REVIEW NOTES FOR FIRST MATH 1B EXAM

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1. Riemann integrals

You should know the definition of the definite integral of a continuous function f on a closed interval $[a, b]$ as a limit of Riemann sums: a Riemann sum is obtained by dividing the interval $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$, by selecting a point x_i^* on the subinterval $[x_i, x_{i+1}]$ and approximating the value of the function on the entire subinterval by the value $f(x_i^*)$: thus we are approximating the area on $[x_i, x_{i+1}]$ by $\Delta x f(x_i^*) = \frac{b-a}{n} f(x_i^*)$; the entire Riemann sum is then $\sum_{i=0}^{n-1} f(x_i^*)\Delta x$. Then, if f is continuous, no matter which sample points x_i^* are chosen, these Riemann sums will converge to a common value in the limit: $\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^*)\Delta x = \int_a^b f(x)dx$. This value is by definition the definite integral.

The Riemann sum can be taken for a function which is both positive and negative; $\int_a^b f(x)dx$ represents the *signed area*, i.e. the positive area - the negative area.

The fundamental theorem of calculus gives us a powerful method for computing these definite integrals (which would be anywhere from needlessly onerous to nigh impossible to compute directly from the definition): if $F(x)$ is an antiderivative of $f(x)$ – that is, a function such that $F' = f$ – then we know $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.

Note that an antiderivative of a continuous function on a closed interval $[a, b]$ is well-determined up to an additive constant: if F_1, F_2 are two functions such that $F_1' = f = F_2'$, then $(F_1 - F_2)' = (F_1' - F_2') = f - f = 0$, so $F_1 - F_2$ has zero derivative on $[a, b]$ hence is constant. It is important to remember this indeterminacy, e.g.

Question: A particle is moving along a line with the velocity function $v(t) = \sin t$. If its initial position is $x(0) = 0$, what is its position at any time t?

Solution: By definition $v(t) = x'(t)$, so that $x(t) = \int v(t) = -\cos t + C$. What is C? It is such that $0 = x(0) = -\cos(0) + C = -1 + C$, i.e. $C = 1$, and the position function is $x(t) = 1 - \cos(t)$. Notice that if we forgot to include the constant C our answer would have been wrong.

Another warning: many students become so enamored of the fundamental theorem of calculus that they forget all about the definition of the definite integral as the limit of Riemann sums. We instructors like to force you to remember the (geometric) definition of the definite integral by giving problems where the definite integral can be easily computed when finding the antiderivative is difficult or impossible. Our favorite example of this is the following

Fact: If $f(x)$ is an odd function: $f(-x) = -f(x)$, or (otherwise put) the graph of f is symmetric about the origin – then $\int_{-a}^a f(x)dx = 0$ for a any finite number, since $\int_{-a}^0 f(x)dx = -\int_0^a f(x)dx$. For example, $\int_{-\pi}^{\pi} (\sin x)e^{x^2} dx$ would be a real pain to find the antiderivative but the definite integral is 0.

Similarly, you should have some idea of how to use Riemann sums to get an approximate value for $\int_a^b f(x)dx$ – although we would not ask you to numerically evaluate a Riemann sum on an exam, we might ask you to argue geometrically about the value of a definite integral using Riemann sums, for example

Question: Give a method for calculating $\int_0^1 e^{x^2} dx$ to within an accuracy of .000001.

Solution: First recall that e^{x^2} is our basic example of a function whose antiderivative we cannot write down, so we must proceed numerically. The idea of course is to divide into a bunch of subintervals and take a Riemann sum, but how will we know how far we are? We exploit the fact that $f(x) = e^{x^2}$ is an increasing function on $[0, 1]$ (check: $f'(x) = 2xe^{x^2} \geq 0$ on this interval), so that the left-endpoint sums will be lower-sums and the right-endpoint sums will be upper-sums (draw a picture!). Here $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$, so the left endpoint sum is $\sum_{i=0}^{n-1} e^{(\frac{i}{n})^2} \frac{i}{n}$, whereas the right-endpoint sum is $\sum_{i=0}^{n-1} e^{(\frac{i+1}{n})^2} \frac{i}{n}$. We know the first quantity is always less than the second, so we get a calculational device (TI-85 or mathematical software package) to spit out these sums for increasing n until we find a value of n for which these two quantities differ by less than .000001.

2. Geometric applications of integration: area, volume, arclength

2.1. Area between curves. If $f(x)$ and $g(x)$ are two functions defined on $[a, b]$, we can, roughly speaking, integrate their difference to find the area between them. However, we must be careful about which function is on top: technically we define the area between f and g to be $\int_a^b |f(x) - g(x)|dx$, but what this really means is that we must break up the integral into regions where $f(x) - g(x)$ has constant sign – always positive or always negative. When $f(x) - g(x)$ is positive, $|f(x) - g(x)| = f(x) - g(x)$, so we integrate $f(x) - g(x)$. But when $f(x) - g(x)$ is negative, $|f(x) - g(x)| = g(x) - f(x)$, so we must integrate the latter.

Example: Find the area enclosed by $y = \sin x$ and $y = 0$ on $[0, 2\pi]$.

Solution: The answer is $\int_0^{2\pi} |\sin x|dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} - \sin x dx$ (because $\sin x < 0$ for $x \in (\pi, 2\pi)$).

2.2. Volumes by slicing. If we want to find the volume of a three-dimensional region whose cross-sectional areas are known to us, then we can simply integrate the cross-sectional area to get the volume:

$$V = \int_a^b A(x)dx$$

2.3. Volumes of revolution. If we are given a "volume of revolution," i.e. a solid obtained by revolving a region bounded by curves about an axis parallel to the x or y axis, we have three methods for finding the volume of this region: the methods of disks, washers, and cylindrical shells.

Disks and Washers really just amount to special cases of 'slicing' like you would slice a loaf of bread. If we are revolving around an axis parallel to the x-axis and we wish to integrate with respect to x, then we use the method of washers or disks. Similarly, if we are revolving around an axis parallel to the y-axis and we wish to integrate with respect to y, then we use washers or disks. (A disk is just a washer whose inner radius is 0.) We need to measure the outer and inner radii: say $r_{max}(x)$ is the outer radius as a function of x and $r_{min}(x)$ is the inner radius as a function of x – when we are revolving around the y-axis we get r_{max} and r_{min} are just the top function and the bottom function, but in general we might need to adjust these two by some constant amount if the axis is of the form $y = a$, so it is better to think of it this way. The formula is then

$$V = \int_a^b \pi(r_{max}(x)^2 - r_{min}(x)^2)dx$$

when we integrate with respect to x and a similar formula is available for integrating with respect to y.

Shells: On the other hand, say we want to revolve around an axis parallel to the y-axis but we want to integrate with respect to x. Then we use the method of cylindrical shells: the idea is that we approximate the volume of revolution by a sequence of concentric hollow cylinders, whose base radius is the distance from the axis of revolution, whose height is the height of the function at that point, and which is "differentially thin": the volume of the cylindrical shell of radius r, height h and thickness Δx is $2\pi rh\Delta x$. This leads to a Riemann sum $\sum_{i=0}^{n-1} 2\pi r(x_i^*)f(x_i^*)\Delta x$ and hence to the expression

$$V = \int_a^b 2\pi r(x)f(x)dx$$

when there is only one curve $y = f(x)$ or, if there are two,

$$V = \int_a^b 2\pi r(x)(f_{max}(x) - f_{min}(x))dx$$

As above, if we revolve around the y-axis precisely, then $r(x) = x$, but in general we could have $r(x) = \pm x + a$ if we revolve around an axis merely parallel to the y-axis – draw a picture to figure out what the radius should be as a function of x.

3. Density and mass

In high school chemistry we learn that mass equals volume times density. This is true if the density is constant; in general, the density of a (one, two or three-dimensional) region can vary on the region according to a density function, and to find the mass we will need to integrate the density function over the region. (In general this would be a multiple integral and hence the subject of 21a; here we study special cases where the density function depends only on a single parameter.) For example, if we had a mass given by horizontal slices from a to b and a density function constant on each slice, $\rho(x)$, then an expression for the mass is

$$M = \int_a^b A(x)\rho(x)dx$$

The other example that we saw lots of was when we had shells: either circular, spherical, or cylindrical, and the density depended only on the radius. In all cases, we adjust the appropriate volume integral by throwing in a factor corresponding to the density function, which we think of as "weighting" the integral, giving us a mass. If we had a radially symmetric density function ρ on a disk of radius R we would then get

$$M = \int_0^R 2\pi r\rho(r)dr$$

Or a radially symmetric density function ρ on a ball of radius R would give rise to an expression

$$M = \int_0^R 4\pi r^2\rho(r)dr$$

(In this last formula we have slipped in the formula for the approximate volume of a spherical shell of radius r and thickness Δr, namely $4\pi r^2\Delta r$. As we discussed in class, this is not literally the volume of a spherical shell, but it is asymptotic to the volume as the thickness goes to 0, so it's an acceptable approximation.) Finally, if we have a volume of revolution and a radially symmetric density function (corresponding to finding the volume using cylindrical shells) $\rho(r)$, then an expression for the mass is

$$M = \int_a^b 2\pi r f(r)\rho(r)dr$$

4. Work

In physics we learn that work equals force times distance (to be precise, it is the component of the force that is in the direction of motion; e.g. whirling a ball around on the end of a string does no work; however this subtlety is for the physicists to sort out: for us it will go without saying that the force is in the direction of motion). As for density , this is only literally true if the force is constant; if the force varies along an interval $[a, b]$ then we define

$$W = \int_a^b F(x)dx$$

When you do rope-lifting, chain-lifting, leaky bucket-lifting, snake-lifting, ... you can think of it this way. Partition the distance the thing (rope, leaky bucket,...) is to be lifted and calculate the force (equal and opposite to the weight, generally) that must be applied over that little distance Δx or Δy. Sum up the work over all the little distance intervals and voila!
(In the case of chain or rope lifting, it is possible to cut up the chain or rope instead of the distance. For some calculations this is no harder than partitioning the distance, but in others it IS more difficult - especially if the rope is being pulled a distance less than its length.)

Let's now look at pumping problems. We have a fluid-filled tank in a certain shape (a cone or a sphere often arise), and we want to pump the fluid out over the top. To calculate the work being done, we give ourselves a coordinate axis. Whatever we choose, we stick with it throughout the problem. Your textbook uses y starting at the top of the tank ($y = 0$) and becoming positive as we go down the tank, up to a maximum value of $y = b$. (I prefer the y axis pointing up in the usual way with $y = 0$ at the bottom, but both are correct.) We imagine the fluid in the tank is divided into thin cylindrical cross-sections with cross-sectional area $A(y)$; the work done to move each cylindrical piece of fluid is equal to its weight, which is $\rho A(y)\Delta y$, times the distance that we have to move that layer, which is y if the textbook's axis convention is used. (Here ρ is a constant which gives the weight density of the fluid. If you're given mass density (e.g. kg/m^3, then you must multiply mass by the acceleration due to gravity to get weight: Force = Mass · acceleration.) So we get a Riemann sum of the form $\sum \rho A(y)(\text{distance moved})\Delta\text{y}$. Taking the limit as the number of slices grows without bound, we get an integral. The endpoints of integration depend upon the coordinate system used. (You can check that your limits of integration give you the maximum and minumum distances you're pumping and that you have partitioned the entire volume of liquid to be moved, not more, not less.

5. Integration by substitution

I will remind you how a u-substitution (obtained by integrating the chain rule $(f(g(x)))' = f'(g(x))g'(x)$) works for definite integrals:

$$\int_a^b f(g(x))g'(x)dx = F(g(x))|_a^b$$

In $u = g(x)$ notation, when we do a definite integral instead of doing the corresponding indefinite integral and back-substituting $u = g(x)$ at the end, it comes to the same thing (but is faster) to change the x-limits to u-limits never look back:

$$\int_a^b f(u)u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

While there is nothing wrong with subtituting back for u – so if you have your heart set against the above equation, you don't need to use it – what you should most definitely *not* write is something like

$$(WRONG!) \int_a^b f(u)u'(x)dx = \int_a^b f(u)du = F(u(b)) - F(u(a))$$

This is not the end of the world, because the first expression is equal to the last expression (so your final answer will be correct), but the middle expression entails a confusion of u and x that drives calculus instructors crazy (and may lead to loss of points).

6. Integration by parts

The basic formula is $\int udv = uv - \int vdu$. If there is a definite integral, don't forget to evaluate the uv part between the limits: $\int_a^b udv = uv|_a^b - \int_a^b vdu$.

Like the u-substitution, learning when and how to integrate by parts is a matter of experience, but it is easier to tell you some patterns in this case. The basics are: Don't let dv be something you can't integrate. That taken into account, it is nice if u is something that gets simpler as you differentiate.

For example, when you have $\int x^n f(x)dx$, where $f(x)$ is a function you could find antiderivatives of all the live-long day. Then you take $u = x^n, dv = f(x)dx$, so $du = nx^{n-1}dx, v = F(x)$, and you get

$$\int x^n f(x)dx = x^n F(x) - n\int x^{n-1}F(x)dx$$

Since the power of x has been reduced by 1, progress has been made. Eventually (by taking more and more antiderivatives of f) there will be no power of x at all. So e.g. you can do $\int x^2 e^x dx$ by integrating by parts twice.

Another pattern is that if you see one chunky mess, like $\ln x$ (or even $(\ln x)^n$ or $\arctan x$) in an expression, it is often a good idea to make it be the u function that gets differentiated. An example of this was on the integration practice list: $\int (\ln x)^2$. Take $u = (\ln x)^2, dv = dx$. Then $du = 2(\ln x)\frac{1}{x}dx$ and $v = x$, to get $\int (\ln x)^2 dx = x(\ln x)^2 - \int (2(\ln x)\frac{1}{x})xdx$; the last term works out nicely to be $2\int(\ln x)dx$, which of course you should integrate by parts. The answer is

$$x(\ln x)^2 - 2x\ln x + 2x$$

Beware that multiple integrations-by-parts introduces multiple minus signs.

The last pattern is that sometimes you integrate by parts twice and return, not to where you started, but to *minus* where you started. This is great – moving the desired integral to the other side, you get that twice the integral you're trying to find is the expression you've gotten so far. I'll leave you to look at the classic example of $\int \sin x e^x dx$. CAUTION: This 'method' works on only a handful of examples - most involving a sine or cosine function and an exponential or logarithmic function. Don't get too carried away with it - one sign error and you'll think you've solved almost anything.

7. Integration of rational functions

When $f(x) = \frac{p(x)}{q(x)}$ is the quotient of two polynomials – a rational function – we have a systematic method for finding the antiderivative, in theory at least (in practice, the algebra involved becomes increasingly intricate). There is a three-step process:
Step 1: Make sure that $\frac{p(x)}{q(x)}$ is a proper rational function, i.e. that the degree of the denominator exceeds that of the numerator. That is, if this is not the case, then long-divide the denominator by the numerator to get a polynomial (great!) plus a rational function with denominator $q(x)$. As an example, our function $\frac{x^7}{1+x^3}$ from above is improper, and dividing we get $f(x) = \frac{x^7}{x^3+1} = x^4 - x + \frac{x}{x^3+1}$.

Step 2: Via a partial fractions decomposition, we can split up our integral into a sum of integrals of the form $\int \frac{C}{(x-b)^n}$ and/or $\int \frac{Ax+B}{(x^2+bx+c)^n}$, where in the second case the quadratic function $x^2 + bx + c$ has no real roots (is "irreducible" over the real numbers). For our purposes in the irreducible quadratic case n will always be 1.

We might first convince ourselves that we know how to integrate these two basic expressions. The first one is easy; just make $u = x - b$ and you're all set. As for the second, by completing the square in the denominator – i.e. writing the denominator in the form $e(x - r)^2 + a^2$ (which can be done since there are no real roots), and then making the substitution $u = x - r$, it's enough to worry about how to find $\int \frac{Ax+B}{(x^2+a^2)^n} dx$. (Don't panic about completing the square. We flew over that this semester!) (In practice, you will probably see no worse than $\int \frac{dx}{x^2+a^2}$, and you should remember that this is going to come out as $\frac{1}{a} \arctan \frac{x}{a}$.)

Step 3: Next we have to discuss the form of the partial fractions decomposition, which you need to memorize (sorry!). But there's really not that much to remember: the denominator will factor into a bunch of factors $(x - r_i)^{n_i}$ for various roots r_i (linear factors, possibly occurring with multiplicity greater than 1) together with factors $(ax^2 + bx + c)^{n_i}$ (irreducible quadratic factors, in theory possibly occurring with multiplicity greater than 1, but in practice in this course, non-repeating). For each factor of the form $(x - r_i)^{n_i}$, you write a sum of terms $\frac{C_1}{x-r_i} + \frac{C_2}{(x-r_i)^2} + \ldots + \frac{C_n}{(x-r_i)^n}$; i.e. the numerator is just a constant, even as the multiplicity increases. For each factor of the form $(ax^2 + bx + c)$ you write $\frac{A_1x+B_1}{ax^2+bx+c}$.
Then you must solve for these coefficients, which you do by multiplying through by the LCD, the denominator of the original fraction, and getting an equality of two polynomials. There are several algebraic options: you could set each of the coefficients of the polynomial on the lefthand side equal to each of the coefficients of the polynomial on the righthand side, obtaining a system of linear equations for the unknown A's, B's and C's. (Yes, this can be painfully tedious.) Easier is to leave everything factored. In case there are linear factors $(x - r_i)$ you will save time by plugging in $x = r_i$ on both sides, since every term on the righthand side except one will contain $(x - r_i)$ as a factor. (In the happy state when the denominator is a product of distinct linear factors $(x - r_1) \cdots (x - r_n)$, plugging in x equals each of the roots will very quickly tell you what all the coefficients are.) To solve for the remaining constants, just evaluate at various values of x.

8. Improper integrals

If f is a continuous function on a closed interval $[a, b]$, then $\int_a^b f(x)dx$ always exists. If however, we try to find the area under a curve over an infinite interval $[a, \infty)$ or $(-\infty, -\infty)$ OR if the function f is discontinuous at at least one point on the interval $[a, b]$, then the finiteness of the area is by no means assured. Integrals of these types are called *improper*, to emphasize that there is an (additional) limiting process taking place, and that the limit may or may not exist. As our basic example, we define $\int_a^\infty f(x)dx$ as $\lim_{b\to\infty} \int_a^b f(x)dx$, i.e. as

$\lim_{b\to\infty} F(b) - F(a)$. That is, what is in question is the limiting behavior of the antiderivative as the variable approaches infinity. Recall that there are three essentially different types of limiting behavior:
1) convergence: $\lim_{x\to\infty} F(x)$ exists and is finite.

2) divergence to infinity: $\lim_{x\to\infty} F(x) = +\infty$ or $-\infty$: the former means that for any fixed y-value N, then the values of $F(x)$ will be at least N for all sufficiently large x. (Divergence to $-\infty$ is the same except that the values of $F(X)$ will be less than or equal to $-N$ for all sufficiently large x.)

Important: If $f(x) \geq 0$ for all x, then the area function $F(x)$ is increasing (why?). If we take the limit of an increasing function at ∞ either 1) or 2) above must occur: the only question is whether the function remains bounded, i.e. whether there exists a fixed constant M such that $F(x) \leq M$ for all x. If $F(x)$ remains bounded, it converges; otherwise it diverges to ∞. This makes improper integrals of positive functions much nicer to deal with; in particular the following cannot occur:

3) Divergence due to oscillation: This means that neither 1) nor 2) above occurs: the values of $F(x)$ do not settle down to any fixed value nor do they get arbitrarily large.

Principle of comparison: Say $0 \leq f(x) \leq g(x)$ are functions defined on $[a, \infty)$. Geometrically we see that the region under the smaller function $f(x)$ is contained in the region under the larger function $g(x)$, so certainly the area under $f(x)$ should be less than or equal to the area under $g(x)$ (and indeed, if f and g are continuous, then the area under f will be strictly less than the area under g unless $f = g$ the whole time). That means, that just as for "proper integrals" we have $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$. Keeping in mind that these improper integrals either converge or are infinite, then (under the reasonable convention that any real number is less than $+\infty$) we can make sense of this inequality even when one or both terms are infinite. We get the following conclusions:
a) If $\int_a^\infty g(x)dx < \infty$, then $\int_a^\infty f(x)dx < \infty$.
b) If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges .

As an example, we know that $\int_1^\infty e^{-x^{100}}dx$ converges, because for all $x \geq 1$, $x \leq x^{100}$, so $e^x \leq e^{x^{100}}$ and taking reciprocals reverses the inequality: $e^{-x} \geq e^{-x^{100}}$. Since we can compute $\int_1^\infty e^{-x}dx = \frac{1}{e}$, we can conclude by comparison that $\int_1^\infty e^{-x^{100}}dx$ converges, and indeed that it is less than $\frac{1}{e}$.

8.1. **Integrals on** $(-\infty, \infty)$**.** If $f(x)$ is defined on $(-\infty, \infty)$ (i.e. on the entire real line), then we can still ask for the area under the curve, but we have to be careful how we define it. The definition we choose is to split up the integral as follows: choose any number a; then

$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

That is, we require *both* of these integrals to converge separately; in particular if one of them is $+\infty$ and the other is $-\infty$, then we regard this as doubly bad – the integral is divergent for "two reasons". (In class we examined $\int_{-\infty}^{\infty} xdx$ and discussed whether or not the answer should be 0; now, while studying for the exam, we just reiterate that this is our definition of improper integral.)

8.2. **Discontinuities at a point.** If $f(x)$ is discontinuous at a point c on the interval $[a, b]$ (c could be an endpoint or a point in the interior of the interval) then again the well-definedness of the area under the curve is not assured. Say for example that $f(x)$ is defined on $[0, 1]$ and is continuous except at 0 – e.g. $\frac{1}{x^p}$ for any $p > 0$ fits the bill. Then $\int_0^1 f(x)dx$ is *improper*, and by this symbol we mean $\lim_{t\to 0^+} \int_t^1 f(x)dx$. As a very important example, recall that we found in class that $\int_0^1 \frac{dx}{x^p}$ was convergent (finite) exactly when $p < 1$. If the discontinuity is in the interior of the interval of definition, then we need to split up the integral and take limits as we approach the bad point both on the left and on the right.

As a fair warning, this is the point in the semester when we sometimes try to sneak in integrals that are improper because the function is discontinuous somewhere on the integral to see whether or not you'll notice this. (We can be pretty mean sometimes.) For example, consider $\int_{-1}^1 \frac{dx}{x}$. This is an odd function on an interval symmetric about 0, so right away the integral is 0 right? Wrong! The function $f(x)$ is discontinuous at 0, and if we approach 0 from either direction we build up infinite area. (Notice that the fact that we have the "same infinite regions" on the left and on the right, one building up infinite positive area and the other building up infinite negative area, is not enough to get us off the hook; BY DEFINITION we require both of these improper integrals to converge.)

9. Probability

A probability density function $f(x)$ is by definition a function defined on $(-\infty, \infty)$ which is continuous, non-negative and such that $\int_{-\infty}^{\infty} f(x)dx$ (converges and) equals 1. (This also suggests that if we had a non-negative function f such that $\int_{-\infty}^{\infty} f(x)dx = C$ (finite!), then we can "renormalize" and get $\frac{1}{C}f(x)$ as a probability density function.)

We then define the probability that $a \leq x \leq b$ to be $\int_a^b f(x)dx$.

Note: We use the probability density function to determine the probability that x is in an interval of positive length. In particular, we don't use it to find the probability, say, of a randomly chosen variable to be 5. The probability of x being greater than 5 plus the probability of x being less than 5 should add up to 1.
Good luck on the exam!