The Diameter of Randomly Perturbed Digraphs and 2 Applications

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Introduction: Smoothed Analysis:

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In short, define a random perturbation and show the hard instances are unlikely with respect to this perturbation.

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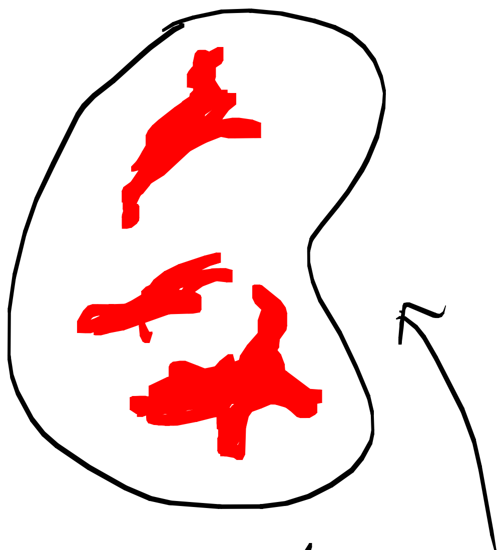
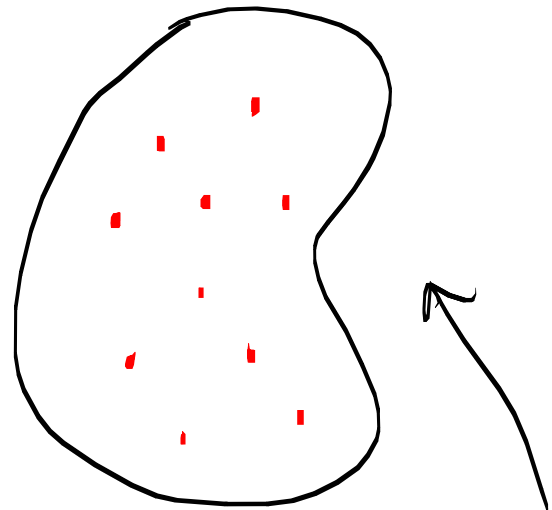


Things like this should be difficult in practice



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Is the perturbation "right"?

(In their case, additive noise, consisting of independent $\mathcal{N}(0, \varepsilon)$.)

Smoothed analysis of discrete problems

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but...

if random graph has too many edge, there will be nothing left of the original instance.

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(Acceptable in certain situations, for example, property testing [Speilman and Teng, 2002])

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But we want to say something about sparse graphs.

So, we perturb by XORing with $G_{n,p}$ where $p = \frac{\varepsilon}{n}$.

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Relationship to semirandom instances.

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Semirandomness

[Santha & Vazirani, 1986]

Semirandom instances

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Semirandom instances

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To go from semirandom instances to smoothed instances, make advisary oblivious.

Central Observation:

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Theorem: Let $\varepsilon > 0$. Let $\overline{G}$ be an n-node connected graph. Let $G = \overline{G} + R$ where $R \sim G_{n, \varepsilon/n}$. Then, whp,

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(In paper, stated for digraphs, but holds for directed/undirected, and for many similar perturbations)

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(Also similar to [Bollobás & Chung, 88] which shows diameter of a cycle + random matching is $O(\log n)$.)

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To do so, we explore neighborhoods around s and t, alternating between edges of $\overline{G}$ and R.

Proof:

First, we find S_0' & T_0', where

$$S_0' = \{ v : d_{\bar{G}}(s,v) \leq O(\varepsilon^{-1} \log n) \}$$

and

T_0' is defined analogously.

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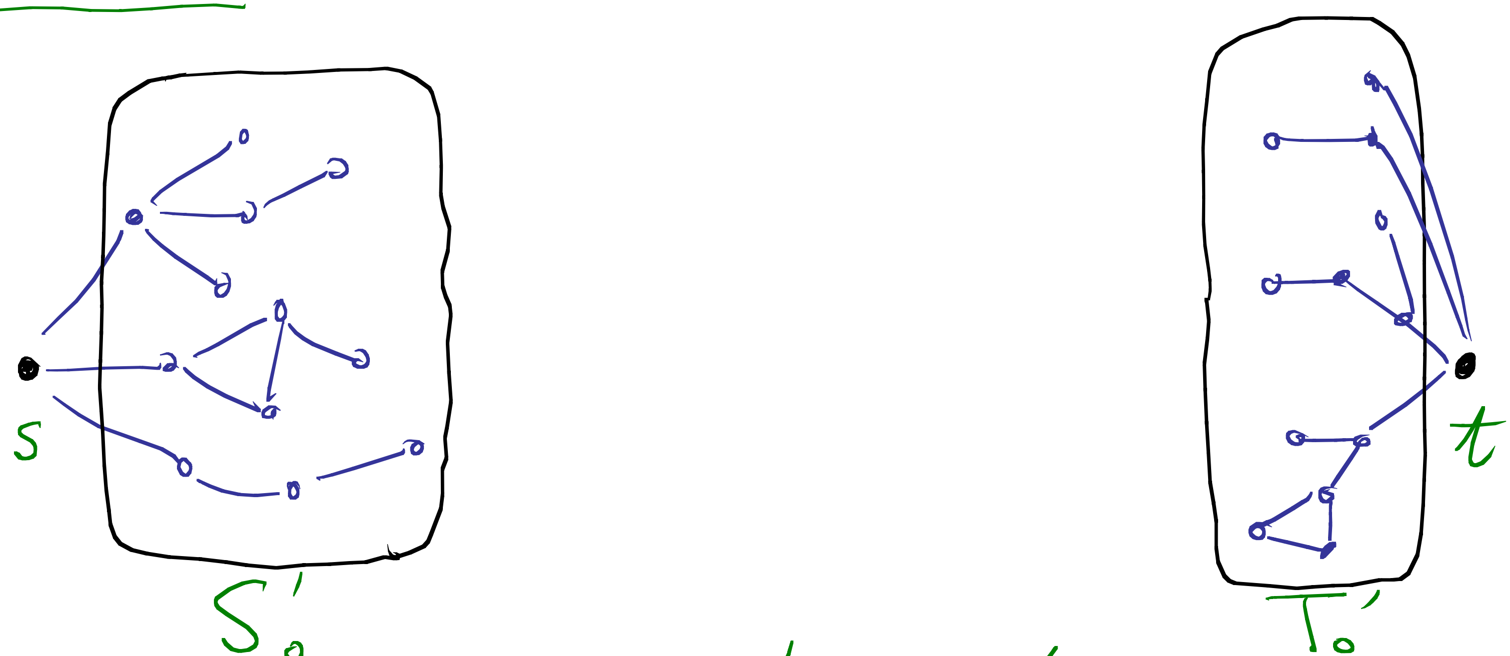

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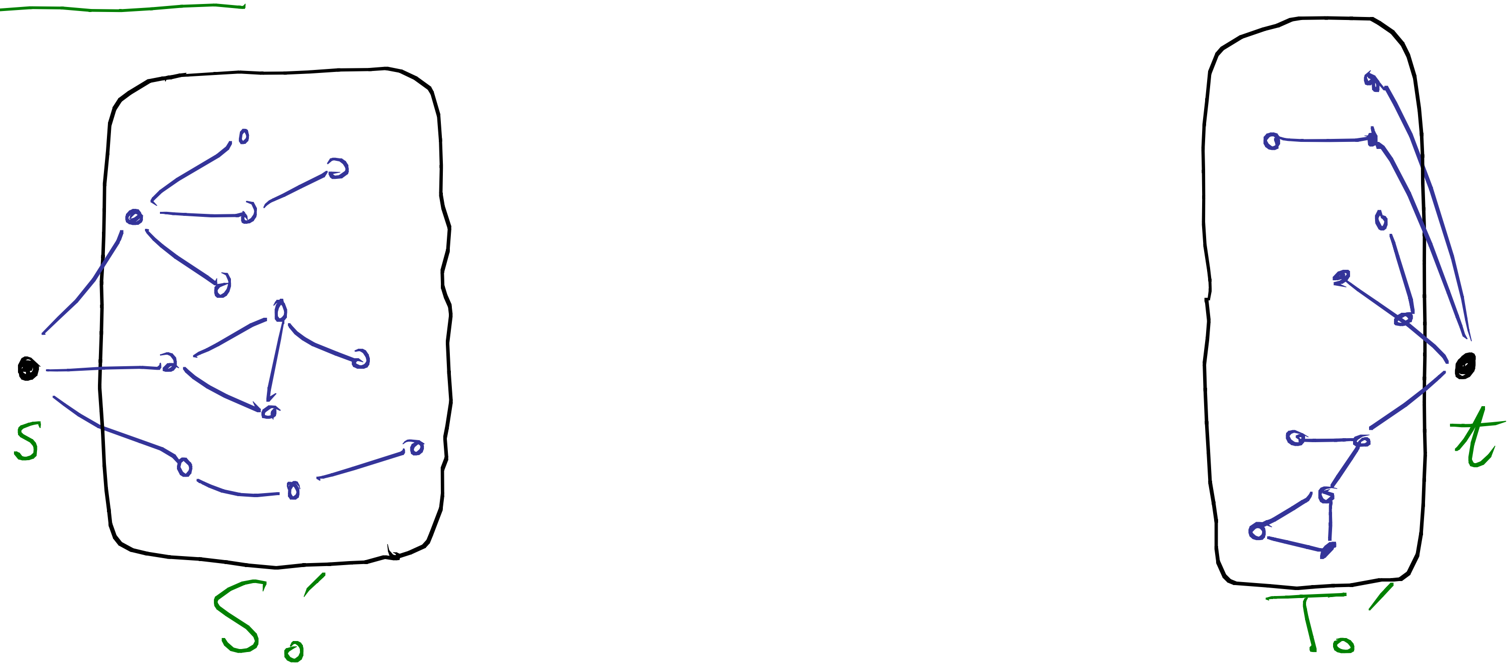

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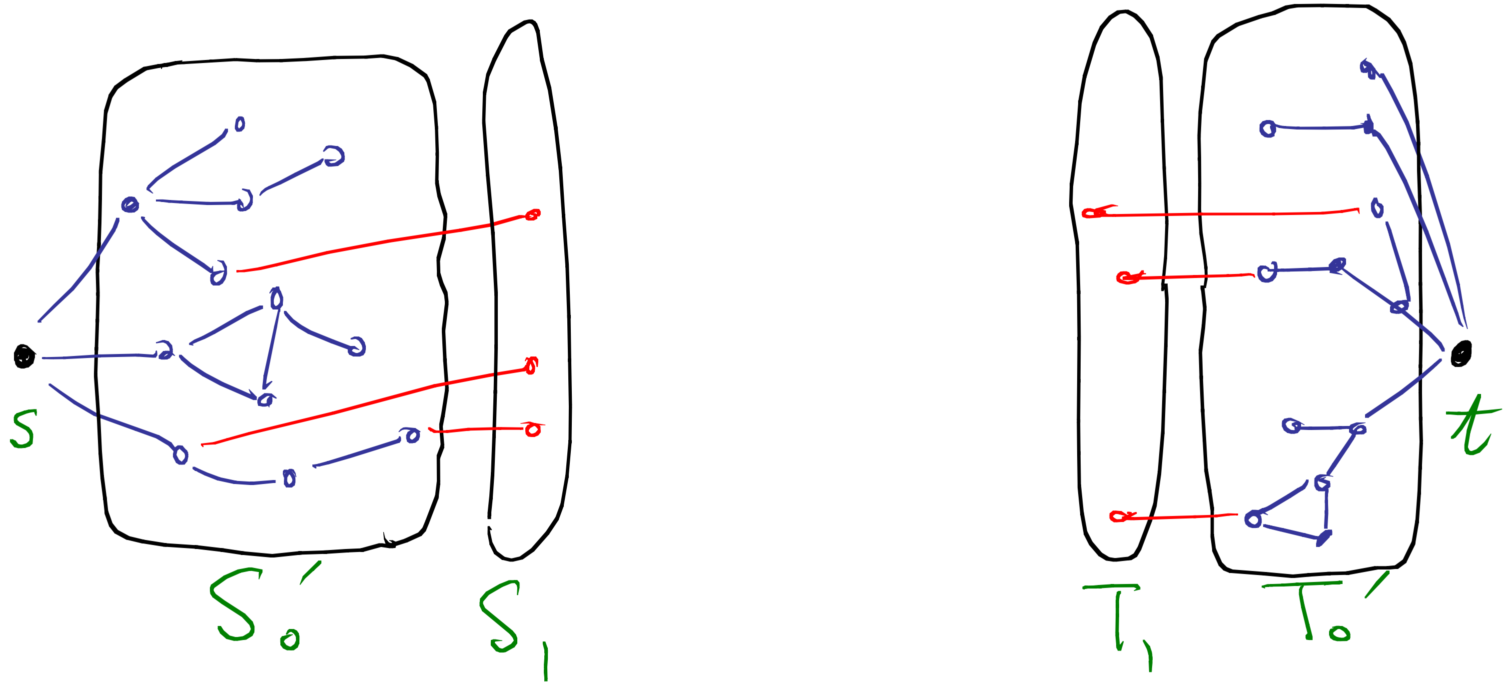

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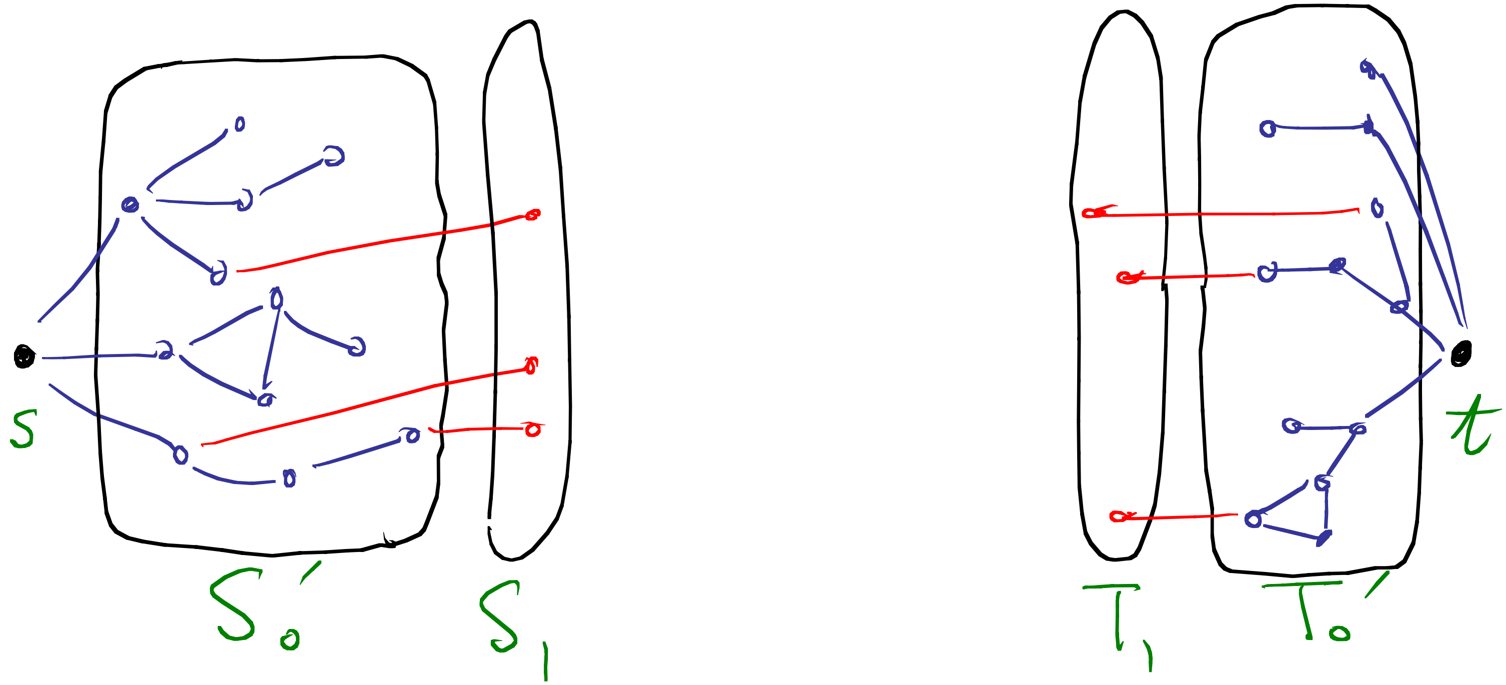

Then we reveal all the random edges of R incident to S_0' and T_0', and denote their other endpoints by S_1 & T_1.

$$\mathbb{E}[|S_1|] \gtrsim |S_0'| \cdot n \cdot \frac{\varepsilon}{n} = \mathcal{O}(\log n).$$

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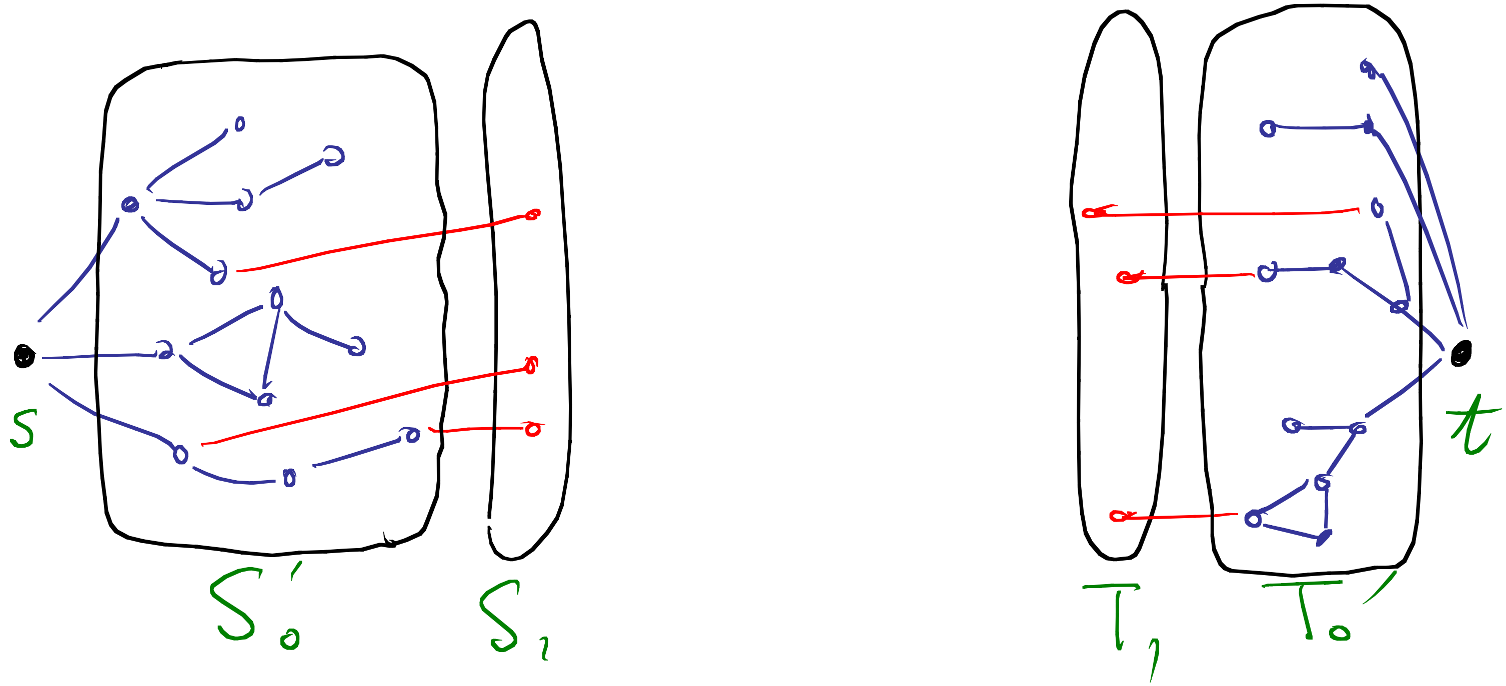

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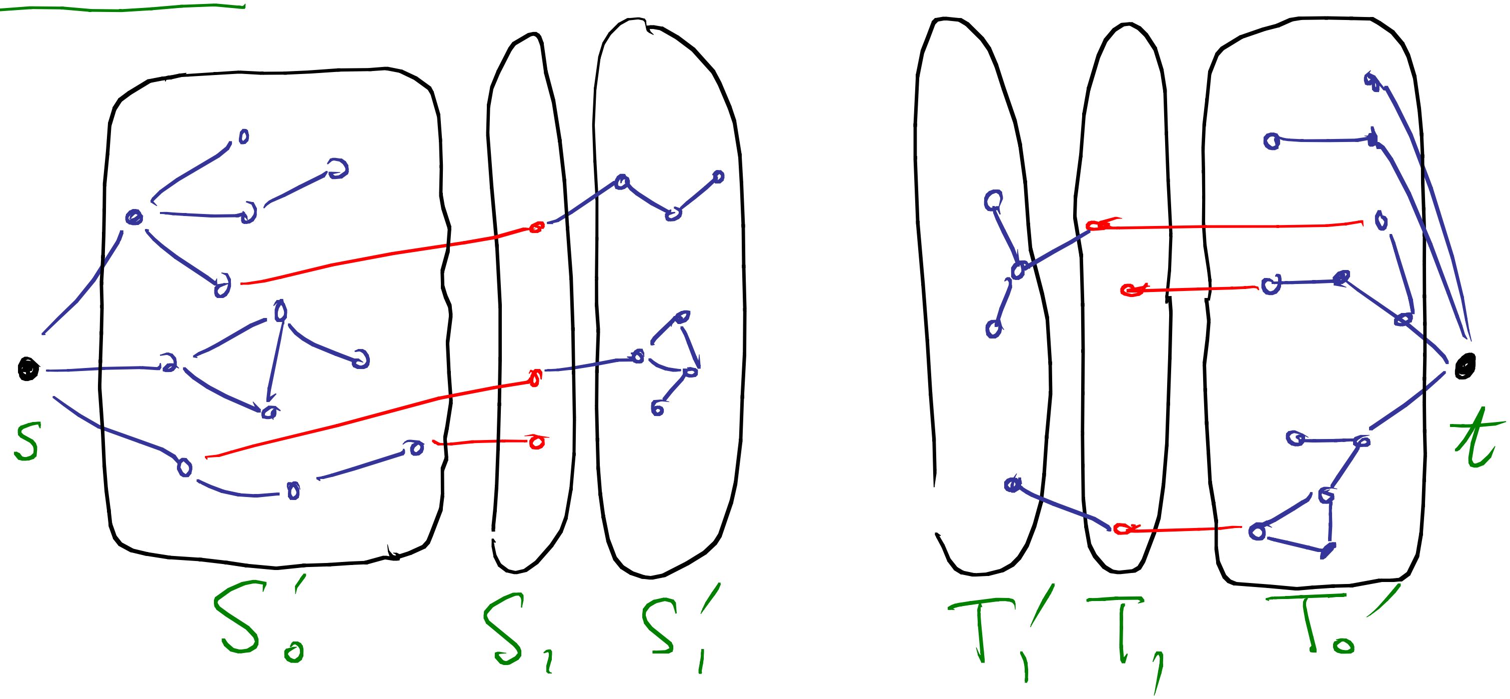

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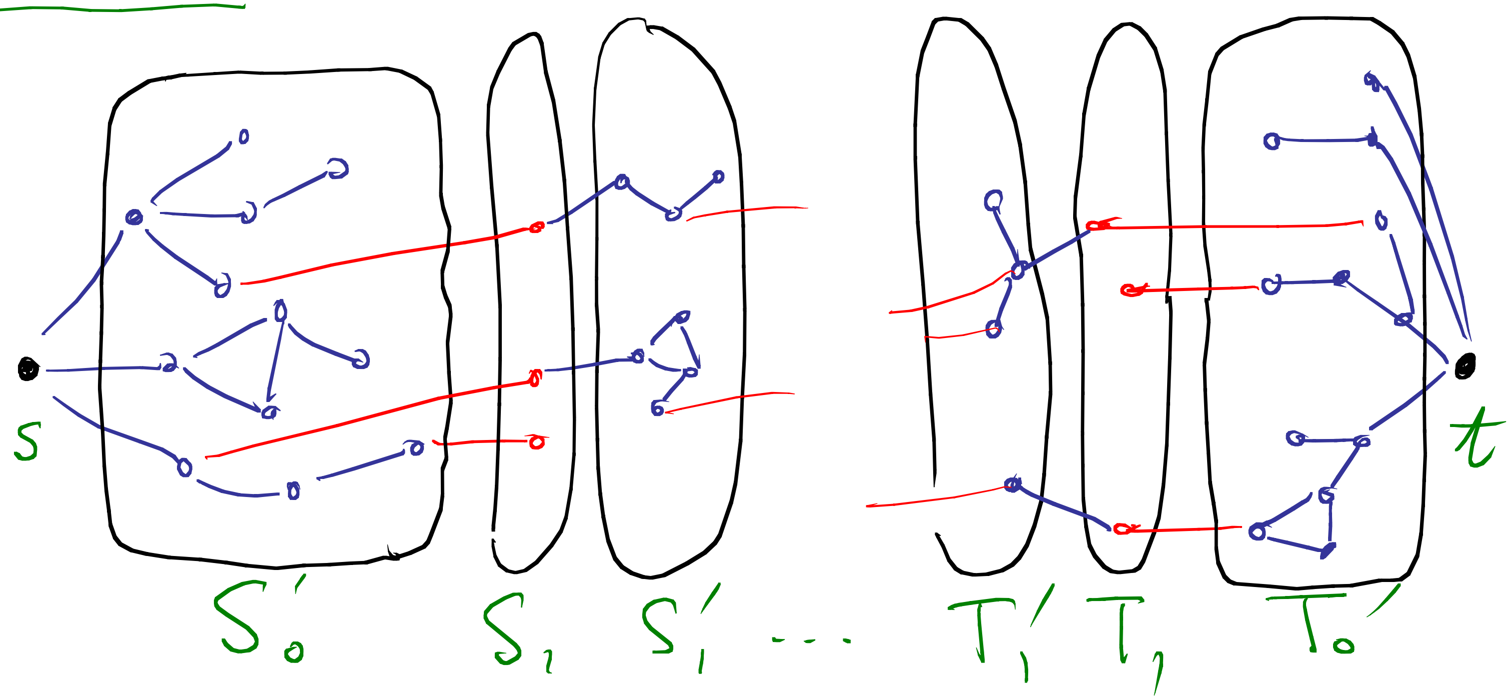

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S_2, T_2 be the R-nbrs of of S_1', T_1'.

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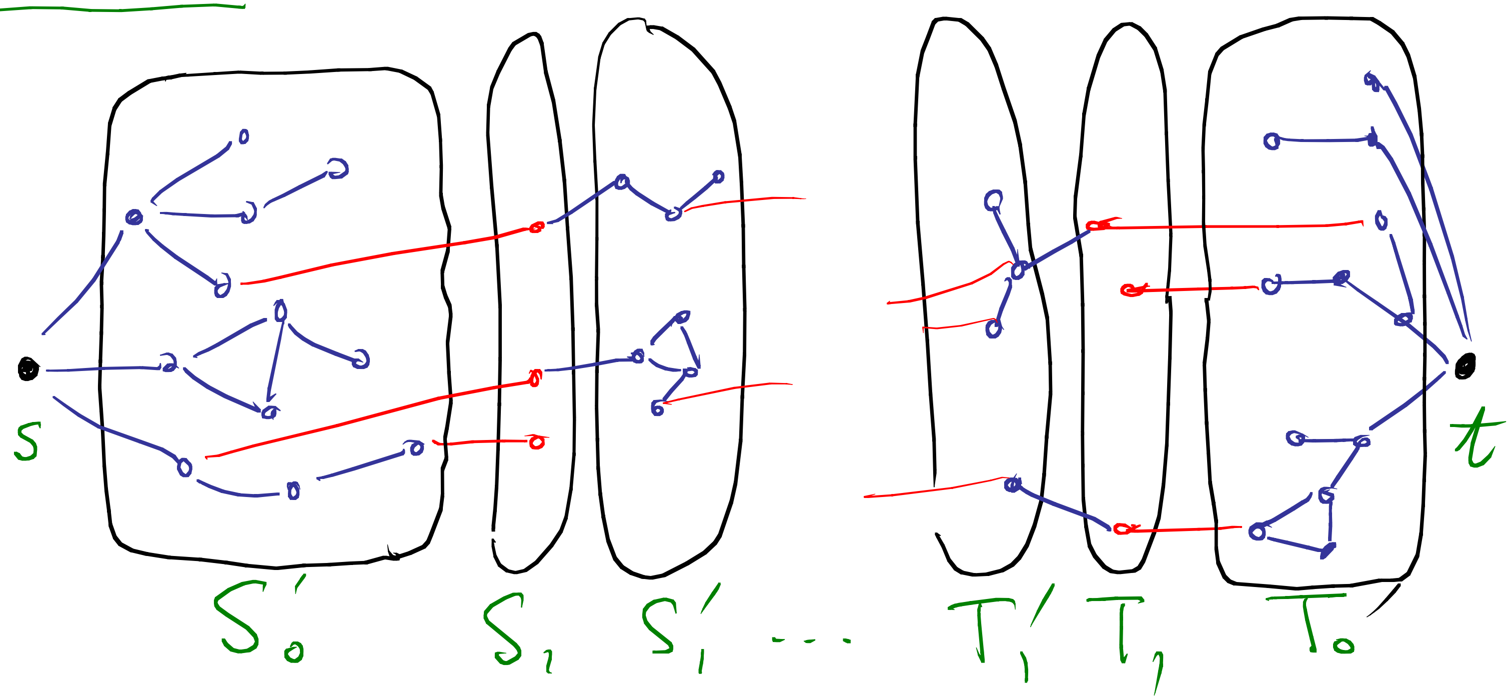

So we have
$$\mathbb{E}[|S_i|] \gtrsim \mathbb{E}[|S'_{i-1}|]\, n \cdot \frac{\varepsilon}{n}$$
$$\geq \left(|S_{i-1}|\, 5\varepsilon^{-1}\right) n \cdot \frac{\varepsilon}{n}$$
$$= 5|S_{i-1}|.$$

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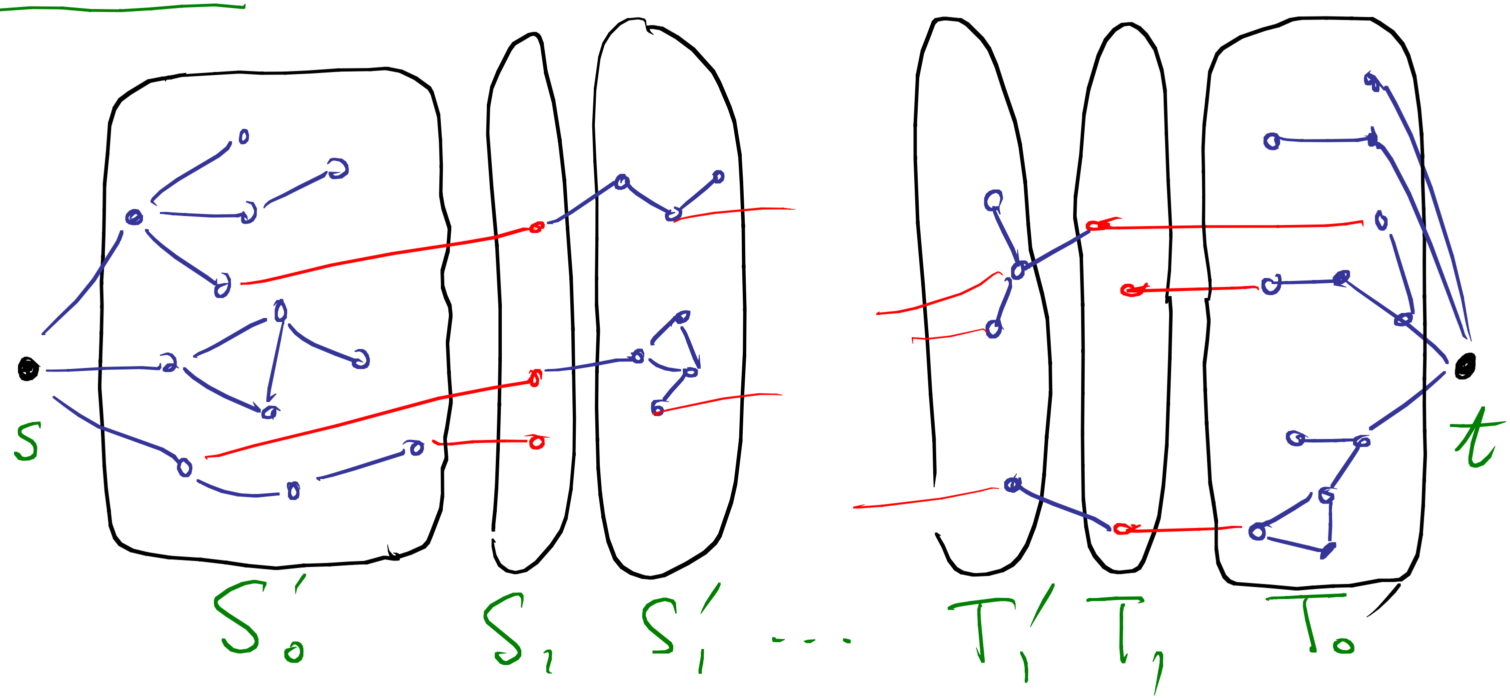

So we have $\mathbb{E}[|S_i|] \gtrsim \mathbb{E}[|S_{i-1}'|] n \cdot \frac{\varepsilon}{n}$

$$\geq (|S_{i-1}| 5 \varepsilon^{-1}) n \cdot \frac{\varepsilon}{n}$$

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S_i and T_i meet before approx gets too inaccurate.

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- Strong connectivity is NL-complete
- Even when restricted to bounded degree instances.

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ALG: Look for short ($O(\log n)$) paths. Accept if you find them between every pair of nodes.

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Elements of Pf: 1. Add edges $\rightarrow$ low diam
2. degree stays "pretty bounded"
3. If $\bar{D}$ not conn, D becomes conn in a good way

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  (More complicated than it sounds in log-space)

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- Recognizing k-linked graphs is NP-hard for $k \geq 2$.
- Want alg for smoothed instances as in previous application. (Doesn't work yet...)

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- Difficulty: Can't rule out possibility: $\bar{D}$ is not k-linked but D is k-linked in a way not witnessed by short paths.

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- This case doesn't come up in property testing: TESTING k-linked:
  - If $\bar{D}$ is k-linked accept
  - If $\bar{D}$ is ε-far from k-linked reject
  - o.w. whatever

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Search all terminal sets for disjoint

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For each node, with pr $\frac{\varepsilon}{2}$ add an arc out to some node chosen u.a.r.

Search all terminal sets for disjoint paths of length $O(\log n)$.

If $\bar{D}$ is k-linked then D is also, and whp we accept. If $\bar{D}$ is ε-far, then D is not k-linked $\Rightarrow$ reject.

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- k-linkedness property is testable in poly-time.