SOLUTION

●. Let $S = [k]$ for some positive integer k. Define P on S such that:

$$P(\{\omega\}) = \frac{\omega}{\alpha}$$

For every $\omega \in S$ for some α. If S is a finite probability space with P. find α.

By property (b) and (c),

$$\sum_{i=1}^{k} P(\{i\}) = 1 \Rightarrow \sum_{i=1}^{k} \frac{i}{\alpha} = 1$$

$$\Rightarrow \frac{1}{\alpha} \sum_{i=1}^{k} i = 1$$

$$\Rightarrow \sum_{i=1}^{k} i = \alpha \qquad \boxed{\alpha = \frac{k(k+1)}{2}}$$

●. Prove that if P is a probability function and A and B are events,

$$A \subseteq B \Rightarrow P(A) \leq P(B)$$

If $A = B$, then $P(A) = P(B)$.

Otherwise, let $C = B - A$. By axiom (a), $0 \leq P(C) \leq 1$.

Since A and C are disjoint, by axiom (c),

$$P(B) = P(A \cup C) = P(A) + P(C) \geq P(A) \checkmark$$

2. What is the expected number of heads in n tosses of a fair coin?

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ coin is head} \\ 0 & \text{else} \end{cases}$$

Then $X = \underset{\text{in } n \text{ tosses}}{\text{number of heads}} = \sum_{i=1}^{n} X_i$

So $E[X] = \underbrace{\sum_{i=1}^{n} E[X_i]}_{\uparrow \text{ by linearity}} = \sum_{i=1}^{n} P(X_i = 1) = \sum_{i=1}^{n} \left(\frac{1}{2}\right) = \boxed{\frac{n}{2}}$ ✓

3. Use it to prove the following:

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

by definition of expected value

$$\frac{n}{2} = E[X] = \sum_{k=0}^{\infty} k\, P(X = k)$$

since $P(X > n) = 0$

$$\frac{n}{2} = \sum_{k=0}^{n} k\, P(X = k)$$

each w/ prob $\frac{1}{2}$

choose coins that's head

multiply both by 2^n

$$\frac{n}{2} = \sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{2}\right)^n$$

$$(2)^n \left(\frac{n}{2}\right) = \sum_{k=0}^{n} k \binom{n}{k}$$

$$(2^{n-1})(n) = \sum_{k=0}^{n} k \binom{n}{k}$$ ✓