Complete the Square

. Prove the following for all real numbers x, y such that $x \geq y$,

$$2x^2 + y^2 \geq 2xy + x + y - 1$$

$$\Leftrightarrow x^2 + x^2 + y^2 \geq 2xy + x + y - 1$$

$$\Leftrightarrow x^2 - 2xy + y^2 + x^2 - x - y + 1 \geq 0$$

$$\Leftrightarrow (x-y)^2 + x^2 - x - y + 1 \geq 0$$

$$\Leftrightarrow (x-y)^2 + x^2 - x - x + x - y + 1 \geq 0$$

$$\Leftrightarrow (x-y)^2 + x^2 - 2x + 1 + x - y \geq 0$$

$$\Leftrightarrow (x-y)^2 + (x-1)^2 + (x-y) \geq 0$$

start with

We know $(x-y)^2 \geq 0$,

$(x-1)^2 \geq 0$

and since $x \geq y$, $x - y \geq 0$ ✓

So sum of nonnegatives is nonnegative.

Induction

●. Remember Fibonacci Numbers?

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n > 1. \end{cases}$$

Now, prove for all natural number n,

$$\sum_{i=0}^{n} iF_i = nF_{n+2} - F_{n+3} + 2$$

Let $P(n) =$ " $\sum_{i=0}^{n} iF_i = nF_{n+2} - F_{n+3} + 2$ ".

Base: $P(1)$ true b/c $(1)(F_1) = 1 = (1)(\overset{2}{F_3}) - \overset{3}{F_4} + 2$

$= 2 - 3 + 2 = 1$ ✓

I.H. Assume $P(k)$ true for some $k \in \mathbb{N}$.

I.S. By IH $\sum_{i=0}^{k} iF_i = kF_{k+2} - F_{k+3} + 2$

$\Rightarrow \sum_{i=0}^{k} iF_i + (k+1)(F_{k+1}) = kF_{k+2} - F_{k+3} + 2 + (k+1)(F_{k+1})$

$= k(F_{k+3} - F_{k+1}) - (F_{k+4} - F_{k+2}) + 2 + kF_{k+1} + F_{k+1}$

$= kF_{k+3} - F_{k+4} + F_{k+2} + F_{k+1} + 2$

$= kF_{k+3} - F_{k+4} + F_{k+3} + 2$

$= (k+1)(F_{k+3}) - F_{k+4} + 2$ ✓

Function

●. Prove that the following function $f:\mathbb{R}\to\mathbb{R}$ is bijective.

$$f(x)=(3(x+1)^3+5)^3$$

Claim $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=x^3$ is bijective.

INJ Let $x,y\in\mathbb{R}$.
Assume $g(x)=g(y)$
$\Rightarrow x^3=y^3$
$\Rightarrow x=y$ ✓

SURJ Let $b\in\mathbb{R}$.
Then take $\sqrt[3]{b}\in\mathbb{R}$
$g(\sqrt[3]{b})=(\sqrt[3]{b})^3=b$ ✓

Claim $h:\mathbb{R}\to\mathbb{R}$ by $h(x)=ax+b$ for real numbers a,b is bijective, where $a\neq 0$

INJ Let $x,y\in\mathbb{R}$
Assume $h(x)=h(y)$
$ax+b=ay+b$
$\Rightarrow x=y$

SURJ Let $c\in\mathbb{R}$.
Take $\frac{c-b}{a}\in\mathbb{R}$
$h\left(\frac{c-b}{a}\right)=a\left(\frac{c-b}{a}\right)+b=c$ ✓

Thus $h'(x)=x+1$, $h''(x)=3x+5$ is bijective,

So $f(x)=g(h''(g(h'(x))))$

Since f is a composition of bijections,
f is bijective.

Counting 2 Ways

4. Prove the following by counting 2 ways.

$$\sum_{i=0}^{n}\binom{n}{i}\cancel{99^i}\,100^i = \cancel{100^n}\,101^n$$

Let S = the set of ways to assign grades (from 0 to 100) to n concepts students.

Clearly, $|S| = 101^n$, so RHS counts S.

Let S_i = the set of ways to assign grades (from 0 to 100) to n concepts students such that $n-i$ students got perfect.

To form S_i, 1) Choose $n-i$ students who got perfect

$$\binom{n}{n-i} = \binom{n}{i}$$

2) For remaining i students, assign grades $0-99$

$$\Rightarrow 100^i$$

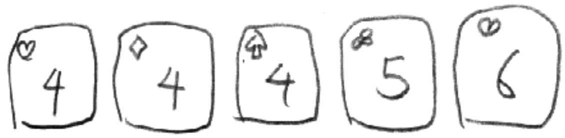
Clearly, $S_0, \ldots, S_n$ partition S.

Thus $|S| = \sum_{i=0}^{n}|S_i| = \sum_{i=0}^{n}\binom{n}{i}(100^i)$ so LHS counts S ✓

~~Counting~~ Probability

- What's the probability of a 5 card poker hand that has the following properties?

    - Has three cards of the same rank.
    - Includes all four suits.
    - Includes three different ranks total



example...

STEPS

1) Choose a rank. $\binom{13}{1}$

2) Choose 3 suits. $\binom{4}{3}$

3) Choose 2 other ranks. $\binom{12}{2}$

4) Of 4^2 ways to choose suits for the last two cards, 3^2 of them do not include unused suit.

So (7) ways

$$\frac{\overset{①}{\binom{13}{1}}\overset{②}{\binom{4}{3}}\overset{③}{\binom{12}{2}}\overset{④}{(7)}}{\binom{52}{5}}$$

↑ total # poker hand

Expectation

[redacted]. There are [redacted] 8 questions on a concepts ~~midterm~~ final. Suppose that you had no idea how to solve any of them. So your strategy for the exam was as follows:

1. Choose a problem at random.
2. Stare at it for 1 minute. Then repeat step 1.

What's the expected number of minutes until you will stare at **all** the questions?

X = the number of minutes until you will stare at all the questions.

X_i = The number of minutes until you stare at a new question when i question remains unstared

Then $X = \sum_{i=1}^{8} X_i$

NOTE: The probability of staring at a new question when i question remains unstared is $\frac{i}{8}$.

by Prop 9.29, the expected number of trials is $\frac{8}{i}$.

So $E[X] = \sum_{i=1}^{8} E[X_i] = \sum_{i=1}^{8} \frac{8}{i}$

$$= \frac{8}{1} + \frac{8}{2} + \frac{8}{3} + \frac{8}{4} + \frac{8}{5} + \frac{8}{6} + \frac{8}{7} + \frac{8}{8}$$

$$= 8 + 4 + \frac{8}{3} + 2 + \frac{8}{5} + \frac{4}{3} + \frac{8}{7} + 1 = 19 + \frac{96}{35}$$