21-715 COMMUTATIVE ALGEBRA LECTURE NOTES 9/7/05

JONATHAN GROSS

Definition 1. *Let R be a ring, suppose I, J are ideals of R. Define*

$$(I : J) = \{r \in R : rJ \subseteq I\}$$

and in particular for $x \in R$, define

$$(I : x) = \{r \in R : rx \in I\}$$

Claim 1. *Let Q be P-primary.*

(1) *If $x \in Q$, then $(Q : x) = R$.*
(2) *If $x \notin Q$, then $(Q : x)$ is P-primary.*
(3) *If $x \notin P$, then $(Q : x) = Q$.*

Proof. (1) Trivial.
(2) Let $x \notin Q$, so $Q \subseteq (Q : x) \subseteq P$ by primaryness. Then,

$$P = \sqrt{Q} \subseteq \sqrt{(Q : x)} \subseteq \sqrt{P} = P$$

so $\sqrt{(Q : x)} = P$. Now, suppose $yz \in (Q : x)$ so $(xy)z \in Q$. By primaryness, either $xy \in Q$, which would imply that $y \in (Q : x)$, or $z \in P = \sqrt{(Q : x)}$.
(3) Let $x \notin P$, note $Q \subseteq (Q : x)$. Take some $y \in (Q : x)$, so $xy \in Q$. As $x \notin \sqrt{Q}$, by primaryness $y \in Q$.

□

Until further notice: Let I be a decomposable ideal with $I = Q_1 \cap \ldots \cap Q_n$ for Q_i primary. Let $P_i = \sqrt{Q_i}$ and suppose this representation is irredundant, that is, $i \neq j$ implies $P_i \neq P_j$ and $Q_i \not\supseteq \bigcap_{j \neq j} Q_j$.

Claim 2. *The set of P_is is the set of prime ideals P such that there exists $x \in R$ with $P = \sqrt{(I : x)}$.*

Note that this tels us that the set of P_is does not depend on the set of Q_is.

Proof. Note that for $x \in R$,

$$(I : x) = \bigcap_i (Q_i : x) = \bigcap_{i:\, x \notin Q_i} (Q_i : x)$$

so taking radicals,

$$P = \sqrt{(I : x)} = \bigcap_{i\colon x \notin Q_i} \sqrt{(Q_i : x)} = \bigcap_{i\colon x \notin Q_i} P_i$$

Since P is prime, we must have $P = P_i$ for some i.

Conversely, choose x such that $x \in \bigcap_{j \neq i} Q_j$ but $x \notin Q_i$, so $\sqrt{(I : x)} = P_i$. □

We say P_i is **minimal** iff there is no j such that $P_j \subsetneq P_i$. Otherwise, we say P_i is **embedded**.

Goal: The set of Q_i corresponding to minimal P_i is determined by I.

Claim 3. *The contraction of a primary ideal is primary.*

Theorem 1. *Let R be a ring, let $S \subseteq R$ be multiplicatively closed, Q a primary ideal, and $P = \sqrt{Q}$.*

1. *If $S \cap P \neq \emptyset$, then $S \cap Q \neq \emptyset$ so $Q^e = S^{-1}R$.*
2. *If $S \cap P = \emptyset$, then $Q^{ec} = Q$ and Q^e is P^e-primary.*

So if P is prime and $P \cap S = \emptyset$, then there is an inclusion-preserving bijection between the P-primary ideals of R and the P^e-primary ideals of $S^{-1}R$, given explicitly by $Q \mapsto Q^e$.

Proof. (2): Suppose $S \cap P = \emptyset$. To show $Q^{ec} = Q$, let $\overline{S}$ be the image of S under the quotient map $R \to R/Q$. We need to show that $\overline{S}$ has no zero-divisors. Suppose for $s \in S$, $s + Q$ is a zero-divisor in R/Q. As Q is primary, $s+Q$ is nilpotent, so there is $n \in \mathbb{N}$ such that $s^n + Q = 0$ so $s^n \in Q$, contradicting the fact that $S \cap P = \emptyset$.

By an old homework, we know that $S^{-1}R/S^{-1}Q \simeq S^{-1}(R/Q)$. The following exercise would prove that $S^{-1}Q$ is primary in $S^{-1}R$:

Exercise: Let A be a ring in which all zero-divisors are nilpotent, let $T \subseteq A$ be multiplicatively closed. Then in $T^{-1}A$ all zero-divisors are nilpotent.

Exercise: $\sqrt{Q^e} = P^e$. □

We now return to the aforementioned situation.

Claim 4.

$$\bigcup_{i=1}^{n} P_i = \{x : I \subsetneq (I : x)\}$$

Proof. Suppose $x \notin \bigcup P_i$. For all i, $x \notin P_i$ so $(Q_i : x) = Q$ by the previous result. Thus, $(I : x) = \bigcap_i (Q_i : x) = \bigcap_i Q_i = I$.

Now suppose $x \in P_i$. Let n be minimal such that $x^n \in Q_i$. Choose y such that $y \in \bigcup_{j \neq i} Q_j$ but $Y \notin Q_i$. Note that $x^n y \in I$ and $x^{n-1}y \notin Q_i$. Thus $x^{n-1}y \in (I : x) \setminus I$. □

Now, let P_i be a minimal prime, let $S = R \setminus P_i$. By minimality, for all $j \neq i$, $P_j \cap S \neq \emptyset$.

Exercise: For any ring A, multiplicatively closed $T \subseteq A$, and ideals $J_1, \dots, J_n$ of A, we have

$$S^{-1}(J_1 \cap \ldots \cap J_n) = S^{-1}J_1 \cap \ldots \cap S^{-1}J_n$$

This implies $I^e = S^{-1}I = S^{-1}Q_i = Q_i^e$, so $I^{ec} = Q_i^{ec} = Q_i$ by the previous result. This accomplished our goal.