Differential Equations Handout A

1. For each of the differential equations below do the following:

   - Sketch the slope field in the $t-x-$ plane (t on the horizontal axis and x on the vertical) and, on the same set of axes, sketch representatives of the family of solutions. Check your answer using the applet labelled "differential equations applet: dfield" and found at the bottom of the supplement page of the website.
   - Guess the general solution to the differential equation and check your answer.
   - Find solutions corresponding to the initial conditions $x(0) = 1$, $x(0) = 0$ and $x(0) = -1$. Will solutions corresponding to different initial conditions intersect?

   (a) $\dfrac{dx}{dt} = -2t$

   (b) $\dfrac{dx}{dt} = -2x$

   (c) $\dfrac{dx}{dt} = t^2$

2. Which of the following is a solution to the first order [1] differential equation

$$\frac{dy}{dt} = y + 1?$$

   (a) $y = Ce^t$ (b) $y = Ce^t - t$ (c) $y = C(t^2 + t)$
   (d) $y = Ce^t - 1$ (e) $y = Ce^{-t} + 1$

3. Which of the following is a solution to the second order [2] differential equation

$$y'' + 9y = 0?$$

   (a) $y = e^{3t} + e^{-3t}$ (b) $y = Ce^t - t$ (c) $y = C(t^2 + t)$
   (d) $y = \sin 3t + 6$ (e) $y = 5\cos 3t$

4. Solutes in the bloodstream enter cells through osmosis, the diffusion of fluid through a semipermeable membrane. Let $C = C(t)$ be the concentration of a certain solute inside a particular cell. The rate at which the concentration inside the cell is changing is proportional to the difference in the concentration of the solute in the bloodstream and the concentration within the cell. Suppose the concentration of a solute in the bloodstream is maintained at a constant level of L gm/cubic cm.

   Write a differential equation involving dC/dt. Your differential equation will involve an undetermined proportionality constant. Specify the sign of that constant.

[1] A first order differential equation involves a first derivative but no higher derivative.
[2] A second order differential equation involves a second derivative but no higher derivative.

5. In this problems you'll construct a model for the spread of contagious disease. Let N denote the total population affected by the epidemic. Make the following assumptions:

   - The size of the population,N, is fixed throughout the time period we are considering.
   - Everyone in the population is susceptible to the disease.
   - The disease is non-fatal but is long in duration, so there are no recoveries during the time period we are modeling. When a person is infected that individual is sick.
   - The rate that people are becoming infected is proportional to the product of infected people and healthy people (since there must be some interaction between the two in order to pass along the disease.)

   (a) Let $I = I(t)$ denote the number of infected people at time t. Write a differential equation involving dI/dt. Your equation will involve a proportionality constant. Determine its sign.

   (b) *Think about it:* Look at the differential equation you wrote in part (a). Is the number of infected people increasing with time, decreasing with time, or sometimes increasing, sometimes decreasing? In the long run, how many people does this model imply stay healthy? Now use the differential equation applet "dfield" to check your answer.

6. In applying differential equations to monetary problems we must make a a continuous model of a discrete phenomenon. We do this in the following two problems.

   Money is deposited in a bank account with a nominal annual interest rate of 4% compounded continuously. This means that the account is set up so that the instantaneous rate of increase of the balance due to interest is 4% of the amount of money in the account at that moment. The initial deposit of $2000 is made at a time we designate as $t = 0$.

   Let $M = M(t)$ be the amount of money in the bank account at time t.

   (a) Assume there are no additional deposits or withdrawals. Write a differential equation involving dM/dt and give the initial condition.

   (b) Solve the differential equation in part (a). Use the initial condition.

   (c) Suppose money is being added to the account continuously at a rate of $1,000 per year and no withdrawals are made. Write a differential equation involving dM/dt and give the initial condition.

   (d) Solve the differential equation in (c) using the initial condition.

7. Elmer takes out a $100,000 loan for a house. He pays money back at a rate of $12,000 per year. The bank charges him interest at a rate of 7.25% per year compounded continuously. Make a continuous model of his economic situation. Use a differential equation involving dB/dt where $B = B(t)$ is the balance he owes the bank at time t.

8. When a population has unlimited resources and is free from disease and strife, the rate at which the population grows is often modeled as being proportional to the population. Assume that both the bee and the mosquito populations described below behave according to this model. We'll make continuous models even though population numbers are discrete.

   In both the following scenarios you are given enough information to find the poportionality constant k. In one case the information allows you to find k solely using the differential equation, without requiring that you solve it. In the other scenario you must actually solve the differential equation in order to find k.

   (a) Let $M = M(t)$ be the mosquito population at time t, t in weeks. At $t = 0$ there are 1000 mosquitoes. Suppose that when there are 5000 mosquitoes the population is growing at a rate of 250 mosquitoes per week. Write a differential equation reflecting the situation. Include a value for k, the proportionality constant.

   (b) Let $B = B(t)$ be the bee population at time t, t in weeks. At $t = 0$ there are 600 bees. When $t = 10$ there are 800 bees. Write a differential equation reflecting the situation. Include a value for k, the proportionality constant.

9. The population in a certain country grows at a rate proportional to the population at time t, with a proportionality constant of 0.03. Due to political turmoil, people are leaving the country at a constant rate of 6000 people per year. Assume that there is no immigration to the country. Let $P = P(t)$ be the population at time t, where t is in years. We'll make a continuous model of the population.

   (a) Write a differential equation reflecting the situation.

   (b) Solve the differential equation for $P(t)$ given the information that at time $t = 0$ there are 3 million people in the country. In other words, find $P(t)$, the number of people in the country at time t.

10. Suppose y satisfies the differential equation $\frac{dy}{dx} = y^2$.

    (a) Describe the behavior of y assuming each one of the following initial conditions

    (i) $y(0) = 0$, (ii) $y(0) = 0.01$ (iii) $y(0) = -0.01$.

    (b) What kind of equilibrium (stable versus unstable) is the solution you found in 4.a(i)?

    (c) Using separation of variables, explicitly solve the differential equation $\frac{dy}{dx} = y^2$ with initial condition $y(0) = 1$.

    (d) What happens to y as x goes to 1?

Note that from the work you did in this problem you can deduce that the solution in (c) not only increases without bound, but increases without bound in finite time! A similar thing happens with the differential equation $\frac{dy}{dx} = (y-1)(y-3)$, the equation that is Example 31.5 on p. 1004 of the Supplement under appropriate initial conditions.

11. A canister contains 10 liters of blue paint. Paint is being used at a rate of 2 liters per hour and the canister is being replenished by pale blue paint that is 80% blue paint and 20% percent white paint. Assume the canister is well-mixed and that paint is both entering and leaving the canister continuously.

    a) Write a differential equation whose solution is $w(t)$, the amount of white paint in the canister at time t. Specify the initial condition. Use qualitative analysis (and common sense) to sketch the solution to the differential equation.

    b) Write a differential equation whose solution is $b(t)$, the amount of blue paint in the canister at time t. Specify the initial condition. Use qualitat ive analysis (and common sense) to sketch the solution to the differential equation.

12. *Getting information from a differential equation:* Suppose that $y = f(x)$ is a solution to the differential equation

$$y'' + y' = -x^2.$$

    Why can't the the graph of $f(x)$ ever be both increasing and concave up?

    Hint: This information is essentially available 'by observation.' If y is both increasing and positive what does that say about y' and y''? With that in mind, look at the differential equation and see why such a function could not possibly satisfy $y'' + y' = -x^2$.

13. Let $x = x(t)$ be the number of thousands of beasts of species X at time t. Let $y = y(t)$ be the number of thousands of beasts of species Y at time t. Suppose that

$$\frac{dx}{dt} = .1x - .05xy$$

$$\frac{dy}{dt} = .1y - .05xy$$

    a) Is the interaction between species X and Y symbiotic, competitive, or a predator/prey relationship?

    b) What are the equilibrium populations?

    c) Find the nullclines and draw directed horizontal and vertical tangents in the phase plane. (Consult the supplement for illustrations.)

d) The nullclines divide the first quadrant of the phase plane into four regions. In ease region, determine the general directions of the trajectories.

e) If $x = 0$, what happens to $y(t)$? How is this indicated in the phase plane?
If $y = 0$, what happens to $x(t)$? How is this indicated in the phase plane?

f) Use the information gathered in parts (b)-(e) to sketch representative trajectories in the phase plane. Include arrows indicating the direction in which the trajectories are travelled.

g) For each of the initial conditions given below, describe how the size of the populations of X and Y change with time and what the situations will look li ke in the long run.

i. $x(0) = 2$ $y(0) = 1.8$
ii. $x(0) = 2$ $y(0) = 2.3$
iii. $x(0) = 2.2$ $y(0) = 2$

h) Does this particular model support or challenge Charles Darwin's Principle of Competitive Exclusion?

14. Consider the following model for the population levels of predators and prey, in a given environment at time t.

$$\frac{dx}{dt} = ax - bx^2 - cxy$$

$$\frac{dy}{dt} = -dy + exy$$

All the parameters a, b, c, d, e are positive constants.

(a) Which of x and y are the predators (and prey)? Explain please.

(b) If you started of with some prey and no predators, what do you expect in the long run?

Suppose, on the other hand, you start with some predators and no prey. What happens in the long run?

(c) Show that there is an equilibrium for the system at the point $(x, y) = \left(\frac{d}{e}, \frac{a}{c} - \frac{bd}{ce}\right)$.

(d) Give some interpretation to the parameters a, b, c, d, and e.

15. Consider the following model for the populations of two species in competition in a given environment at time t (say,.. mice and rats).

$$\frac{dx}{dt} = x - x^2 - axy$$

$$\frac{dy}{dt} = y - y^2 - axy$$

The parameter a is a positive constant.

(a) If you started out with some mice, no rats, what would you expect to see in the long run? (What about some rats, no mice)

(b) Sketch the phase portrait (label the equilibrium points and null clines) of the system for the cases $a = 1/2$ and $a = 3/2$ and note the difference in the behavior of the system. Please describe the difference between the cases $a < 1$ and $a > 1$.

16. Consider a population A of aphids and L of ladybugs in a large broccoli patch.

(a) If the aphids are alone, they grow according to the differential equation

$$\frac{dA}{dt} = A(100,000 - A).$$

Suppose we start with 100 aphids. What happens in the long run? Sketch a rough graph showing A as a function of time. (You need not put units on the $t$axis, but you should label important value(s) on the A axis.)

(b) If the ladybugs are alone, L satisfies

$$\frac{dL}{dt} = L(-1000 - L).$$

Suppose we start with 100 ladybugs. What happens in the long run? Sketch a rough graph showing L as a function of time. (You need not put units on the $t$axis, but you should label important value(s) on the L axis.)

(c) When the aphids and ladybugs are together, the populations satisfy

$$\frac{dA}{dt} = A(100,000 - A - 50L).$$

$$\frac{dL}{dt} = L(-1000 - L + \frac{1}{25}A).$$

Do a qualitative phase plane analysis for these differential equations showing the equilibrium point(s), the null clines, and the direction of the trajectories in each region.

Note: In fact, the trajectories spiral in. Make sure your results are consistent with this.

17. Use series to solve the differential equation $y' = y$ with the initial condition $y(0) = 1$. Do you recognize this as a familiar function?

18. Use series to solve the differential equation $y'' = -y$. Can you express your answer in terms of familiar functions?

19. Solve the following differential equations for $y(t)$. Give the general solution.

    a) $y'' + 6y' = 7y$
    b) $y'' + 6y' + 9y = 0$
    c) $y'' + 5y' + 6y = 0$

20. For each of the differential equations in the problem above, suppose that the initial conditions are $y(0) = -2$ and $y'(0) = 0$.

    (i) Use the initial conditions to find $y(t)$.
    (ii) Find $\lim_{t\to\infty} y(t)$.

21. Interpret $x(t)$ as the position of a mass on a spring at time t where $x(t)$ satisfies
$$x'' + 4x' + 3x = 0.$$
Suppose the mass is pulled out, stretching the spring one unit from its equilibrium position, and given an initial velocity of $+2$ units per second.

    (a) Find the position of the mass at time t.
    (b) Determine whether or not the mass ever crosses the equilibrium position of $x = 0$.
    (c) When (at what time) is the mass furthest from its equilibrium position? Approximately how far from the equilibrium position does it get?

22. (a) Suppose that $x(t) = C_1e^{at} + C_2e^{bt}$. Show that $x(t) = 0$ at most once. Find the value of t for which $x(t) = 0$ if such a value exists.
    (b) Suppose that $x(t) = C_1e^{at} + C_2te^{at}$. Show that $x(t) = 0$ at most once. Find the value of t for which $x(t) = 0$ if such a value exists.
    (c) Conclude from parts (a) and (b) that if the characteristic equation of $x'' + bx' + cx = 0$ has either one real root or two real roots then the differential equation cannot model a mass at the end of a spring in the scenario that the mass oscillates back and forth around the equilibrium point.

23. Let's try to make sense of the expression $e^{(a+bi)t}$, that is, e raised to a complex number $a + bi$ where $i = \sqrt{-1}$. To do this, first observe that $e^{(a+bi)t} = e^{at} \cdot e^{bit}$, where a and b are real numbers. The part we must make sense of is e^{bit}. Use the Taylor Series for e^x about $x = 0$ to expand e^{bti}. Gather all terms with i and all terms without i. (Factor out the i from the terms with i.) Now rewrite e^{bit} in terms of familiar functions.

    Given your work above, what's $e^{\pi i}$?

24. Solve the following differential equations for $y(x)$.

    (a) $y'' - 9y' = 0$
    (b) $y'' - 9y = 0$
    (c) $y'' + 9y = 0$
    (d) $y'' - 9 = 0$
    (e) $y'' - 2y' - y = 0$
    (f) $y'' - 2y' + 2y = 0$

25. Suppose that $x'' + bx' + cx = 0$ is used to model the position of a block at the end of a vibrating spring.

    (a) What can you say about the signs of b and c? Explain.
    (b) As long as friction plays a role, we expect that regardless of the initial conditions $\lim_{t\to\infty} x(t) = 0$. Explain how your answer to part (a) guarantees this.

        Hint: it is necessary to do three different cases.

26. Write a differential equation of the form $x'' + bx' + cx = 0$ such that if $x(0) = 1$ and $x'(0) = 2$ then $x(t)$ has the property that

    (a) $\lim_{t\to\infty} x(t) = 0$
    (b) $\lim_{t\to\infty} x(t) = \infty$
    (c) $\lim_{t\to\infty} x(t)$ does not exist.

    Note: there are not unique answers to these problems!

27. *Extra Credit:* The goal of what we did in class was to familiarize you with the idea behind Euler's method as opposed to teaching you to use Eueler's method. For extra credit you can do a simple problem using Euler's method - given below. (Euler's method in the textbook in section 7.2 and on page 1022 in the supplement but you can do this problem without the formulas from Stewart.)

    Consider the differential equation

    $$\frac{dy}{dx} = y - x + 1.$$

    Suppose $y(0) = 1$.

    (a) What is the slope of the solution curve passing through (0, 1) at the point (0,1)?
    (b) Use Euler's method to estimate $y(1)$ in one step.
    (c) Same as (b), but now estimate $y(1)$ in two steps, by first estimating $y(0.5)$.
    (d) Verify that $y = e^x + x$ is a solution to the differential equation that satisfies the initial condition $y(0) = 1$, and compare the true value of y at $x = 1$ to the values you got in parts (b) and (c).