COMMUTATIVE ALGEBRA HW 4

JC

Due in class Mon 12 September.

(1) (A and M III.6) Let $A \neq \{0\}$ be a ring. let Σ be the set of all multiplicatively closed sets S with $0 \notin S$. Show that
  (a) Σ has maximal elements.
  (b) S is maximal in Σ iff $A \setminus S$ is a minimal prime ideal.

Clearly $\{1\}$ is in Σ and Σ is closed under union of chains. Now use ZL to see there exist maximal elements.

Suppose first that S is maximal and let $P = A \setminus S$. Let r be a ring element. If $r \in P$ then the least multiplicatively closed set containing r and S is

$$S \cup rS \cup r^2 S \cup \dots$$

which must contain 0 by maximality of S, that is for some $n > 0$ and $s \in S$ we have $r^n s = 0$. Conversely if $r^n s = 0$ for some $n > 0$ and $s \in S$ then $r \in P$ as S is multiplicatively closed and $0 \notin S$.

We claim P is an ideal. Clearly $0 \in P$. If a and b are in P find $m, n > 0$ and $s, t \in S$ such that $a^m s = b^n t = 0$. Then $(a+b)^{m+n-1} st = 0$ so that $a + b \in P$. Also for any r we have $(ra)^m s = 0$ so that $ra \in P$. Since $1 \in S$ we see that $1 \notin P$. So P must be prime as S is multiplicatively closed, and must be a minimal prime by the maximality of S.

Finally suppose that P is a minimal prime and let $S = A \setminus P$. Certainly $S \in \Sigma$. If S is not maximal in Σ find $S \subsetneq T$ with T maximal in Σ, then by the work we just did $A \setminus T$ is a prime ideal strictly contained in P, contradiction!

(2) (A and M I.12) Recall that a *local ring* is a ring with a unique maximal ideal, equivalently a ring where the non-units form an ideal. Recall also that e is *idempotent* iff $e^2 = e$. Show that in a local ring the only idempotents are 0 and 1.

Let A be a local ring and M be the maximal ideal of non-units. If $E = e + M$ then $E^2 = E$ in the field A/M, so $E = 0$ or $E = 1$. That is either $e \in M$ or $1 - e \in M$. Now $1 = e + (1 - e) \notin M$ so either $1 - e \notin M$ or $e \notin M$, that is to say that e or $1 - e$ is a unit.

Now $e(1-e) = 0$ so if e is a unit then $1-e = 0$, while if $1-e$ is a unit then $e = 0$.

(3) True or false? If $R[x]$ is Noetherian then R is Noetherian.

$R[x]/(x) \simeq R$. True.

(4) A *topological space* is a set X equipped with a collection $\mathcal{O}$ of subsets of X (the "open sets") satisfying the axioms:

(a) $\emptyset$ and X are in $\mathcal{O}$.

(b) The intersection of any two elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

(c) The union of any set of elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

(A homely example: Let (X, d) be a metric space and $\mathcal{O}$ be the set of open sets for this metric)

Let R be an arbitrary ring and let $Spec(R)$ (the *spectrum of* R) be the set of prime ideals of R. For each $a \in R$ let $O_a = \{P \in Spec(R) : a \notin P\}$. Define $\mathcal{O}$ to be the set of all subsets of $Spec(R)$ which are unions of sets of the form O_a, or more explicitly $X \in \mathcal{O}$ iff for every $P \in X$ there is $a \in R$ so that $P \in O_a \subseteq X$.

Show that this choice of $\mathcal{O}$ makes $Spec(R)$ into a topological space.

Every prime ideal contains 0 and fails to contain 1 so $O_1 = Spec(R)$ and $O_0 = \emptyset$. It is immediate from the definition that $\mathcal{O}$ is closed under unions. To finish we need to show that $O_a \cap O_b$ is a union of sets of the form O_c (the rest is elementary set theory).

Now for P prime we know that $a \notin P$ and $b \notin P$ implies $ab \notin P$, and for any ideal $a \in P$ or $b \in P$ implies $ab \in P$. So for P prime in fact $ab \notin P$ iff $a \notin P$ and $b \notin P$, that is $O_a \cap O_b = O_{ab}$.