n,d_k} \approx \mathcal{G}_{n,d}$$

provided $d_1 + d_2 + \cdots + d_k = d \geq 3$.

There is one failure:

$$\mathcal{G}_{n,1} \oplus \mathcal{G}_{n,1} \not\approx \mathcal{G}_{n,2}$$



Contiguity with $\mathcal{H}_n$

$\mathcal{H}_n$ = random Hamilton cycle on n vertices

Thm 4 (Frieze, Jerrum, Molloy, Robinson & Wormald)

$$\mathcal{G}_{n,d-2} \oplus \mathcal{H}_n \approx \mathcal{G}_{n,d} \qquad \text{(fixed } d \geq 3\text{).}$$

Thm 5 (Kim & Wormald)

$$\mathcal{H}_n \oplus \mathcal{H}_n \approx \mathcal{G}_{n,4}$$

Thus all equations involving various $\mathcal{G}_{n,d}$ and $\mathcal{H}_n$ respecting degree sums are true, provided the total degree is at least 3.

Contiguity with $\mathcal{F}_n$

$\mathcal{F}_n$= graph formed from uniformly chosen random permutation of n vertices (with no loops or multiple edges).

Thm 6 (Greenhill, Janson, Kim & Wormald)

$\mathcal{F}_n \oplus \mathcal{F}_n \approx \mathcal{G}_{n,4}$, $\quad \mathcal{G}_{n,d-2} \oplus \mathcal{F}_n \approx \mathcal{G}_{n,d}$ $(d \geq 3)$.

Thus equations involving various $\mathcal{G}_{n,d}$, $\mathcal{F}_n$ and $\mathcal{H}_n$ respecting degree sums are true, provided the total degree is at least 3.

Corollaries

$$\mathcal{F}_n \oplus \mathcal{F}_n \approx \mathcal{G}_{n,4} \approx \mathcal{G}_{n,d-2} \oplus \mathcal{H}_n$$

implies that $\mathcal{F}_n \oplus \mathcal{F}_n$ is a.a.s. hamiltonian. (Also proved algorithmically by Frieze.)

$$\mathcal{G}_{n,d-2} \oplus \mathcal{H}_n \approx \mathcal{G}_{n,d}$$

implies that $\mathcal{G}_{n,d}$ is a.a.s. hamiltonian.

$$d\,\mathcal{G}_{n,1} \approx \mathcal{G}_{n,d} \quad (n \text{ even})$$

implies that $\mathcal{G}_{n,d}$ is a.a.s. decomposable into d perfect matchings (so is d-edge-colourable).

Other corollaries

Each model is a.a.s. decomposable into $d/2$ edge-disjoint Hamilton cycles (even $d \geq 4$).

Bipartite version of this is also true (Greenhill, Kim & Wormald). So there exist 4-regular bipartite graphs with a hamiltonian decomposition and arbitrarily large girth (=length of shortest cycle).

This gives examples of complexes with incoherent fundamental group (McCammond & Wise).

Other corollaries

Anything a.a.s. true in one of the models is also a.a.s. true in the others.

Handles for analysis

1. Pairing model (esp. for small d)

2. Switchings (esp. for moderate d)

3. Enumeration results (esp. for large d)

For this talk: only 1 in detail.

Pairing model - $\mathcal{P}_{n,d}$

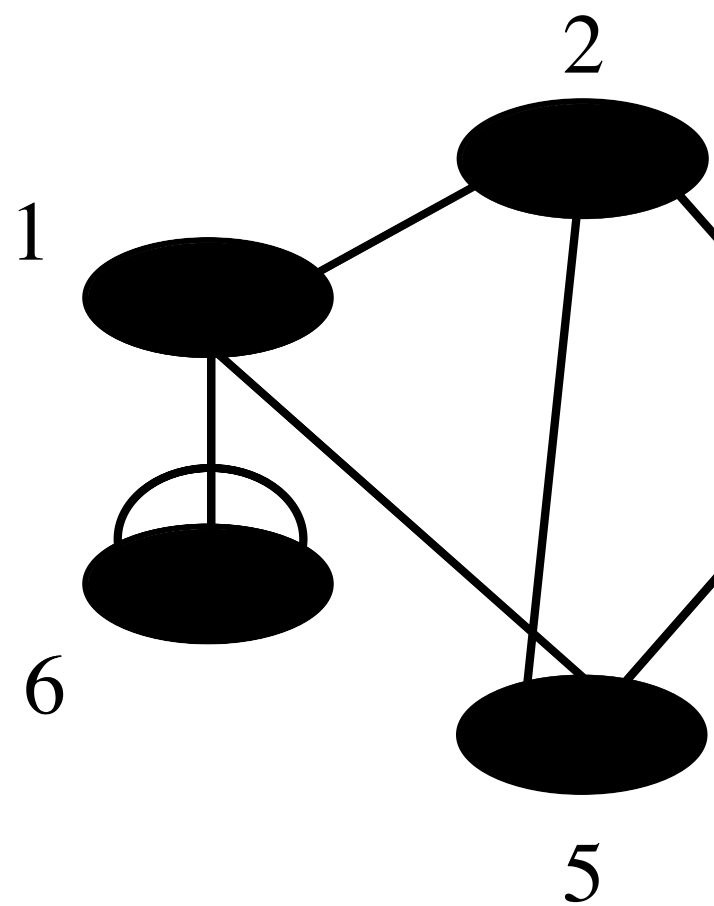
To stand for a vertex, take d points in a “cell” or bucket.

vertex 1 vertex 2 ...

Then take a random pairing (perfect matching) of all the points. The pairs determine the edges of the graph.

Pairing model - $\mathcal{P}_{n,d}$

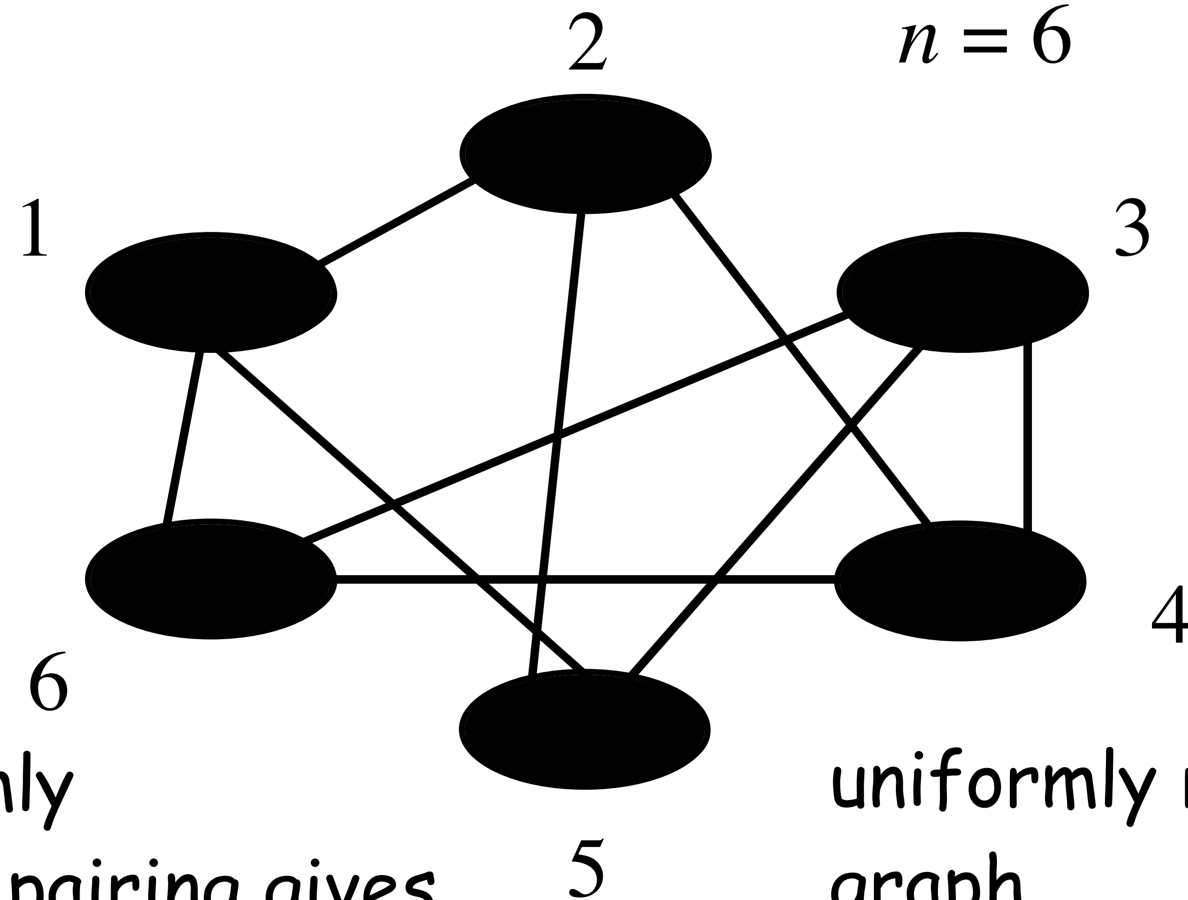
$n = 6 \qquad d = 3$



Reject if loops or multiple edges

Pairing model - $\mathcal{P}_{n,d}$

$n = 6$ $d = 3$



Uniformly random pairing gives uniformly random graph

Analysis of random pairings

For analysis, permit loops and multiple edges.

Example: distribution of number of triangles.

For this we use the method of moments as in Lecture 4. But now pairs are dependent. (c.f. $G(n,p)$, where edges are independent.)

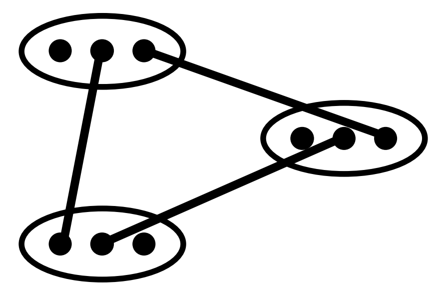
Create an indicator variable I_j for each triple of pairs that induce a triangle in the graph. Call such a triple a triangle of the pairing.

Expected number of triangles

If X_3 is the total number of triangles in the random pairing then since I_j is an indicator

$$\mathbf{E}X_3 = \sum_j \mathbf{E}\, I_j = \sum_j \mathbf{P}(I_j = 1)$$

$$\mathbf{P}(I_j = 1) = M(dn - 6)/M(dn)$$

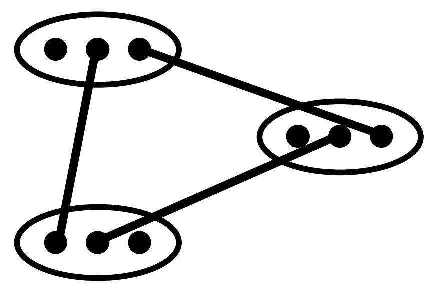


where $M(k)$ is the number of perfect matchings of k points.

triangles (cont.)

We easily get $M(k) = (k-1)(k-3)\cdots 1$ and then $\mathbf{P}(I_j = 1) \sim (dn)^{-3}$. The number of ways to choose a triangle in the pairing is

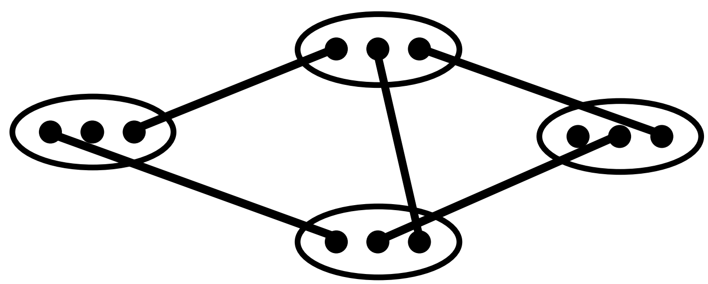
$$(d(d-1))^3 \binom{n}{3}$$



Thus $\mathbf{E}X_3 \sim (d-1)^3/6$.

ASIDE: Easy exercise

Show that if F has more edges than vertices then it a.a.s. does not occur as a subgraph of $\mathcal{G}_{n,d}$.



Distribution of cycle counts

Higher moments easily computed in a similar way. Conclusion:

X_3 has asymptotically Poisson distribution with expectation $\lambda_3 = (d-1)^3/6$.

X_r - cycles of length r - can be done similarly and again the distribution is asymptotically Poisson with expectation $\lambda_r = (d-1)^r/2r$.

Joint distribution

Joint moments also computed in the same fashion. For instance

$$\mathbf{E}(X_1)_i(X_2)_j \sim \lambda_1{}^i \lambda_2{}^j$$

from which we may conclude X_1 and X_2 are asymptotically jointly independent Poisson.

One implication of this is

$$\mathbf{P}(X_1 = X_2 = 0) \sim \exp(-\lambda_1 - \lambda_2)$$
$$= e^{(1-d^2)/4}.$$

Simple graphs

Let $\mathbf{simple}$ denote the event that the random pairing produces no loops or multiple edges. Then we have found

$$\mathbf{P}(\mathbf{simple}) \sim \mathrm{e}^{(1-d^2)/4}.$$

Joint moments of X_1, X_2 and other X_r 's give the asymptotic distribution of X_r, X_s, . . . , in $\mathcal{G}_{n,d}$.

a.a.s. properties of simple graphs

Since $\mathbf{P}(\mathbf{simple})$ is bounded below, if for any event A we show that $\mathbf{P}(A) = 1 - o(1)$ in $\mathcal{P}_{n,d}$ then $\mathbf{P}(A) = 1 - o(1)$ also in $\mathcal{G}_{n,d}$.

This is the basis of attack for many problems.

Example: "1+1" is not "2" because the probability of $\mathcal{G}_{n,2}$ having no odd cycle of of length less than $2g$ is asymptotically

$$\exp(-\lambda_3 - \lambda_5 - \cdots - \lambda_{2g-1}),$$

which tends to 0 as g goes to infinity.

Variance and Hamilton cycles

Now let Y be the total number of Hamilton cycles in (the graph of) the random pairing. Use an indicator variable I_j for each possible set of pairs inducing a Hamilton cycle to find $\mathbf{E}(Y)$ and $var(Y)$. For $d \geq 3$we find in $\mathcal{P}_{n,d}$

$$var(Y)/\mathbf{E}(Y)^2 \sim d/(d-2) - 1,$$

a positive constant. By second moment method this is an upper bound on $\mathbf{P}(Y = 0)$.

Small subgraph conditioning - for proving contiguity

Implicit in work of Robinson & Wormald, distilled by Janson, also Molloy, Robalewska, R&W.

The technique may apply when $var(Y)$ is of the order of $\mathbf{E}(Y)^2$ and the variability signified by the large variance is "induced" by some variables describing local properties.

Often these are $X_1, X_2, \ldots$ (short cycle counts).

The hypotheses

Let Y count decompositions of a graph of a specific type. For example, Hamilton cycle + $(d-2)$-regular graph.

1. $X_1, X_2, \ldots, X_k$ are asymptotically independent Poisson with expectations λ_i.

The hypotheses (PART 2)

2. $$\mathbf{E}\,Y(X_1)_{j_1}(X_2)_{j_2}\cdots(X_k)_{j_k}\to \mathbf{E}Y\prod_{i=1}^{k}(\lambda_i(1+\delta_i))^{j_i}$$

for every finite sequence $j_1, j_2, \ldots, j_k$ of non-negative integers, where all $\delta_i > -1$.

3. $$\mathbf{E}\,Y^2 \sim (\mathbf{E}\,Y)^2 \exp\left(\sum_{i=1}^{\infty}\lambda_i\delta_i{}^2\right)$$

and the sum converges.

Small subgraph conditioning - conclusion

$$\mathcal{R}_n \approx \mathcal{G}_{n,d}$$

where $\mathcal{R}_n$ is the space of random regular graphs each with probability proportional to the number of decompositions of the specified type.

Superposition models relate to decompositions!

Calculations in the Hamilton cycle example verify the conditions for $d \geq 3$, so

$$\mathcal{G}_{n,d-2} \oplus \mathcal{H}_n \approx \mathcal{G}_{n,d}$$

and so on

All the contiguity results stated before are proved by that method.

If we have time, let's look at the differential equation method.

Greedy algorithms

Problem: what is the size of the largest independent set in a random regular graph? Largest dominating set?

Greedy algorithms often achieve good a.a.s. bounds.

How do we analyse them?

Differential equation method

A randomised algorithm is applied to a graph.

When the algorithm is applied to a random regular graph, its steps depend on some variables that a.a.s. follow close to the solutions of some system of differential equations. (Justification by martingale techniques.)

DE method for random pairings

We will need to compute the expected changes in these variables, in each step.

Consider the algorithm applied to a random pairing. In each step of the algorithm, one may generate at random just those pairs involving whichever points are relevant for the next step of the algorithm.

example: max independent set

An independent set is a set of vertices, no two of which are adjacent. $\alpha\,(G)$ denotes the largest size of an independent set in G.

From expectation arguments, $\alpha(G) < \beta\,(d)n$ a.a.s., where e.g. (McKay)
$\beta\,(3) =$ 0.4554,
$\beta\,(4) =$ 0.4163.

Lower bounds come from greedy algorithms.

Greedy alg for max independent set

Simple algorithm: select vertices consecutively at random to build an independent set. Upon selecting a vertex, delete it and its neighbours.

$Y_i(t)$: number of vertices of degree i after t steps of the algorithm. In pairing model, find (asymptotically) expected change in Y_i in one step, as function of Y_j 's.

Writing the expected change as a derivative gives a differential equation:

d.e. for independent set algorithm

$$y_i' = f(y_0, y_1, \dots, y_d)$$

where $y_i(x)$ approximates $Y_i(t)/n$ at time t, $x = t/n$. (Details omitted!)

The d.e. method includes general results for showing that a.a.s. the Y_i stay close to the scaled solutions of the d.e.:

$$Y_i(t) = n y_i(t/n) + o(n) \quad \text{a.a.s.}$$

Conclusion of simple algorithm

Let x_o be the solution of $\sum y_i(x) = 0$. Then a.a.s. the process lasts for $x_o n + o(n)$ steps. Thus $\alpha(G) > x_o n + o(n)$ a.a.s.

We find for $d \geq 3$ that

$$x_o = (1/2)(1 - (d-1)^{-2/(d-2)})$$

d	x_o
3	0.3750
4	0.3333

Degree greedy algorithm

Give priority to vertices with minimum degree in the ever-shrinking graph.

d	x_{o}	the upper bounds again
3	0.4327	0.4554
4	0.3901	0.4163

(Analysis requires extra bells and whistles.)

The case d=3 also obtained by Frieze and Suen (analytically as 6 log(1.5) - 2) analysing the same algorithm another way.

Colouring

Easy exercise that $\chi(\mathcal{G}_{n,3}) = 3$ a.a.s.

(Hints: Brooks Thm, and short odd cycles.)

Greedy algorithm: assign colours randomly to vertices: a.a.s. requires $d+1$ colours.

Better algorithm: higher priority to vertices with more colours already on their neighbours. Achlioptas and Moore showed in this way that $\mathbf{P}(\chi(\mathcal{G}_{n,4}) = 3) > c + o(1)$ for some $c > 0$.

Colouring - even better algorithm

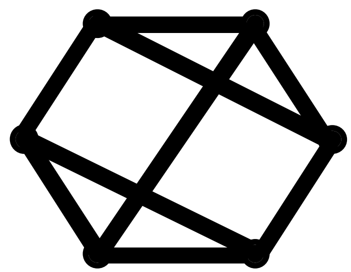
Modified better algorithm: first colour the short odd cycles, then proceed as before. This shows above with $c = 1$ (Shi & Wormald). So $\chi(\mathcal{G}_{n,4}) = 3$ a.a.s.

Similarly, we get $\chi(\mathcal{G}_{n,5}) = 3$ or 4 a.a.s.,

$\chi(\mathcal{G}_{n,6}) = 4$ a.a.s., etc.

Analysis for upper bounds by d.e. method using bells, whistles and flashing lights. Lower bounds proved earlier (Molloy & Reed) using expectation.

Unsolved Problems



Conjecture that a random d-regular graph with an even number of vertices a.a.s. has a perfect 1-factorisation ($d \geq 3$).

Does a random d-regular directed graph a.a.s. have d edge-disjoint Hamilton cycles ($d \geq 3$)?

More unsolved Problems

Is the uniform random d-regular graph contiguous to the algorithmically defined model (add edges at random subject to maximum degree d)?

Is a random 5-regular graph a.a.s. 3-colourable?

References

"Recent publications" 16, 17, 27, 36, 54, 66 on my web page:

http://www.math.uwaterloo.ca/~nwormald/abstracts.ht