CA LECTURE 9

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Recall that every ideal of $S^{-1}R$ is an extension of some ideal of R. Now we analyse which ideals of R are contractions.

Let I be an ideal of R. Then $I^e = \{a/s : a \in I, s \in R\}$. So if $a \in I^{ec}$ then $a/1 = b/s$ for some $b \in I$ and $s \in S$, and (by the definition of the ER defining $S^{-1}R$) there is $t \in S$ such that $t(b - as) = 0$. It follows that $a(st) = bt \in I$, so $I^{ec} \subseteq \{a : \exists u \in S \; au \in I\}$. On the other hand if $au \in I$ for some $u \in S$ then $a/1 = (au)/u \in I^e$, so that $a \in I^{ec}$. In conclusion

$$I^{ec} = \{a \; \exists u \in S \; au \in I\}.$$

Now we ask which ideals extend to $S^{-1}R$. Easily

$$I^e = S^{-1}R \iff 1 \in I^e \iff 1 \in I^{ec} \iff I \cap S \neq 0,$$

so the ideals with non-trivial extensions are the ones avoiding S.

By the general theory of extension and contraction, I is a contraction of an ideal in $S^{-1}R$ iff $I^{ec} \subseteq I$. By the calculation above this amounts to saying

$$\forall a \in R \, \forall u \in S \; au \in I \implies a \in I,$$

equivalently going to R/I

$$\forall a \in R \, \forall u \in S \; (a + I)(u + I) = 0 \implies a + I = 0,$$

So if $\bar{S} = \{u + I : s \in S\}$ then we see that $I = I^{ec}$ iff $\bar{S}$ contains no zero-divisors in R/I.

Recall that we have a dichotomy for $S^{-1}R$ when R is an ID: EITHER $0 \in S$ and $S^{-1}R$ is the zero ring, OR $S^{-1}R$ is isomorphic to the subring of the field of fractions of R consisting of quotients a/s where $a \in R$, $s \in S$. In the latter case $S^{-1}R$ is an ID.

Theorem 1. *Let $S \subseteq R$ be a MC set in the ring R. Then there is a 1-1 inclusion preserving correspondence between prime ideals of $S^{-1}R$ and prime ideals of R which are disjoint from S.*

Proof. Let Q be a prime ideal of $S^{-1}R$. All ideals of $S^{-1}R$ are extensions so $Q = Q^{ce}$. A contraction of a prime ideal is prime so Q^c is prime, and $Q \neq S^{-1}R$ so that $Q^c \cap S = \emptyset$. So every prime ideal of $S^{-1}R$ is the extension of a prime ideal disjoint from S.

It remains to show that for every prime ideal P of R which is disjoint from S, P is a contraction (that is $P = P^{ec}$) and $P^e = S^{-1}P$ is prime in $S^{-1}R$. We use HW6 Q2 which tells us that if $\bar{S} = \{u + P : u \in S\}$ then

$$\bar{S}^{-1}R/P \simeq S^{-1}R/S^{-1}P.$$

The LHS is a ring of fractions of the ID R/P, and $0 \notin \bar{S}$ since P is disjoint from S. Hence the LHS is an ID, so that $S^{-1}P$ is prime in $S^{-1}R$. Also since R/P is an ID it has no nonzero zero-divisors, so $\bar{S}$ contains no zero-divisors and thus $P = P^{ec}$. □

Most important special case: P is prime in R and $S = R \setminus P$. We write R_P for $S^{-1}R$ in this case. We get a 1-1 inclusion-preserving correspondnece between prime ideals of R_P and prime ideals of R contained in P. In particular P^e is the unique maximal ideal of R_P, which is therefore a local ring.

A sample application:

Theorem 2. *Let $\phi : R \to S$ be a HM. Then any prime ideal P in R which is a contraction of some ideal in S is a contraction of a prime ideal.*

Proof. As P is a contraction, we have[1] $P = P^{ec}$.

Let $X = R \setminus P$ and $Y = \phi[X]$, so that Y is a MC subset of S. We claim that P^e is disjoint from Y; to see this observe that if $b \in Y$ then $b = \phi(a)$ for some $a \in X$, and if $b \in P^e$ then $a \in P^{ec} = P$.

Let J be the extension of P^e in $Y^{-1}S$, let $K \supseteq J$ be a prime ideal of $Y^{-1}S$ and let Q be the contraction of K in S. Then Q is prime with $P^e \subseteq Q \subseteq S \setminus Y$, hence $P \subseteq Q^c \subseteq P$ as required. □

Remark: our old way of building prime ideals boils down to an argument that maximal ideals of $S^{-1}R$ correspond to prime ideals of R. We could have used that idea here.

Modules of fractions: let M be an R-module and S MC in R. We define an $S^{-1}R$-module $S^{-1}M$ as follows: we have an ER on $M \times S$ in which (m_1, s_1) is equivalent to (m_2, s_2) iff there is $u \in S$ such that $u(s_1 m_2 - s_2 m_1) = 0$. We let m/s be the class of (m, s) and $S^{-1}M$ the set of all classes. Then we define addition by the formula $m_1/s_1 + m_2/s_2 = (m_1 s_2 + m_2 s_1)/(s_1 s_2)$ and scalar multiplication by the formula $a/s m/t = am/st$. It is routine to check that this works.

[1] P^e is not necessarily the prime ideal we want, but note that if $P = Q^c$ then $P^e = Q^{ce} \subseteq Q$