FINAL FOR FIELD THEORY

JAMES CUMMINGS

You may consult any books or papers you wish. Please make a note of any works consulted. You may not collaborate but may consult with me if you have any questions. This final is due by email before 11:59pm on May 10.

(1) (20 points total) Let F be a subfield of $\mathbb{C}$ with $[F : \mathbb{Q}]$ finite. Prove that F contains only finitely many roots of unity (recall that a root of unity is a complex number ζ such that $\zeta^n = 1$ for some $n > 0$).

(2) (60 points total) Let K be the subfield of $\mathbb{C}$ generated by the complex roots of $x^5 + 2$.
   - (a) (5 points) Find $[K : \mathbb{Q}]$.
   - (b) (10 points) Describe the group $G = Aut(K/\mathbb{Q})$.
   - (c) (10 points) Give an explicit isomorphism between G and a subgroup H of S_5.
   - (d) (25 points) Find all the subgroups of G and all the fields intermediate between $\mathbb{Q}$ and K. Which ones are Galois extensions of $\mathbb{Q}$? Identify $K \cap \mathbb{R}$.
   - (e) (10 points) Find all the roots of unity in K.

(3) (25 points total)
   - (a) (15 points) Let F be a finite degree extension of E, and assume F is a Galois extension of E. Say that F_1 is an "abelian extension of E" if F_1 is a subfield of F containing E, F_1 is a Galois extension of E and $Aut(F_1/E)$ is abelian. Show that there is a unique abelian extension F_{max} of E which contains all the abelian extensions. Hint: use the fundamental theorem to see which group this field must correspond to!
   - (b) (10 points) Let $E = \mathbb{Q}$ and K the field considered in the preceding question. Find K_{max}.

(4) (25 points total) Prove that for any finite group G there is a pair of fields E and F such that F is a Galois extension of E and $Aut(F/E)$ is isomorphic to G. Hint: any finite group is isomorphic to a subgroup of S_n.

(5) (10 points total) Find the minimal polynomial of $1 + \sqrt{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.

(6) (40 points total) Let α be a real irrational number which is algebraic over $\mathbb{Q}$, let f be the minimal polynomial of α over $\mathbb{Q}$ and let n be the degree of f.
   (a) (10 points) Prove that $f'(\alpha) \neq 0$, and that $f(r) \neq 0$ for all rational r.
   (b) (10 points) Prove that there is a constant $D > 0$ such that for every $p, q \in \mathbb{Z}$ with $q > 0$, $|f(p/q)| \geq D/q^n$. Hint: you may find it easier to think about multiplying f by some integer to get a polynomial in $\mathbb{Z}[x]$.
   (c) (10 points) Prove there is a constant $C > 0$ such that for every $p, q \in \mathbb{Z}$ with $q > 0$, $|\alpha - p/q| \geq C/q^n$. Hint: the mean value theorem.
   (d) (10 points) Prove that $\sum_{n=0}^{\infty} 10^{-n!}$ is transcendental over $\mathbb{Q}$.

(7) (50 points total) Let $F_0 = \mathbb{Q}$. For each $n > 0$ let $\alpha_n = \sqrt[2^n]{2}$, and let $F_n = \mathbb{Q}(\alpha_n)$.
   (a) (10 points) Prove that $F_n \subseteq F_{n+1}$ and find $[F_{n+1} : F_n]$.
   (b) (20 points) Let $F_\infty = \bigcup_n F_n$. Find a basis for F_∞ as a vector space over F_0, and use it to show that F_∞ has countably infinite dimension as an F_0-vector space.
   (c) (20 points) Is $Aut(F_\infty/F_0)$ finite, countably infinite or uncountably infinite?