(1) Let R be unital. The *units* of R are the elements which have a multiplicative inverse. They form a group $U(R)$ under multiplication.
(2) A *left (resp right) ideal* of R is an additive subgrup $I \leq (R, +)$ such that $RI \subseteq I$ (resp $IR \subseteq I$). A *two sided ideal* is an additive subgroup with both these properties.
(3) If I is a two sided ideal then the quotient abelian group R/I has a natural ring structure, where $(a + I)(b + I) = ab + I$.
(4) A *ring HM* is a map $\phi : R \to S$ which preserves both $+$ and $\times$, that is $\phi(a+b) = \phi(a)+\phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$. The *kernel* is $\{r : \phi(r) = 0\}$ and the *image* is $\{\phi(r) : r \in R\}$.
(5) The image of a ring under a HM is a ring, the kernel of a HM is a two sided ideal.
(6) If $\phi : R \to S$ is a HM then $R/ker(\phi) \simeq im(\phi)$.
(7) For the rest of this handout R is a commutative ring with 1. An ideal I is *prime* iff $I \neq R$ and $ab \in I$ implies $a \in I$ or $b \in I$, and *maximal* if $I \neq R$ and I is maximal under inclusion among proper ideals (that is $I \subseteq J \subseteq R$ implies $J = I$ or $J = R$ for any ideal J).
(8) I is prime iff R/I is an ID and *maximal* iff R/I is a field. In particular R is an ID iff the zero ideal is prime, and is a field iff it is maximal.
(9) If R is an ID we can embed it in an essentially unique way into a field F such that every element of F is of form ab^{-1} for $a, b \in R$. This is called the *field of fractions*.
(10) Two nonzero elements of an ID are *associates* iff each divides the other, equivalently one is a unit times the other. This is an ER.
(11) Let R be an ID. Then $a \in R$ is *irreducible* iff a is a nonzero nonunit and $a = bc$ implies one of b, c is a unit (note that b is a unit iff c is an associate of a). a is *prime* iff a is a nonzero nonunit and Ra is a prime ideal, that is $a|bc$ implies that $a|b$ or $a|c$. Prime elements are irreducible.
(12) R is a *unique factorisation domain (UFD)* iff R is an ID and every element has a factorisation into irreducibles, which is unique up to order and associates. In a UFD irreducible elements are prime.
(13) R is a *principal ideal domain (PID)* iff R is an ID and every ideal is *principal* (that is of form Ra for some a). A PID is a UFD. In a PID Ra is maximal for a irreducible, and if R is not a field the converse holds.

(14) R is a *Euclidean domain* iff R is an ID and there is a *Euclidean function* ϕ from R to $\mathbb{N}$. That is for all a and all $b \neq 0$ there exist q, r such that $a = bq + r$ and either $r = 0$ or $\phi(r) < \phi(b)$. Euclidean domains are PIDs. $\mathbb{Z}$ is Euclidean, and so is $F[x]$ for F any field.