MATRIX THEORY HOMEWORK 5

(1) The *rowspace* of an $m \times n$ matrix A is the set of linear combinations of the rows of A. Prove that the rowspace of A is the set of row vectors of the form yA. Prove that if E is an invertible $m \times m$ matrix then the rowspace of A is equal to the rowspace of EA.

By elementary matrix arithmetic, if $Y = (\mu_1 \dots \mu_m)$ then YA is the row vector $\sum_i \mu_i R_i$ where R_i is the i row of A.

Since $Y(EA) = (YE)A$, the rowspace of EA is a subset of the rowspace of A for any E. If E is invertible then $ZA = (ZE^{-1})EA$, so that the rowpsace of A is a subset of the rowspace of EA.

(2) Let A be a square matrix. Prove that if A is invertible then the transpose A^T is invertible.

$AA^{-1} = 1$, so transposing $(A^{-1})^T A^T = 1^T = 1$. By basic facts about inverses A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

(3) Let A be an invertible square matrix. For $n > 0$ we already defined A^n to be the product of n copies of A. We let $A^0 = 1_{n \times n}$ and for $n > 0$ we define $A^{-n} = (A^{-1})^n$. Prove that for all integers m, n we have $A^{m+n} = A^m A^n$ and $(A^m)^n = A^{mn}$.

Step 1: For all $n \geq 0$, $A^n A^{-n} = 1$ and so $(A^n)^{-1} = A^{-n}$.

Proof by induction on n. Easy for $n = 0$. $A^{n+1} A^{-n-1} = AA^n A^{-n} A^{-1} = AA^{-1} = 1$, using that $A^n A^{-n} = 1$ by induction.

Step 2: For all m, $A^{m+1} = A^m A = AA^m$.

There are 3 cases: $m < 0$, $m = 0$ and $m > 0$. Each is immediate from the definitions.

Step 3: For all m and all $n \geq 0$, $A^m A^n = A^{m+n}$.

Proof: By induction on n for all m simultaneously (that is to say the induction hypothesis asserts of n that "for all m we have $A^m A^n = A^{m+n}$"). For $n = 0$: $A^m A^0 = A^m = A^{m+0}$. For the induction step: $A^m A^{n+1} = A^m A A^n = A^{m+1} A^n = A^{m+1+n} = A^{m+n+1}$, where we used Step 2 to see $A^m A = A^{m+1}$ and the induction hypothesis for $m+1$ and n to see $A^{m+1} A^n = A^{m+1+n}$.

Step 4: For all m and n, $A^{m+n} = A^m A^n$.

Proof: Step 3 covers it unless m, n are both negative, and in this case we have $A^{m+n} = (A^{-n-m})^{-1} = (A^{-n} A^{-m})^{-1} = (A^{-m})^{-1} (A^{-n})^{-1} = A^m A^n$ by Step 1, Step 3, the general fact that $(AB)^{-1} = B^{-1} A^{-1}$ for invertible A, B and Step 1 again.

Step 5: For all m and all $n \geq 0$, $A^{mn} = (A^m)^n$.

Proof: By induction on n. $n = 0$ is easy. For the induction step: $(A^m)^{n+1} = (A^m)^n A^m = A^{mn} A^m = A^{mn+m}$, by the definition of matrix powers, the induction hypothesis and Step 4.

Step 6: For all m and n, $A^{mn} = (A^m)^n$.

Proof: we are done by Step 5 unless $n < 0$. In this case $(A^m)^n = ((A^m)^{-n})^{-1} = (A^{-mn})^{-1} = A^{mn}$ by Step 1, Step 5 and Step 1 again.

(4) Let A be an $m \times m$ matrix, and for $t > 0$ let X_t be the columnspace of A^t. Prove that $X_{t+1} \subseteq X_t$ for all $t > 0$. Prove that there is a number $T > 0$ such that $X_t = X_T$ for all $t \geq T$.

For any column vector Y, $A^{t+1}Y = (A^t A)Y = A^t(AY)$, so that the columnspace of A^{t+1} is a subspace of the columnspace of A^t.

To finish, it is useful to prove a couple of facts about spaces and dimensions.

Fact: Let X, Y be spaces of column vectors of height m, and let $X \subseteq Y$. Let $s = dim(X)$ and $t = dim(Y)$. Then $s \leq t$, and if $s = t$ then $X = Y$.

Proof of fact: Let B be a basis for X, so that $|B| = s$. Now $B \subseteq X \subseteq Y$, so that B is an independent subset of Y. By a general fact about independent sets and bases, there is a basis C for Y such that $B \subseteq C$. So $s = |B| \leq t = |C|$, and if $s = t$ then $B = C$ and so $X = span(B) = span(C) = Y$.

Returning to the problem, we see by the fact above that $dim(X_t)$ is decreasing with t, so that there is some T such that $dim(X_t)$ is constant for $t \geq T$. In particular for $t \geq T$ we have $X_t \subseteq X_T$ and $dim(X_t) = dim(X_T)$, so by the fact again $X_t = X_T$.

Note: With more work we can show that $T \leq m$, can you see why?