Integration Handout A

Looking at Integration from a Graphical Perspective

1. This set of problems encourages you to make estimations, use symmetry, and generally to take a graphical look at definite integrals. For each of the following claims use an appropriate graph to evaluate the truth or falsehood of each claim. (Feel free to use a graphing calculator or computer to produce graphs for you.) Your answers should include pictures as well as your line of reasoning.

   - Claim 1: $0 < \int_0^a e^{-x^2}\,dx < a$
   - Claim 2: $\int_0^{\sqrt{\pi}} \sin(x^2)\,dx < 0$
   - Claim 3: $\int_{-\pi}^{\pi} e^{-x^2/\sqrt{2}}\,dx = 2\int_0^{\pi} e^{-x^2/\sqrt{2}}\,dx$
   - Claim 4: $\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx < \int_1^2 \frac{1}{\sqrt{1+x^4}}\,dx$
   - Claim 5: $\int_{-3}^{3} \frac{x}{1+x^4}\,dx > 0.001$
   - Claim 6: The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is greater than $2ab$ and less than $4ab$.
   - Claim 7: If $f(x)$ is continuous for all x then $\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$.

   On the time interval $[a, b]$ a car's velocity, $v(t)$, is positive and increasing. The velocity is increasing at a decreasing rate on this interval. Suppose we partition the interval $[a.b]$ into 10 equal subintervals, each of length Δt. Let $t_k = a + k\Delta t$ where $k = 0, 1, 2, \ldots, 10$. Your job is to make sense out of the Riemann Sums in the next three claims and use pictures to figure out the veracity of the claims.

   - Claim 8: $\sum_{k=1}^{10} v(t_{k-1})\,\Delta t >$ the distance traveled on $[a, b]$.
   - Claim 9: $\sum_{k=1}^{10} v(t_k)\,\Delta t >$ the distance traveled on $[a, b]$.
   - Claim 10: $\frac{1}{2}\left[\sum_{k=1}^{10} v(t_k)\,\Delta t + \sum_{k=1}^{10} v(t_{k-1})\,\Delta t\right] <$ the distance traveled on $[a, b]$.

Methods of Integration

2. Substitution can be used to do the integral $\int \sin^4 x \cos x\,dx$. (Let $u = \sin x$.) Similarly, substitution can be used to do the integrals below, but you must first prepare the integral using the trigonometric identity $\sin^2 x + \cos^2 x = 1$. This identity lets you convert even powers of $\sin x$ to $\cos x$ and vice-versa. For instance, in problem (a) you can express the integrand as $\sin^2 x \cos^2 x \cos x$, convert the $\cos^2 x$ to $(1 - \sin^2 x)$ and integrate using the substitution $u = \sin x$. (Notice that it would not work to convert the $\sin^2 x$ to $(1-\cos^2 x)$ in this problem since that would make the integrand into powers of $\cos x$ but you would not have a $\sin x\,dx$ to serve as the du.)

   (a) $\int \sin^2 x \cos^3 x\,dx$

   (b) $\int \sin^5 x \cos^4 x\,dx$

3. Find $\int \sin^2 \theta\,d\theta$.

   Hint: The identities

   $$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta) \quad \text{and} \quad \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

   can be useful in integrating $\int \sin^2\theta\,d\theta$ and $\int \cos^2\theta\,d\theta$. (If you forget these formulas you can use parts to do the integrals. You might want to check this out on your own.)

4. Show that $\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}9\pi$ - in other words, show analytically that the area of a circle of radius 3 is 9π by doing the following:

   We'd like to eliminate $\sqrt{9-x^2}$ by making a substitution that makes the integrand a perfect square. We will exploit the trig identity $\sin^2 t + \cos^2 t = 1$, or, equivalently, $9\sin^2 t + 9\cos^2 t = 9$ We know that $9 - 9\sin^2 t$ is a perfect square, so we'll use the substitution $x = 3\sin t$. Now we need to write the entire integral in terms of t.

   a) If $x = 3\sin t$ then what is dx in terms of t and dt?
   b) If $x = 3\sin t$ then what is $\sqrt{9-x^2}$ in terms of t?
      (Notice that what's inside the square root is now a perfect square so the square root can be eliminated.)
   c) If $x = 3\sin t$ then what are the new endpoints of integration in terms of t?
   d) Write the integral in terms of t.
   e) Evaluate the integral in (d).
   f) Conclude that the area of a circle of radius 3 is 9π.

5. In the following set of integrals your job is to determine which method of integration - substitution, parts, or partial fractions - is the simplest to use in order to evaluate the integral.

   If you answer 'substitution', indicate the substitution you would use.

   If you answer 'parts', indicate 'u' and 'dv'.

   If you answer 'partial fractions', set up the partial fractions decompostion (without solving for the constants).

   You need not complete the integrals unless you want to just for practice.

   (a) $\int x \cos x\,dx$
   (b) $\int \cos x \sin^2 x\,dx$
   (c) $\int \frac{x}{x^2-4x-5}\,dx$
   (d) $\int \frac{x-2}{x^2-4x+5}\,dx$
   (e) $\int \frac{\ln x}{x}\,dx$
   (f) $\int \ln x\,dx$

6. Evaluate the integrals below. All can be done by trigonometric substitution, but only one requires trigonometric substitution; the others do not.

   (a) $\int \frac{x}{\sqrt{4+x^2}}\,dx$
   (b) $\int_0^1 \sqrt{4-t^2}\,dt$
   (c) $\int_0^1 x^3\sqrt{4-x^2}\,dx$
   (d) $\int_0^1 \frac{x^3}{\sqrt{9-x^2}}\,dx$

7. Show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are positive, is given by πab. (Recall that in a previous homework you've already shown that the area is between $2ab$ and $4ab$.

Applications of Integration

8. Write an integral (or sum of difference of integrals) giving the area of the region bounded above by the graph of $y = -x^2 + 2$ and below by the graph of $y = x$. (You need not evaluate.)

9. Find the area in the first quadrant bounded by $y = \arcsin x$, $y = \pi/2$, and $x = 0$.

   Hint: To get an exact answer it will be simplest to integrate with respect to y.

10. The following definite integrals can be computed exactly without knowing the antiderivative of $\arctan x$. The point of this problem is to interpret the definite integral given as the area under a curve and then to either use the symmetry of $\arctan x$ to evaluate the definite integral or chop the area into horizontal strips to arrive at different definite integral that is easy to evaluate.

    (a) $\int_{-2}^{2} \arctan x\, dx$

    (b) $\int_{0}^{1} \arctan x\, dx$

11. Write an integral that gives the volume generated by revolving the region bounded by $y = x^2$ and $y = 4$ about

    (a) the $y-$axis
    (b) the vertical line $x = 2$
    (c) the horizontal line $y = 4$
    (d) the horizontal line $y = -1$

    You need not evaluate these integrals.

12. A parfait cup is formed by revolving the curve $y = x^3, 0 \leq x \leq 2$, about the $y-$axis. The parfait cup is filled to the brim with hot chocolate. If you plan to drink exactly half the hot chocolate in the cup, what height should the liquid be when you stop drinking?

Feeling Blue, Looking Bluer

Spraying a piece of pottery with cobalt will result in a blue color when the piece is fired. The shade of blue is determined by the density of cobalt; the greater the density of the cobalt application, the darker the blue of the pot. You can get gradations of blue by applying cobalt glaze with a spray gun and varying the density of the application. Makoto and Wasma are professional potters at the Radcliffe Pottery Studio on Western Avenue and the next four problems are about glazing pieces of pottery shades of blue.

13. Makoto has made a rectangular sushi platter from a slab of clay 14 inches by 6 inches. He applies cobalt such that the density of the application increases with the distance from one of the long sides of the platter. The density of cobalt glaze is given by $\rho(x)$ mg/square inch where x is the distance (in inches) from one long side of the sushi platter.

    a) How can you approximate the amount of cobalt Makoto used?
    b) Give an expression in terms of $\rho(x)$ that gives the amount of cobalt used.

14. Makoto decides to try a more symmetric glaze application on his next sushi platter. The platter is again 14 inches by 6 inches. This time the deepest blue is in a stripe along the long center line of the platter and the intensity of the blue fades with the distance from this central line. The density of cobalt glaze is given by $\rho(x)$ mg/square inch where x is the distance (in inches) from the longitudinal center of the sushi platter.

    a) How can you approximate the amount of cobalt Makoto used?
    b) Give an expression in terms of $\rho(x)$ that gives the amount of cobalt used.

15. Wasma is glazing a large round plates 16 inches in diameter. For one plate she decides to have a deep blue center fading out into pale blue along the rim. She applies cobalt glaze such that its density is given by $\rho(x)$ mg/square inch where x is the distance (in inches) from the center of the plate.

    a) How can we approximate the amount of cobalt Wasma used?
    b) Write an expression in terms of $\rho(x)$ that gives the amount of cobalt used.

16. For her next round plate (16 inches in diameter) Wasma decides to have a deep blue line 16 inches long running through the center of the plate. She has the shade of blue fade into paler and paler blue as the distance from this deep blue line increases. She applies cobalt glaze such that its density is given by $\rho(x)$ mg/square inch where x is the distance (in inches) from dark blue diameter of the plate. Write an expression in terms of $\rho(x)$ that gives the amount of cobalt used.

17. The density of dart holes on an old dartboard is given by $\rho(x) = \frac{1010}{\pi(x^2+1)}$ holes per square inch, where x is the distance, in inches, from the center of the board. If the board is a circle with diameter 20 inches, find the total number of holes in the board.

18. Given a disk of radius R, suppose you partition the interval $[0, R]$ into n equal pieces, each of length $\Delta x = \frac{R}{n}$

    Typically in our work we have let $x_k = k \cdot \Delta x$ so $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2 \cdot \Delta x$, $\cdots$ and $x_n = n \cdot \Delta x = R$ and then approximated the area of the kth ring by $2\pi x \Delta x$. There are several ways to justify the validity of using that approximation in our work. This problem asks you to work through one of them.

    Let x_k^* be the midpoint of the kth interval. Then the left and right hand endpoints of the kth interval can be written as $x_k^* - \frac{1}{2}\Delta x$ and $x_k^* + \frac{1}{2}\Delta x$ respectively. Show that the area of the kth ring (computed using $x_k^* - \frac{1}{2}\Delta x$ as the inner radius and $x_k^* + \frac{1}{2}\Delta x$ as the outer radius) is exactly $2\pi x_k^* \Delta x$. Conclude that the approximation we have been using is a valid one.

19. The density of a ball of ice is greatest at the center and decreases with the distance from the center of the ball. The ball is 10 centimeters in radius and the density is given by $\rho(x)$ grams per cubic centimeter. What is the mass of the ball?

20. A chocolate truffle is a wonderfully decadent chocolate concoction. Truffles tend to be spherical or hemispherical.

    (a) Consider a truffle made by dipping a round hazelnut into various chocolates, building up a delicious chocolate delicacy. The number of calories per cubic millimeter varies with x, the distance from the center of the hazelnut. If $\rho(x)$ gives the calories per cubic millimeter at a distance x millimeters from the center, write an integral that gives the number of calories in a truffle of radius R.

    (b) Another truffle is made in a hemispherical mold of radius R. (The mold looks like a tiny hemispherical bowl.) Different layers of chocolate are poured into the mold, one at a time, and allowed to set. The number of calories per cubic millimeter varies with x, the distance from the top of the mold. The caloric density is given by $\delta(x)$ calories per cubic millimeter. Write an integral that gives the number of calories in this hemispherical truffle.

21. In the town of Lybonrehc there has been a nuclear reactor meltdown which released radioactive iodine 131. Fortunately, the reactor has a containment building which kept the iodine from being released into the air. The containment building is hemispherical with a radius of 100 feet. The density of iodine ina the building was 6 x $10^{-5}(200 - h)$ g/cubic feet, where h is the height from the floor (in feet). (It ranges from 12 x 10^{-3} g/cubic feet at the floor to 6 x 10^{-3} g/cubic feet by the top.)

    (a) Derive an integral that gives the amount of iodine in the building. Explain your reasoning fully and clearly.

    (b) Calculate the amount of iodine in the building.

22. Rocket fuel is stored in a hemispherical tank of radius 5 m. The base of the tank is the disk of radius 5 m. Rocket fuel has a density of 100 kg/m^3. How much work is done if the full tank is emptied out via a pipe located 2 m above the top of the tank? (In other words, the fuel must be pumped 2 meters higher than the top of the dome of the tank.)

23. Between December and July the Serengeti in Tanzania is the scene of a mass animal migration as over 1 million wildebeest, 200,000 zebra and 300,000 Thomson's gazelle journey across the plains in search of new grazing lands and water. Suppose $f(x)$ gives the rate at which zebra are entering/leaving the Seronera region of the Serengeti, where $f(x)$ is given in tens of thousands of zebra per month and $t = 0$ corresponds to January. How can we interpret $\int_{-1}^{1} f(x)\,dx$?

24. Compare the average values of the following functions on the interval $[-1, 1]$. Which is largest? Smallest? Try to do this problem by graphing each of the functions below on the interval $[-1, 1]$ and solving by visual inspection.

    (a) $f(x) = \sqrt{1 - x^2}$

    (b) $g(x) = -|x| + 1$

(c) $h(x) = e^{-|x|}$

25. Let f be a continuous function on $[a, b]$. We are interested in comparing the average value, f_{ave}, of f to the value $f(\frac{a+b}{2})$ of f at the midpoint of the interval.

   (a) Assume $f''(x) = 0$ for all x on $[a, b]$. Show that $f_{\text{ave}} = f(\frac{a+b}{2})$. [Hint: what kind of function has second-derivative always equal to zero?]

   (b) Draw the rectangle on the interval $[a, b]$ whose height is $f(\frac{a+b}{2})$, the value of the function on the midoint. Now draw the trapezoid bounded by $x = a$, $x = b$, the $x-$ axis, and the line tangent to the curve $f(x)$ at $x = \frac{a+b}{2}$. Show that the area of the rectangle and of the trapzoid are the same. (Need help? See the supplement pp. 809-810).

   (c) Assume $f''(x) > 0$ for all x on $[a, b]$, i.e. f is concave up. Show that $f_{\text{ave}} > f(\frac{a+b}{2})$. Draw a picture to illustrate your reasoning.

   (d) Assume $f''(x) < 0$ for all x on $[a, b]$, i.e. f is concave down. Show that $f_{\text{ave}} < f(\frac{a+b}{2})$. Draw a picture to illustrate your reasoning.

26. For what values of p does the integral $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ converge? For what values of p does it diverge?

   (Let p run though all the real numbers, not just the integers! You'll have to break your work up into cases.)