MATRIX THEORY AUTUMN 2011: HOMEWORK 8

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Homework due at start of class on *Friday* 18 November.

Recall that if $x, y \in \mathbb{C}^n$ then

$$x.y = \sum_{i=1}^{n} x_i \overline{y_i}$$

and

$$\|x\|^2 = x.x$$

Some of these questions are about real linear algebra and some are about complex linear algebra. To avoid any confusion I have marked each question as “real” or “complex”.

(1) (Complex) Exactly as for $\mathbb{R}^n$, a set X of vectors in $\mathbb{C}^n$ is *orthonormal* if for all $x, y \in X$ we have $x.y = 1$ for $x = y$ and $x.y = 0$ for $x \neq y$.

Prove that if $v_1, \dots v_n$ enumerates an ON set in $\mathbb{C}^n$ then for every $v \in \mathbb{C}^n$ we have $v = \sum_i (v.v_i)v_i$. What would be wrong with the formula “$v = \sum_i (v_i.v)v_i$”?

(2) (Complex) Find an ON basis for the subset of $\mathbb{C}^4$ spanned by the vectors $(1, i, 0, 1)$, $(0, 2, -i, 3)$ and $(i, 1, -1, 2)$.

(3) (Real) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal linear map. Prove that T is either a rotation about the origin or a reflection in a line through the origin.

Hint: T is determned by Te_1 and Te_2.

(4) (Real) Prove that if n is odd then every $n \times n$ real matrix has at least one eigenvector.

Hint: The characteristic polynomial has odd degree. Ask Paul or consult the internet if you don't see why this is helpful.

(5) (Real) Let A be an orthogonal $n \times n$ matrix. Prove that every eigenvalue of A is either 1 or -1, and that $det(A)$ is 1 or -1.

(6) (Real) Let $v \in \mathbb{R}^3$ be a nonzero vector. Describe (with proof) in geometrical terms the orthogonal linear maps $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Tv = v$.

Hint: knowing what the orthogonal maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ are like may be useful.

(7) (Real) Describe (with proof) in geometrical terms all the orthogonal linear maps $T : \mathbb{R}^3 \to \mathbb{R}^3$.

Hint: Start by combining some of the previous exercises to prove that if R is the orthogonal map $R(x, y, z) = (-x, -y, -z)$, then one of the orthogonal maps T and $R \circ T$ fixes at least one nonzero vector v.