Applications of Donaldson's Diagonalization Theorem

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Introduction

When a link L has non-zero determinant, the maximum Euler characteristic of a surface with L as its boundary is 1. However, much is still unknown about which links bound surfaces of maximal Euler characteristic in the 4-ball.

The goal of our project is to characterize **which four stranded pretzel links with non-zero determinant bound a maximal Euler characteristic surface in B^4.** We call these links χ-slice.

We obstruct most 4-stranded pretzel links from being χ-slice using Donaldson's Diagonalization theorem. We then prove many of the remaining links are χ-slice using ribbon moves.

A **knot** is defined as the image of a smooth embedding $S^1 \to S^3$. A **link** of n **components** is the disjoint union of n knots.

A **four stranded pretzel link** is written in the form $P(p, q, r, s)$ where p, q, r, and s represent the the twists in each strand.

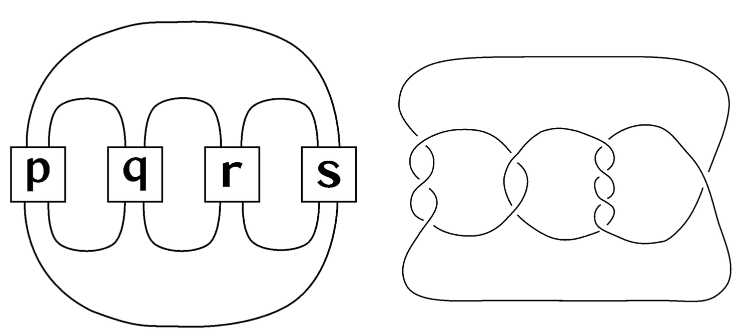

Fig. 1: **Left:** General four stranded pretzel link $P(p, q, r, s)$ **Right:** $P(3, 2, -4, -1)$

A link $L \subseteq S^3$ is **χ-slice** if L bounds a smoothly properly embedded surface $F \subseteq B^4$ with Euler characteristic 1. If L is a knot and is χ-slice, we say L is **slice**.

The **determinant** of a link is an integer-valued invariant.

We call a link **χ-ribbon** if it bounds a surface that can be obtained via ribbon moves (see construction). If L is a knot and is χ-ribbon, we say L is **ribbon**.

Our work builds upon several notable papers including the work of Greene-Jabuka classifying 3-stranded slice pretzel knots [3], the work of Lecuona classifying slice pretzel knots more generally [4], and the work of Lisca classifying χ-slice 2-bridge links [5].

Big Picture

Our work has applications towards the following unsolved questions in low dimensional topology:

1. The **slice ribbon conjecture** states that all slice knots are ribbon. We can generalize this to links by asking whether all χ-slice links are χ-ribbon. Our work indicates the answer to this question is yes for 4-stranded pretzel links.
2. A **rational homology 3-sphere** or $\mathbb{Q}S^3$ is a 3-manifold with the same rational homology as S^3. Likewise, a **rational homology 4-ball** or $\mathbb{Q}B^4$ is a 4-manifold with the same rational homology as B^4.

   An outstanding question in low-dimensional topology is which $\mathbb{Q}S^3$s bound $\mathbb{Q}B^4$s. By Donald-Owens [1], the *double branched cover* of a χ-slice link with non-zero determinant is a $\mathbb{Q}S^3$ that bounds a $\mathbb{Q}B^4$. Hence, our work serves to give examples of such $\mathbb{Q}S^3$s.

How to show a link is χ-slice

Use Ribbon Moves

How to show a link is not χ-slice

Analyze Lattice Embeddings

Construction

One way to show that a link is χ-slice is by searching for the right **ribbon moves**. These ribbon moves consist of adding a band with any number of twists in it and then adjusting the link accordingly.

Any number of ribbon moves can be made until the resulting link is an unlink. Then in order for the link to be χ-slice, we want $\chi = 1$ where

$$\chi = \#\text{ of unknots} - \#\text{ of ribbon moves}.$$

Looking for ribbon moves like this can be helpful in some cases, but becomes difficult to use in generality due the large number of possible bands.

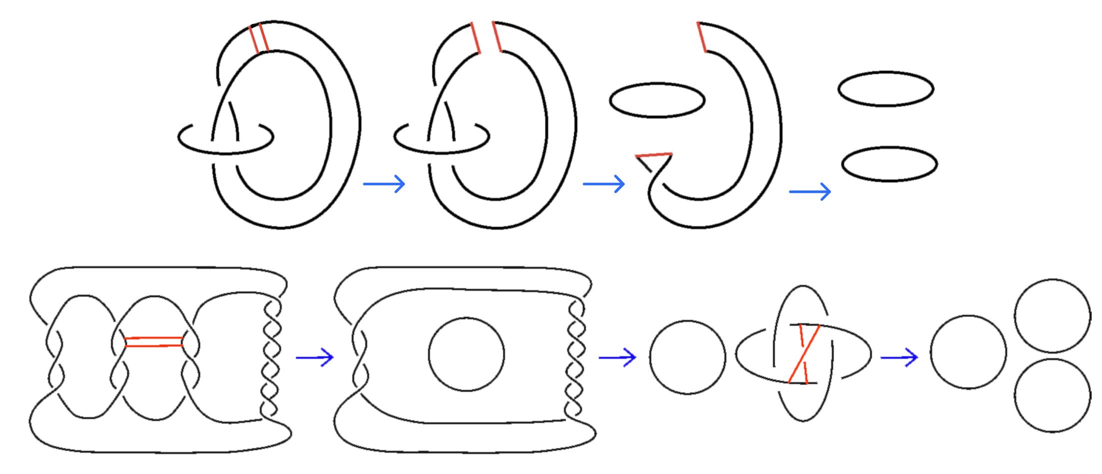

Fig. 3: Examples of using ribbon moves to prove two links are χ-slice

Obstruction

Fact: Associated to any oriented 4-manifold X is a symmetric, integer $n \times n$ matrix called the **intersection form**, denoted by Q_X. This matrix is **definite** if it has all positive or all negative eigenvalues.

Donaldson's Diagonalization Theorem: [2] The intersection form of a closed, definite, oriented, smooth 4-manifold is diagonalizable over $\mathbb{Z}$.

A consequence of Donaldson's theorem is that there exists a lattice embedding $(\mathbb{Z}^n, Q_X) \to (\mathbb{Z}^n, \pm I)$ when X is a closed, definite, oriented, smooth 4-manifold.

A **lattice** is a pair $(\mathbb{Z}^n, Q)$, where Q is a symmetric matrix with non-zero determinant. Let $(\mathbb{Z}^n, Q_1)$ and $(\mathbb{Z}^n, Q_2)$ be lattices. A map $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$ satisfying

$$\phi(v + w) = \phi(v) + \phi(w)$$

$$v^T Q_1 w = \phi(v)^T Q_2 \phi(w)$$

for all $v, w \in \mathbb{Z}^n$ is called a **lattice embedding**. We write $\phi : (\mathbb{Z}^n, Q_1) \to (\mathbb{Z}^n, Q_2)$.

Theorem: [1, 2, 5] Let $L \subseteq S^3$ be a link with $\det L \neq 0$. Suppose that L is χ-slice and that $\Sigma_2(L)$ bounds a negative-definite 4-manifold X. Then there is a lattice embedding

$$\phi : (\mathbb{Z}^n, Q_X) \to (\mathbb{Z}^n, -I). \quad (\star)$$

Start with a pretzel link $L = P(p, q, r, s)$, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} > 0$.

1. Suppose that L bounds a surface $F \subseteq B^4$ with $\chi(F) = 1$.
2. Construct the *double branched covers* $\Sigma_2(L)$ and $\Sigma_2(F)$ of L and F (3- and 4-manifolds respectively).
3. Build a negative-definite 4-manifold X with boundary $\Sigma_2(L)$.
4. Glue X to $\Sigma_2(F)$ along their boundary $\Sigma_2(L)$ to get a closed, negative-definite, oriented, smooth 4-manifold.

$$\begin{array}{ccccc} & & X \cup -\Sigma_2(F) & & \\ & \nearrow & & \nwarrow & \\ X & \xrightarrow{\partial} & \Sigma_2(L) & \xleftarrow{\partial} & \Sigma_2(F) \\ & & \downarrow & & \downarrow \\ & & L & \xleftarrow{\partial} & F \end{array}$$

By Donaldson's theorem, there must be an embedding $(\star)$.

Obstruction. If there is no embedding $(\star)$, then L is not χ-slice.

Findings

Throughout, we suppose $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} > 0$. The following cases arise:

- If p, q, r, and s are greater than 0, the only χ-slice links are $P(1, 1, 1, 1)$ and $P(5, 1, 1, 1)$.
- In fact, an n-stranded pretzel link with all positive twists is χ-slice if and only if it is of the form
  (i) $P(n + 1, 1, 1, ...1)$
  (ii) $P(n - 3, 1, ..., 1)$
- If $p, q, r > 0$, $s < 0$, and L is χ-slice, then either:
  (i) Two of p, q, r are 2, the other one is any integer $t \geq 1$, and $s \in \{-1, -t, -(t + 4)\}$, or
  (ii) One of p, q, r is 1, another is 3, the third is any integer $t \geq 1$, and $s \in \{-t, -(t + 3)\}$.

  Indeed, L is χ-slice in each case for any permutation of p, q, r, s, except possibly $P(2, t, 2, -t)$ and $P(2, t, 2, -(t+4))$ in (i). These are χ-slice when $t = 1$ and are not χ-slice if $t > 1$ is odd or if $t \equiv 0 \mod 4$.

In the future, we would like to classify the remaining 4-stranded pretzel links and continue the general n-stranded case. Here are some specific questions we still have:

1. Are $P(2, t, 2, -t)$ and $P(2, t, 2, -(t + 4))$ χ-slice when $t \equiv 2 \mod 4$?
2. For what values of p, q, r, and s is $P(p, q, r, s)$ χ-slice when two of these values are negative? This case has posed some complications because Donaldson's theorem leaves several infinite families unobstructed. We have begun to use some other invariants to narrow down the list of potentially χ-slice links.

Acknowledgements

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