FIELD THEORY HOMEWORK SET II

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You may collaborate on this homework set, but must write up your solutions by yourself. Please contact me by email if you are puzzled by something, would like a hint or believe that you have found a typo.

(1) Let F be a field, and let $Aut(F)$ be the set of automorphisms of F.
   (a) Show that $Aut(F)$ forms a group under composition.
   (b) Show that if $X \subseteq Aut(F)$ and we define $Fix(X) = \{a \in F : \forall \sigma \in X \; \sigma(a) = a\}$ then $Fix(X)$ is a subfield of F.
   (c) Show that $Fix(X) = Fix(\langle X \rangle)$, where as usual $\langle X \rangle$ is the subgroup generated by X.

(2) Let $E = \mathbb{Z}/2\mathbb{Z}$.
   (a) Find an irreducible element in $E[x]$ of degree 3.
   (b) Construct a finite field with 8 elements.
   (c) Find all the subfields of the field you just constructed, and also find its automorphism group.

(3) What is the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over each of the fields
   (a) $\mathbb{Q}$?
   (b) $\mathbb{Q}(\sqrt{2})$?
   (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$?

(4) Let $\mathbb{Z}$ be a subring of S and let $a \in S$. a is *integral* over $\mathbb{Z}$ iff there is $g \in \mathbb{Z}[x]$ such that g is monic and $g(a) = 0$.

   Show that a is integral over $\mathbb{Z}$ iff $(\mathbb{Z}[a], +)$ is a finitely generated abelian group, where as usual $\mathbb{Z}[a] = \{f(a) : f \in \mathbb{Z}[x]\}$ is the least subring of S containing $\mathbb{Z} \cup \{a\}$.

(5) Prove that if $f \in \mathbb{R}[x]$ has odd degree then f has at least one root. Hint: use calculus.

(6) Prove that $\mathbb{Z}[x]$ is not a PID. (Harder) What are the prime ideals?

(7) Prove that $\mathbb{C}$ is a vector space over $\mathbb{R}$. What is its dimension? Find a basis. For each $z \in \mathbb{C}$ prove that the map which takes a to za is linear; also find its trace and determinant.