The evolution of $G_{n,m}$

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OUTLINE

1. **Background.** Definitions, history and a brief account of the evolution.

2. **Methods.** An example illustrating expected value and concentration.

3. **The giant component.** An explanation of the sudden emergence of the giant component.

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1. **Background.** Definitions, history and a brief account of the evolution.

2. **Methods.** An example illustrating expected value and concentration.

3. **The giant component.** An explanation of the sudden emergence of the giant component.

$G_{n,m}$: A graph chosen uniformly at random from the set of all graphs on vertex set $[n]$ with m edges.

So, if G is a graph on vertex set $[n]$ with m edges

$$\Pr(G_{n,m} = G) = \frac{1}{\binom{\binom{n}{2}}{m}}$$

e.g.

$$\Pr\left(G_{4,p} = H\right) = p^3(1-p)^3$$

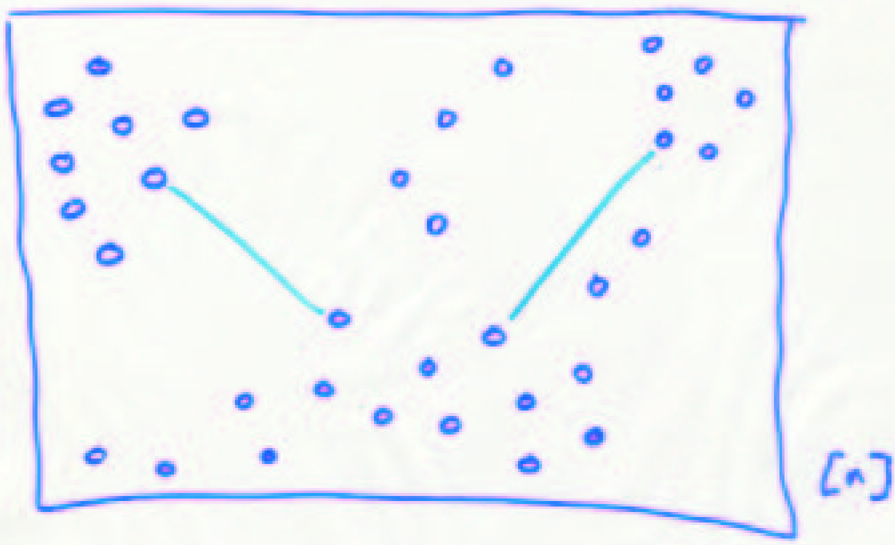
$$\Pr\left(G_{4,3} = H\right) = \frac{1}{\binom{6}{3}} = \frac{1}{20}$$

note: An event $\mathcal{E}_n$ is a collection of graphs on vertex set $[n]$. We are interested in

$$\lim_{n \to \infty} \Pr(\mathcal{E}_n)$$

where p, m are functions of n

$G_{n,m}$ can be viewed as a random graph process



$E(G_{n,m}) = \{e_1, \ldots, e_m\}$ where

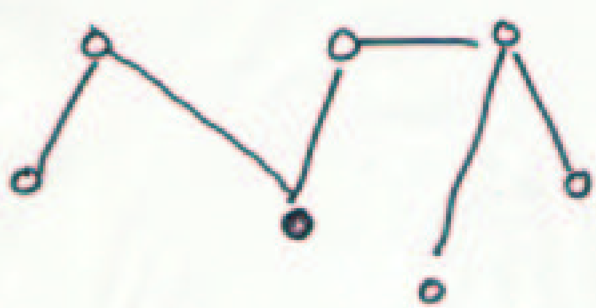
e_i is chosen uniformly at random

from $\binom{[n]}{2} \setminus \{e_1, \ldots, e_{i-1}\}$

Definition: The girth of a graph G is the size of the shortest cycle in G.

The chromatic number of G is the smallest number of colors in a proper coloring of G.

Note: large girth $\Rightarrow$ G is locally 2-colorable



Theorem (Erdös, 1959):

For all k, ℓ there exists a graph G with girth $> \ell$ and chromatic number $> k$.

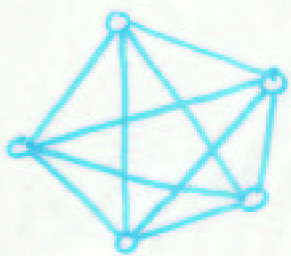
Pf: Consider $G_{n,p}$ for n sufficiently large and p carefully chosen.

$G_{n,p}$ can be altered to give a graph with the desired properties with nonzero probability.

$\square$

Theorem (Erdös, Rényi 1960)

$$\lim_{n\to\infty} \Pr\left(G_{n,m} \text{ contains a copy of } K_k\right) = \begin{cases} 0 & \text{if } m = o\left(n^{2-\frac{2}{k-1}}\right) \\ 1 & \text{if } m = \omega\left(n^{2-\frac{2}{k-1}}\right) \end{cases}$$



K_5

$f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$f(n) = \omega(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty$

Theorem (Erdös, Rényi 1959)

(i) If $m = \frac{n \log n}{2} + f(n)$ where $\frac{f(n)}{n} \to +\infty$ then

$$\lim_{n\to\infty} \Pr(G_{n,m} \text{ is connected}) = 1$$

$$\lim_{n\to\infty} \Pr(G_{n,m} \text{ has a perfect matching}) = 1$$

(ii) If $m = \frac{n \log n}{2} + f(n)$ where $\frac{f(n)}{n} \to -\infty$ then

$$\lim_{n\to\infty} \Pr(G_{n,m} \text{ is connected}) = 0$$

$$\lim_{n\to\infty} \Pr(G_{n,m} \text{ has a perfect matching}) = 0$$

Theorem (Erdös, Rényi 1960)

(i) If $m = cn + o(n)$ and $c < 1/2$ then a.a.s. the largest component of $G_{n,m}$ has $O(\log n)$ vertices.

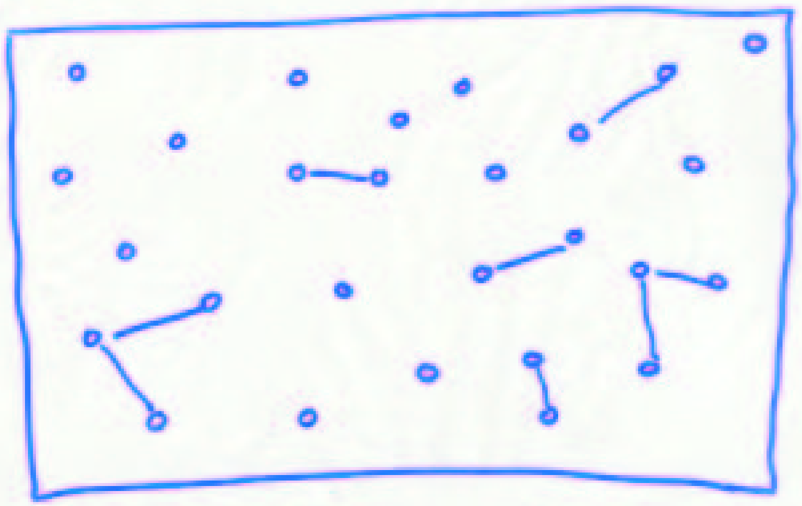
(ii) If $m = cn + o(n)$ and $c > 1/2$ then a.a.s. the largest component of $G_{n,m}$ has $\Omega(n)$ vertices and the second largest component has $O(\log n)$ vertices.

$f(n) = o(n)$ if $\frac{f(n)}{n} \to 0$

$f(n) = O(\log n)$ if $f(n) < C \log n$ for some constant C.

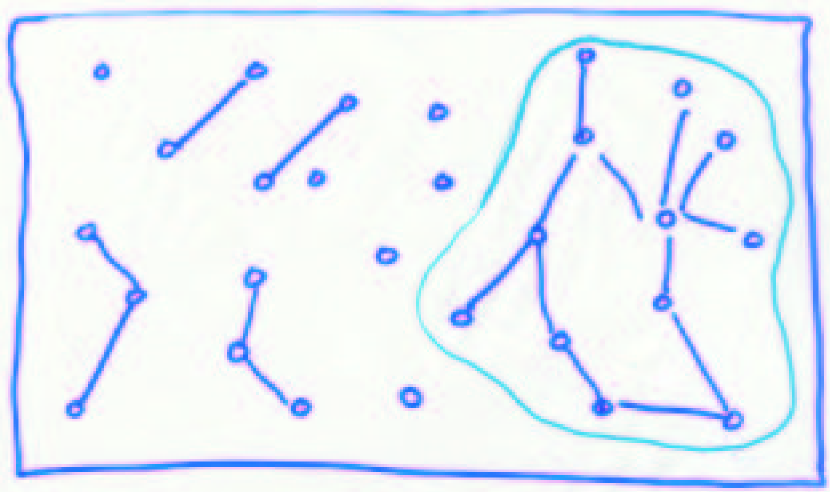
$f(n) = \Omega(n)$ if $f(n) > C \log n$ for some constant C.

$m < n/2$: all components are small



- $m = o(\sqrt{n})$. Isolated vertices and isolated edges.
- $m = \omega(\sqrt{n})$, $m = o(n^{2/3})$. Isolated vertices, isolated edges and paths with 2 edges.
- $m = cn$, $c < 1/2$. Every component has $O(\log n)$ vertices. No K_4's.

$\underline{m > n/2}$: $\exists$ a unique "giant component."



- $m = cn$, $c > 1/2$. There is one connected component that has $\Omega(n)$ vertices. All other components have $O(\log n)$ vertices.
- $m = cn \log n$, $c < 1/2$. Isolated vertices remain.
- $m = cn \log n$, $c > 1/2$. $G_{m,n}$ is connected and has a perfect matching.

An example

Claim: Let $p = n^{-\alpha}$, $4/3 < \alpha < 3/2$.

A.a.s. $G_{n,p}$ has at least one 2-path but no component with 4 or more vertices.

2-path — this appears.

3-path, 3-star — these do not appear

Pf: Let

X = # of 2-paths in $G_{n,p}$

Y = # of 3-paths

Z = # of 3-stars

For each 2-path A in K_n (complete graph on n vertices) define

$$X_A = \begin{cases} 1 & \text{if } A \text{ is in } G_{n,p} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$X = \sum_A X_A$$

there are $\frac{n(n-1)(n-2)}{2}$ terms in this sum

$$E[X] = E\left[\sum_A X_A\right]$$

$$= \sum_A E[X_A]$$

$$= \sum_A \Pr(X_A = 1)$$

$$= \sum_A p^2$$

$$= \frac{n(n-1)(n-2)}{2} p^2$$

Simillarly,

$$E[Y] = \frac{n(n-1)(n-2)(n-3)}{2} p^3$$

$$E[Z] = n \binom{n-1}{3} p^3$$

Now,

$$E[Y] \le n^4 p^3 = n^{\overbrace{4-3\alpha}} \to 0$$

$\alpha > 4/3$

$\Rightarrow 4 - 3\alpha < 0$

$$E[Z] \le n^4 p^3 = n^{4-3\alpha} \longrightarrow 0$$

<u>Markov's inequality:</u> If X takes only non-negative values and $\lambda > 0$ then

$$\Pr[X \ge \lambda] \le \frac{E[X]}{\lambda}$$

So, $\Pr[Y \ge 1] \le E[Y] \to 0$

$\Pr[Z \ge 1] \le E[Z] \to 0$

$$E[X] = \frac{n(n-1)(n-2)}{2} p^2$$

assuming $n \geq 4$

$$\geq \frac{n^3}{8} p^2$$

$$= \frac{n^{3-2\alpha}}{8}$$

$$\longrightarrow +\infty$$

$$\boxed{\begin{array}{l} \alpha < 3/2 \\ \Rightarrow 3-2\alpha > 0 \end{array}}$$

Does this imply

Pr($\exists$ a 2-path in $G_{n,p}$)

$\longrightarrow 1$

?

Recall: $\mathrm{Var}(X) = E[(X - E[X])^2]$

Claim: $\Pr(X=0) \leq \dfrac{\mathrm{Var}[X]}{E^2[X]}$

Proof:

$$\Pr(X=0) \leq \Pr\left((X-E[X])^2 \geq E^2[X]\right)$$

$$\leq \frac{E[(X-E[X])^2]}{E^2[X]}$$

$$= \frac{\mathrm{Var}[X]}{E^2[X]}$$

$\square$

To Show: $\mathrm{Var}[X] = o\left(E^2[X]\right)$

Since $X = \sum_A X_A$

$$
\begin{aligned}
\mathrm{Var}[X] &= E\left[(X - E[X])^2\right] \\
&= E[X^2] - E^2[X] \\
&= E\left[\left(\sum_A X_A\right)^2\right] - \left(\sum_A E[X_A]\right)^2 \\
&= \sum_{A,B} E[X_A X_B] - E[X_A]\,E[X_B] \\
&= \sum_{A \neq B} \underbrace{E[X_A X_B] - E[X_A]\,E[X_B]}_{\mathrm{Cov}(X_A, X_B)} \\
&\quad + \underbrace{\sum_A \mathrm{Var}[X_A]}
\end{aligned}
$$

This term is bounded above by $E[X]$ and $E[X] = o(E^2[X])$

If A and B share an edge then

$$\mathrm{Cov}(X_A, X_B) = p^3 - p^4$$

If A and B do not share an edge then

$$\mathrm{Cov}(X_A, X_B) = 0$$

$$\sum_{A \neq B} \mathrm{Cov}(X_A, X_B) = \sum_{A} \sum_{\substack{B: \\ A \cap B \neq \emptyset}} p^3 - p^4$$

$$\leq \sum_{A} 2np^3$$

$$\leq n^4 p^3$$

Since $a > 4/3$

$$\rightarrow 0$$

Differential Equations

$X_0, X_1, X_2, \ldots$ Sequence of random variables

$f: D \to \mathbb{R}$ function Lipshitz on D

$D \subseteq \mathbb{R}^2$ domain bounded, connected, open

If

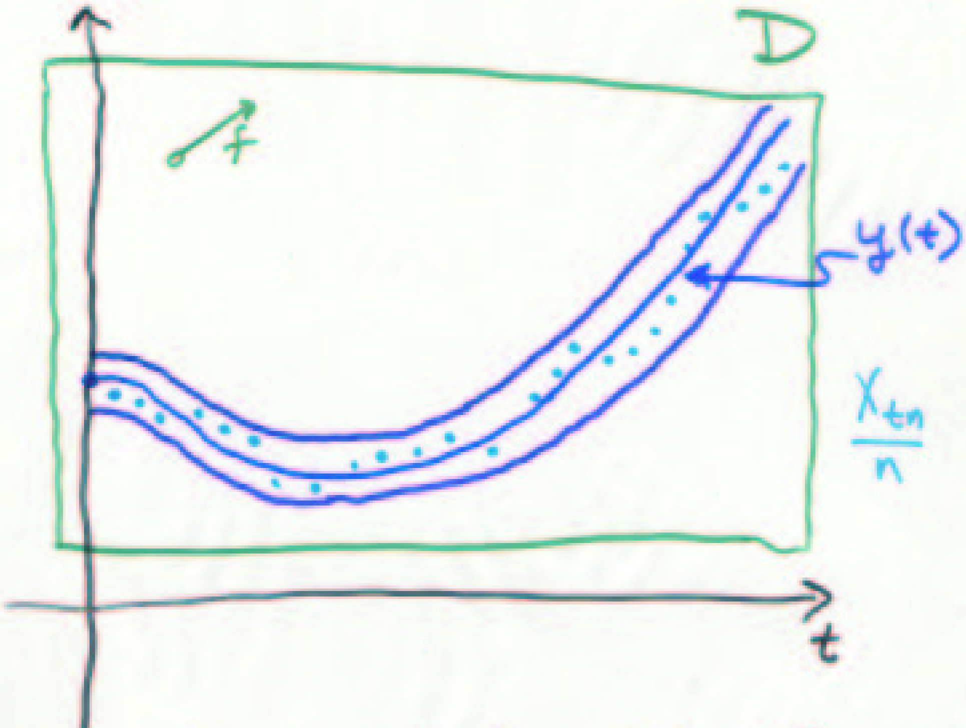
$$E[X_{i+1} - X_i \mid X_i] = f\left(\frac{i}{n}, \frac{X_i}{n}\right)$$

$$|X_{i+1} - X_i| \leq b$$

$$\frac{dy}{dt} = f(t, y) \qquad y_0 = \frac{X_0}{n}$$

Then

X_i is nearly $n\,y(i/n)$



$$E[X_{i+1} - X_i \mid X_i] = f\left(\frac{i}{n}, \frac{X_i}{n}\right)$$

(if $(i/n, X_i/n) \in D$)

$$\frac{dy}{dt} = f(t, y) \qquad y_0 = \frac{X_0}{n}$$

(We assume X_0 is a fixed constant times n)

An Argument

$$
\begin{aligned}
X_{(t+\varepsilon)n} &= X_{tn} + \sum_{i=0}^{\varepsilon n-1} X_{tn+i+1} - X_{tn+i} \\
\left.\begin{array}{r}\text{by}\\ \text{probability}\\ \text{theory}\end{array}\right\} &\approx X_{tn} + \varepsilon n \, E\left[X_{tn+1} - X_{tn} \mid X_{tn}\right] \\
&= X_{tn} + \varepsilon n \, f\left(t, \frac{X_{tn}}{n}\right) \\
\left.\begin{array}{r}\text{by}\\ \text{induction}\end{array}\right\} &\approx n y(t) + \varepsilon n \, f(t, y(t)) \\
&= n y(t) + \varepsilon n y'(t) \\
&= n \left(y(t) + \varepsilon y'(t) \right) \\
\left.\begin{array}{r}\text{by}\\ \text{calculus}\end{array}\right\} &\approx n y(t+\varepsilon)
\end{aligned}
$$

The Giant Component

Let G be a graph on vertex set $[n]$ with connected components $C_1, C_2, \dots, C_\ell$.

Let

$$X(G) = \sum_{i=1}^{\ell} |C_i|^2$$

$$G^+ = G + e$$

an edge chosen uniformly at random from $\binom{[n]}{2} \setminus E(G)$

$$E\left[X(G^+) - X(G)\right] = \sum_{i \neq j} \frac{|C_i||C_j|}{n^2} \, 2|C_i||C_j|$$

$$= \frac{2}{n^2} \sum_{i \neq j} |C_i|^2 |C_j|^2$$

$$= \frac{2}{n^2}\left[\left(\sum_i |C_i|^2\right)^2 - \underbrace{\sum_i |C_i|^4}_{\text{ignore}}\right]$$

$$= 2X^2(G)\frac{1}{n^2}$$

$|C_i|^2 + |C_j|^2$

$\to \cancel{2|C_i||C_j|}$

$(|C_i| + |C_j|)^2$

Let $e_1, e_2, \ldots$ be a sequence of edges where e_i is chosen uniformly at random from $\binom{[n]}{2} \setminus \{e_1, \ldots, e_{i-1}\}$.

Let G_i be the graph with vertex set $\{e_1, \ldots, e_i\}$.

Let $X_i = X(G_i)$.

X_i is nearly $ny(i/n)$ where

$$\frac{dy}{dt} = 2\cancel{t^2}y^2 \qquad y(0) = 1$$

That is,

$$y(t) = \frac{1}{1-2t}$$

Obstacles to making this a proper proof:

1. Does $X_i = (\text{large constant})n$ imply that a giant component will soon appear?

2. $|X_{i+1} - X_i|$ could be very large (e.g. if we connect two components with $\sqrt{n}$ vertices then $|X_{i+1} - X_i| = 2n$).

B., Kravitz
Spencer, Wormald
} gave techniques to overcome these obstacles.