FIELD THEORY HOMEWORK SET IV

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You may collaborate on this homework set, but must write up your solutions by yourself. Please contact me by email if you are puzzled by something, would like a hint or believe that you have found a typo.

(1) An *ordered field* is a field F together with a set $P \subseteq F$ of elements such that (defining $-P = \{-a : a \in P\}$)
   (a) P is closed under $+$ and $\times$
   (b) $P \cap -P = \{0\}$.
   (c) $P \cup -P = F$.

   Intuitively, you can think of P as the set of field elements which have been designated as non-negative.

   Prove that
   (a) There is a unique set $P \subseteq \mathbb{R}$ such that $(\mathbb{R}, P)$ is an ordered field.
   (b) There is no set $P \subseteq \mathbb{C}$ such that $(\mathbb{C}, P)$ is an ordered field.
   (c) There are at least two sets $P \subseteq \mathbb{Q}(\sqrt{2})$ such that $(\mathbb{Q}(\sqrt{2}), P)$ is an ordered field.
   (d) If (F, P) is an ordered field then F has characteristic zero and -1 is not a sum of squares in F.

(2) Let F be a field extending the field E and let $a \in F$ be algebraic over E. Show that the following are equivalent:
   (a) a is separable over E.
   (b) There is a field F' extending E such that there exist $[E(a) : E]$ distinct monomorphisms $\sigma : E(a) \to F'$ with $\sigma \restriction E = id$.

(3) Let $f = x^4 - 2$.
   (a) Show that f is irreducible in $\mathbb{Q}[x]$.
   (b) Find the complex roots of f, and show that f splits over the complex numbers.
   (c) Let E be the subfield of $\mathbb{C}$ generated by the roots of f (so that E is a splitting field for $\mathbb{Q}$). Find $[E : \mathbb{Q}]$.
   (d) Compute the group $Aut(E/\mathbb{Q})$, and describe how each element of this group permutes the roots of f.
   (e) Find all the subgroups of $Aut(E/\mathbb{Q})$ and their fixed fields.

(4) Let F have characteristic p. Use the binomial theorem to show that the map $a \mapsto a^p$ is a monomorphism from F to F. Show that if F is finite this map is an automorphism of F, and in that case describe its fixed field.