MATRIX THEORY AUTUMN 2011: HOMEWORK 1

JAMES CUMMINGS

Homework due at start of class on Wed 7 September.

1. Sets and logic

(1) Using truth tables or otherwise, prove that:
  (a) "A implies B" is logically equivalent to "not-A or B".
  (b) "not-(A or B)" is logically equivalent to "not-A and not-B".
  (c) "not-(A implies B)" is logically equivalent to "A and not-B".

(2) Recall that the basic operations on sets are intersection, union and set difference. Let X be a set, let $\phi(x)$ and $\psi(x)$ be properties that elements $x \in X$ may have. Let $A = \{x \in X : \phi(x)\}$ and let $B = \{x \in X : \psi(x)\}$. Describe (with proof) each of the following sets in terms of the basic operations:
  (a) $\{x \in X : \text{not } \phi(x)\}$.
  (b) $\{x \in X : \phi(x) \text{ or } \psi(x)\}$.
  (c) $\{x \in X : \phi(x) \text{ implies } \psi(x)\}$.

  For example: $\{x \in X : \phi(x) \text{ and } \psi(x)\} = A \cap B$.

2. Linear equations

(1) Consider linear systems in two variables x, y of the general form

$$\begin{aligned} x + 2y &= 1 \\ ax + y &= b \end{aligned}$$

where a, b are constants.

  For which values of a, b does this system have
  (a) Exactly one solution?
  (b) No solutions?
  (c) Infinitely many solutions?

3. Matrices and vectors

(1) A function T from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called a *linear transformation* if there exists a 2×2 matrix A such that for all x and y we have $T(x, y) = (x_1, y_1)$, where

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

  (a) Prove that if T is a linear transformation then there is exactly one matrix A as above.
  (b) Prove that a rotation by ϕ radians about the origin $(0, 0)$ is a linear transformation and describe the corresponding matrix.

(c) Prove that the translation $(x, y) \mapsto (x+1, y+1)$ is *not* a linear transformation.
(d) Consider the line L with equation $y = ax$. Is reflection in L a linear transformation? If yes find the corresponding matrix.
(e) Let A_ϕ be the matrix for rotation by ϕ. Prove that $A_\phi A_\psi = A_{\phi+\psi}$.

(2) Let X be the set of 2×2 real matrices. Prove or give a counterexample to each of the following statements:
(a) $A + B = B + A$ for all $A, B \in X$.
(b) $AB = BA$ for all $A, B \in X$.
(c) $A^2 = 0$ implies $A = 0$ for all $A \in X$.
(d) $AA^2 = A^2 A$ for all $A \in X$.

4. Maple

This section is *not for credit*, *you may collaborate* and *you need not hand anything in.* But you still should do it because probably none of this will be true of the Maple sections in later homeworks.

(a) Either install Maple (CMU has a license) on your personal computer or find a public workstation which has it installed.
(b) Start Maple.
(c) Open up a new Maple worksheet.
(d) Load the "LinearAlgebra" package with the command `with(LinearAlgebra);`
(e) Locate the online help for the LinearAlgebra package.
(f) Create matrices

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & 7 \\ 6 & 8 \end{pmatrix}.$$

(g) Use Maple to compute $A + B, AB, BA, A^{100}$.
(h) Export your results to a PDF file.